CHAPTER

10

Trigonometric Identities

10.1 Introduction

In this section, we shall first establish the **fundamental law of trigonometry** before discussing the **Trigonometric Identities**. For this we should know the formula to find the distance between two points in a plane.

10.1.1 The Law of Cosine

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points. If " d " denotes the distance between them,

then, $d = |\overline{PQ}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

i.e., square root o f the sum of square the difference of x-coordinates and square the difference o f y-coordinates.

Example 1: Find distance between the following points:

i) $A(3,8)$, $B(5,6)$

ii) $P(\cos x, \cos y)$, $Q(\sin x, \sin y)$

Solution:

i) Distance $= |\overline{AB}| = \sqrt{(3-5)^2 + (8-6)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$$= \sqrt{(5-3)^2 + (6-8)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

ii) Distance $= \sqrt{(\cos x - \sin x)^2 + (\cos y - \sin y)^2}$

$$= \sqrt{\cos^2 x + \sin^2 x - 2\cos x \ \sin x + \cos^2 y + \sin^2 y - 2\cos y \ \sin y}$$

$$= \sqrt{2 - 2\cos x \sin x - 2\cos y \ \sin y}$$

$$= \sqrt{2 - 2(\cos x \sin x + \cos y \ \sin y}$$

10.1.2 Fundamental Law of trigonometry

Let α and β any two angles (real numbers), then

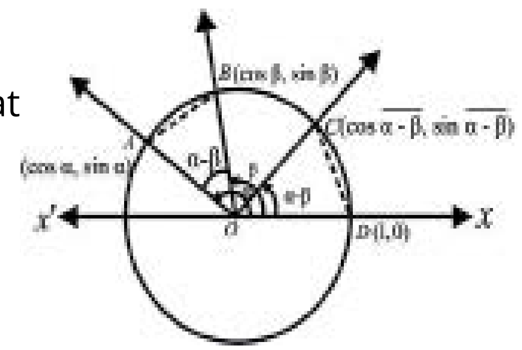
$$\boxed{\cos(\alpha - \beta) \ = \cos\alpha \ \cos\beta + \sin\alpha \ \sin\beta}$$

which is called the **Fundamental Law of Trigonometry**.

Proof: For our convenience, let us assume that α > β > 0.

Consider a unit circle with centre at origin O.

Let terminal sides o f angles α and β cut the unit circle at A and B respectively.



Evidently $\angle AOB = \alpha - \beta$

Take a point C on the unit circle so that

$$\angle XOC = m\angle AOB \ = \ \alpha - \beta.$$

Join A,B and C,D.

Now angles α β and α − β are in standard position.

∴ The coordinates o f A are (cos α, sin α)

the coordinates o f B are (cos β, sin β)

the coordinates o f C are $(\cos\overline{\alpha-\beta}, \sin\overline{\alpha-\beta})$

and the coordinates o f D are (1, 0).

Now ΔAOB and ΔCOD are congruent. [(SAS) theorem]

$\therefore \quad |AB| = |CD|$

$\Rightarrow \quad |AB|^2 = |CD|^2$

Using the distance formula, we have:

$$(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = [(\cos(\alpha-\beta) - 1]^2 + [\sin(\alpha-\beta) - 0]^2$$

$$\Rightarrow \quad \cos^2\alpha + \cos^2\beta - 2\cos\alpha \ \cos\beta + \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta$$

$$= \cos^2(\alpha-\beta) + 1 - 2\cos(\alpha-\beta) + \sin^2(\alpha-\beta)$$

$$\Rightarrow \quad 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \ = \ 2 - 2\cos(\alpha-\beta)$$

Hence $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

Note: Although we have proved this law for $\alpha > \beta > 0$. it is true for all values of α and β

Suppose we know the values of sin and cos of two angles α and β. we can find $\cos(\alpha - \beta)$ using this law as explained in the following example:

Example 1: Find the value of $\cos\frac{\pi}{12}$.

Solution: **As** $\frac{\pi}{12} = 15^\circ = 45^\circ - 30^\circ = \frac{\pi}{4} - \frac{\pi}{6}$

$$\therefore \cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

10.2 Deductions from Fundamental Law

1) We know that:

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Putting $\alpha = \frac{\pi}{2}$ in it, we get

$$\cos\left(\frac{\pi}{2}-\beta\right) = \cos\frac{\pi}{2}\cos\beta + \sin\frac{\pi}{2}\sin\beta$$

$$\Rightarrow \cos\left(\frac{\pi}{2}-\beta\right) = 0.\cos\beta + 1.\sin\beta \qquad \left(\because \cos\frac{\pi}{2} = 0, \sin\frac{\pi}{2} = 1\right)$$

$$\therefore \boxed{\cos\left(\frac{\pi}{2}-\beta\right) = \sin\beta}$$

2) We know that:

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Putting $\beta = -\frac{\pi}{2}$ in it, we get

$$\cos\left[\alpha - \left(-\frac{\pi}{2}\right)\right] = \cos\alpha.\cos\left(-\frac{\pi}{2}\right) + \sin\alpha\sin\left(-\frac{\pi}{2}\right)$$

$$\Rightarrow \cos\left(\alpha+\frac{\pi}{2}\right) = \cos\alpha.0 + \sin\alpha.(-1) \qquad \left\{\begin{array}{l}\sin\left(-\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1 \\ \cos\left(-\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0\end{array}\right.$$

$$\therefore \boxed{\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha} \qquad \text{(ii)}$$

3) We known that

$$\cos\left(\frac{\pi}{2}-\beta\right) = \sin\beta \qquad \text{[(i) above]}$$

Putting $\beta = \frac{\pi}{2} + \alpha$ in it, we get

$$\cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2}+\alpha\right)\right] = \sin\left(\frac{\pi}{2}+\alpha\right)$$

$$\Rightarrow \cos(-\alpha) = \sin\left(\frac{\pi}{2}+\alpha\right)$$

$$\Rightarrow \cos\alpha = \sin\left(\frac{\pi}{2}+\alpha\right) \qquad \{\because \cos(-\alpha) = \cos\alpha\}$$

$$\therefore \boxed{\sin\left(\frac{\pi}{2}+\alpha\right) = \cos\alpha} \qquad \text{(iii)}$$

4) We known that

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

replacing β by $-\beta$ we get

$$\cos[\alpha-(-\beta)] = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$$

$$\{\because \cos(-\beta) = \cos\beta, \sin(-\beta) = -\sin\beta\}$$

7 $\Rightarrow \boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$

5) We known that

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

replacing α by $\frac{\pi}{2}+\alpha$ we get

$$\cos\left[\left(\frac{\pi}{2}+\alpha\right)+\beta\right] = \cos\left(\frac{\pi}{2}+\alpha\right)\cos\beta - \sin\left(\frac{\pi}{2}+\alpha\right)\sin\beta$$

$$\Rightarrow \cos\left[\frac{\pi}{2}+(\alpha+\beta)\right] = -\sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\Rightarrow -\sin(\alpha+\beta) = -[\sin\alpha\cos\beta + \cos\alpha\sin\beta]$$

$$\therefore \boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

6) We known that

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ [from (v) above]

replacing β by $-\beta$ we get

$$\sin(\alpha-\beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$

$$\because \begin{cases} \sin(-\beta) = -\sin\beta \\ \cos(-\beta) = \cos\beta \end{cases}$$

$$\therefore \boxed{\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta} \quad \text{(vi)}$$

7) We known that

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha \,.\, \sin\beta$$

Let $\alpha = 2\pi$ and $\beta = \theta$

$$\therefore \cos(2\pi-\theta) = \cos 2\pi \,.\, \cos\theta + \sin 2\pi \sin\theta$$

$$= 1 \,.\, \cos\theta + 0 \,.\, \sin\theta \qquad \because \begin{cases} \cos 2\pi = 1 \\ \sin 2\pi = 0 \end{cases}$$

$$= \cos\theta$$

8) We known that $\sin(\alpha-\beta) = \sin\alpha \,.\, \cos\beta - \cos\alpha \,.\, \sin\beta$

$$\therefore \sin(2\pi-\theta) = \sin 2\pi \,.\, \cos\theta - \cos 2\pi \sin\theta$$

$$= 0 \,.\, \cos\theta - 1 \,.\, \sin\theta \qquad \because \begin{cases} \sin 2\pi = 0 \\ \cos 2\pi = 1 \end{cases}$$

$$= -\sin\theta \quad \text{(viii)}$$

9) $$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

$$= \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \quad \left[\begin{array}{c}\text{Dividing} \\ \text{neumerator and} \\ \text{denuminator} \\ -\cos\alpha\cos\beta\end{array}\right]$$

$$\therefore \boxed{\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}} \quad \text{(ix)}$$

10) $$\tan(\alpha-\beta) = \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} = \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta}$$

$$= \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} - \frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} + \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \quad \left[\begin{array}{c}\text{Dividing} \\ \text{neumerator and} \\ \text{denuminator}\end{array}\right]$$

$$\therefore \boxed{\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}} \quad \text{(x)}$$

10.3 Trigonometric Ratios of Allied Angles

The angles associated with basic angles of measure θ to a right angle or its multiples are called **allied angles**. So, the angles of measure $90° \pm \theta$, $180° \pm \theta$, $270° \pm \theta$, $360° \pm \theta$, are known as **allied angles**.

Using fundamental law, $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ and its deductions, we derive the following identities:

$$\begin{cases}\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta\ , & \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta\ , & \tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta\\ \sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta\ , & \cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta\ , & \tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta\end{cases}$$

$$\begin{cases}\sin(\pi-\theta)=\sin\theta\ , & \cos(\pi-\theta)=-\cos\theta\ , & \tan(\pi-\theta)=-\tan\theta\\ \sin(\pi+\theta)=-\sin\theta\ , & \cos(\pi+\theta)=-\cos\theta\ , & \tan(\pi+\theta)=\tan\theta\end{cases}$$

$$\begin{cases}\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta\ , & \cos\left(\frac{3\pi}{2}-\theta\right)=-\sin\theta\ , & \tan\left(\frac{3\pi}{2}-\theta\right)=\cot\theta\\ \sin\left(\frac{3\pi}{2}+\theta\right)=-\cos\theta\ , & \cos\left(\frac{3\pi}{2}+\theta\right)=\sin\theta\ , & \tan\left(\frac{3\pi}{2}+\theta\right)=-\cot\theta\end{cases}$$

$$\begin{cases}\sin(2\pi-\theta)=-\sin\theta\ , & \cos(2\pi-\theta)=\cos\theta\ , & \tan(2\pi-\theta)=-\tan\theta\\ \sin(2\pi+\theta)=\sin\theta\ , & \cos(2\pi+\theta)=\cos\theta\ , & \tan(2\pi+\theta)=\tan\theta\end{cases}$$

Note: The above results also apply to the reciprocals of sine, cosine and tangent. These results are to be applied frequently in the study of trigonometry, and they can be remembered by using the following device:

1) If θ is added to or subtracted from **odd multiple** of right angle, the trigonometric ratios change into **co-ratios** and vice versa.

i.e, $\sin \rightleftarrows \cos,\quad \tan \rightleftarrows \cot,\quad \sec \rightleftarrows \operatorname{cosec}$

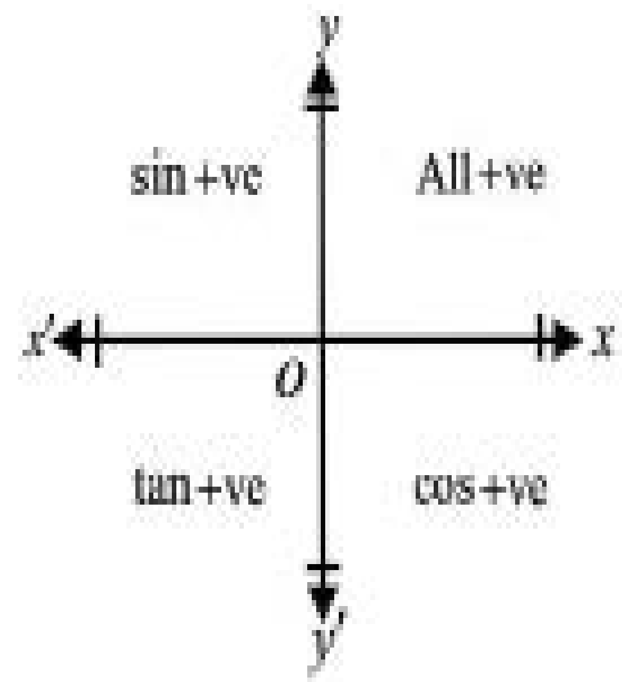
e.g. $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$ and $\cos\left(\frac{3\pi}{2}+\theta\right)=\sin\theta$

2) If θ is added to or subtracted from an even multiple of $\frac{\pi}{2}$ the trigonometric ratios shall remain the same.

3) So far as the sign of the results is concerned, it is determined by the quadrant in which the terminal arm of the angle lies.

e.g. $\sin(\pi-\theta)=\sin\theta,\quad \tan(\pi+\theta)=\tan\theta,\quad \cos(2\pi-\theta)=\cos\theta$

Measure of the angle	Quad.
$\frac{\pi}{2}-\theta$	I
$\frac{\pi}{2}+\theta$ or $\pi-\theta$	II
$\pi+\theta$ or $\frac{3\pi}{2}-\theta$	III
$\frac{3\pi}{2}+\theta$ or $2\pi-\theta$	Iv



a) In $\sin\left(\frac{\pi}{2}-\theta\right), \sin\left(\frac{\pi}{2}+\theta\right), \sin\left(\frac{3\pi}{2}-\theta\right)$ and $\sin\left(\frac{3\pi}{2}+\theta\right)$

odd multiples of $\frac{\pi}{2}$ are involved.

$\therefore$ sin will change into cos.

Moreover, the angle of measure

i) $\left(\frac{\pi}{2}-\theta\right)$ will have terminal side in Quad.I,

So $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta;$

ii) $\left(\frac{\pi}{2}+\theta\right)$ will have terminal side in Quad.II,

So $\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta;$

iii) $\left(\frac{3\pi}{2}-\theta\right)$ will have terminal side in Quad.III, So $\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta;$

iv) $\left(\frac{3\pi}{2}+\theta\right)$ will have terminal side in Quad.IV,

So $\sin\left(\frac{3\pi}{2}+\theta\right)=-\cos\theta.$

b) In $\cos(\pi-\theta)$, $\cos(\pi+\theta)$, $\cos(2\pi-\theta)$ and $\cos(2\pi+\theta)$ even multiples of $\frac{\pi}{2}$ are involved.

$\therefore$ cos will remain as cos.

Moreover, the angle of measure

i) $(\pi-\theta)$ will have terminal side in Quad. II,
$\therefore\ \cos(\pi-\theta)=-\cos\theta;$

ii) $(\pi+\theta)$ will have terminal side in Quad. III,
$\therefore\ \cos(\pi+\theta)=-\cos\theta;$

iii) $(2\pi-\theta)$ will have terminal side in Quad. IV:
$\therefore\ \cos(2\pi-\theta)=\cos\theta;$

iv) $(2\pi+\theta)$ will have terminal side in Quad. I
$\therefore\ \cos(2\pi+\theta)=\cos\theta.$

Example 2: Without using the tables, write down the values of:

i) cos 315° ii) sin 540° iii) tan (–135°) iv) sec(–300°)

Solution:

i) $\cos 315^\circ = \cos(270+45)^\circ = \cos(3\times 90+45)^\circ = +\sin 45^\circ = \frac{1}{\sqrt{2}}$

ii) $\sin 540^\circ = \sin(540+0)^\circ = \sin(6\times 90+0)^\circ = +\sin 0 = 0$

iii) $\tan(-135^\circ) = -\tan 135^\circ = -\tan(180-45)^\circ = -\tan(2\times 90-45)^\circ = -(-\tan 45^\circ) = 1$

iv) $\sec(-300^\circ) = \sec 300^\circ = \sec(360-60)^\circ = \sec(4\times 90-60)^\circ = \sec 60^\circ = 2$

Example 3: Simplify:

$$\frac{\sin(360^\circ-\theta)\cos(180^\circ-\theta)\tan(180^\circ+\theta)}{\sin(90^\circ+\theta)\cos(90^\circ-\theta)\tan(360^\circ+\theta)}$$

Solution: $\because$ $\begin{cases}\sin(360^\circ-\theta)=-\sin\theta, & \cos(180^\circ-\theta)=-\cos\theta\\ \tan(180^\circ+\theta)=\tan\theta\ , & \sin(90^\circ+\theta)=\cos\theta\\ \cos(90^\circ-\theta)=\sin\theta\ , & \tan(360^\circ+\theta)=\tan\theta\end{cases}$

$$\therefore\ \frac{\sin(360^\circ-\theta)\cos(180^\circ-\theta)\tan(180^\circ+\theta)}{\cos\theta.\sin\theta\ .\tan\theta} = \frac{(-\sin\theta)(-\cos\theta)\tan\theta}{\cos\theta\ .\ \sin\theta\ .\tan\theta} = 1$$

Exercise 10.1

1. Without using the tables, find the values of:

i) $\sin(-780^\circ)$ ii) $\cot(-855^\circ)$ iii) $\csc(2040^\circ)$

iv) $\sec(-960^\circ)$ v) $\tan(1110^\circ)$ vi) $\sin(-300^\circ)$

2. Express each of the following as a trigonometric function of an angle of positive degree measure of less than 45°.

i) $\sin 196^\circ$ ii) $\cos 147^\circ$ iii) $\sin 319^\circ$

iv) $\cos 254^\circ$ v) $\tan 294^\circ$ vi) $\cos 728^\circ$

vii) $\sin(-625^\circ)$ viii) $\cos(-435^\circ)$ ix) $\sin 150^\circ$

3. Prove the following:

i) $\sin(180^\circ+\alpha)\sin(90-\alpha) = -\sin\alpha\cos\alpha$

ii) $\sin 780^\circ\sin 480^\circ+\cos 120^\circ\sin 30^\circ = \frac{1}{2}$

iii) $\cos 306^\circ+\cos 234^\circ+\cos 162^\circ+\cos 18^\circ = 0$

iv) $\cos 330^\circ\sin 600^\circ+\cos 120^\circ\sin 150^\circ = -1$

4. Prove that:

i) $$\frac{\sin^2(\pi+\theta)\tan\left(\frac{3\pi}{2}+\theta\right)}{\cot^2\left(\frac{3\pi}{2}-\theta\right)\cos^2(\pi-\theta)\operatorname{coses}(2\pi-\theta)}=\cos\theta$$

ii) $$\frac{\cos(90^\circ+\theta)\sec(-\theta)\tan(180^\circ-\theta)}{\sec(360^\circ-\theta)\sin(180^\circ+\theta)\cot(90^\circ-\theta)}=-1$$

5. **If α, β, γ are the angles of a triangle ABC, then prove that**

i) $\sin(\alpha+\beta) = \sin\gamma$ ii) $\cos\left(\frac{\alpha+\beta}{2}\right) = \sin\frac{\gamma}{2}$

iii) $\cos(\alpha+\beta) = -\cos\gamma$ iv) $\tan(\alpha+\beta)+\tan\gamma=0.$

10.4 Further Application of Basic Identities

Example 1: Prove that

$$\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha-\sin^2\beta \quad \text{(i)}$$
$$= \cos^2\beta-\cos^2\alpha \quad \text{(ii)}$$

Solution: L.H.S. $= \sin(\alpha+\beta)\sin(\alpha-\beta)$

$$= (\sin\alpha\cos\beta+\cos\alpha\sin\beta)(\sin\alpha\cos\beta-\cos\alpha\sin\beta)$$
$$= \sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta$$
$$= \sin^2\alpha(1-\sin^2\beta)-(1-\sin^2\alpha)\sin^2\beta$$
$$= \sin^2\alpha-\sin^2\alpha\sin^2\beta-\sin^2\beta+\sin^2\alpha\sin^2\beta$$
$$= \sin^2\alpha-\sin^2\beta \quad \text{(i)}$$
$$= (1-\cos^2\alpha)-(1-\cos^2\beta)$$
$$= 1-\cos^2\alpha-1+\cos^2\beta$$
$$= \cos^2\beta-\cos^2\alpha \quad \text{(ii)}$$

Example 2: Without using tables, find the values of all trigonometric functions of 75°.

Solution: As $75^\circ = 45^\circ+30^\circ$

$$\sin 75^\circ = \sin(45^\circ+30^\circ)=\sin 45^\circ\cos 30^\circ+\cos 45^\circ\sin 30^\circ$$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\cos 75^\circ = \cos(45^\circ+30^\circ)=\cos 45^\circ\cos 30^\circ-\sin 45^\circ\sin 30^\circ$$
$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)=\frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\tan 75^\circ = \tan(45^\circ+30^\circ)=\frac{\tan 45^\circ+\tan 30^\circ}{1-\tan 45^\circ\tan 30^\circ}$$
$$= \frac{1+\frac{1}{\sqrt{3}}}{1-1.\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$\cot 75^\circ = \frac{1}{\tan 75^\circ}=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$\operatorname{coses} 75^\circ = \frac{1}{\sin 75^\circ}=\frac{2\sqrt{2}}{\sqrt{3}+1} \quad \text{and} \quad \sec 75^\circ = \frac{1}{\cos 75^\circ}=\frac{2\sqrt{2}}{\sqrt{3}-1}$$

Example 3: Prove that: $\frac{\cos 11^\circ+\sin 11^\circ}{\cos 11^\circ-\sin 11^\circ} = \tan 56^\circ.$

Solution: Consider

$$\text{R.H.S.} = \tan 56^\circ = \tan(45^\circ+11^\circ) = \frac{\tan 45^\circ+\tan 11^\circ}{1-\tan 45^\circ\tan 11^\circ}$$

$$= \frac{1+\tan 11^\circ}{1-\tan 11^\circ} = \frac{1+\frac{\sin 11^\circ}{\cos 11^\circ}}{1-\frac{\sin 11^\circ}{\cos 11^\circ}}=\frac{\cos 11^\circ+\sin 11^\circ}{\cos 11^\circ-\sin 11^\circ} = \text{L.H.S.}$$

Hence $\dfrac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ} = \tan 56^\circ.$

Example 4: If $\cos\alpha = -\dfrac{24}{25}$, $\tan\beta = \dfrac{9}{40}$ the terminal side of the angle of measure α is in the II quadrant and that of β is in the III quadrant, find the values of:

i) $\sin(\alpha+\beta)$ ii) $\cos(\alpha+\beta)$

In which quadrant does the terminal side of the angle of measure $(\alpha+\beta)$ lie?

Solution: We know that $\sin^2\alpha + \cos^2\alpha = 1$

$$\therefore \quad \sin\alpha = \pm\sqrt{1-\cos^2\alpha} = \pm\sqrt{1-\frac{576}{625}} = \pm\sqrt{\frac{49}{625}} = \pm\frac{7}{25}$$

As the terminal side o f the angle of measure of α is in the II quadrant, where sin α is positive.

$$\therefore \quad \sin\alpha = \frac{7}{25}$$

Now $\sec\beta = \pm\sqrt{1+\tan^2\beta} = \pm\sqrt{1+\frac{81}{1600}} = \pm\frac{41}{40}$

As the terminal side of the angle of measure of β in the III quadrant, so sec β is negative

$$\therefore \quad \sec\beta = -\frac{41}{40} \quad \text{and} \quad \cos\beta = -\frac{40}{41}$$

$$\sin\beta = \pm\sqrt{1-\cos^2\beta} = \pm\sqrt{1-\frac{1600}{1681}} = \pm\frac{9}{41}$$

As the terminal arm of the angle of measure β is in the III quadrant, so sin β is negative

$$\therefore \quad \sin\beta = -\frac{9}{41}$$

$$\therefore \quad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \left(\frac{7}{25}\right)\left(-\frac{40}{41}\right) + \left(-\frac{24}{25}\right)\left(-\frac{9}{41}\right)$$

$$= \frac{-280+216}{1025} = -\frac{64}{1025}$$

and $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \left(-\frac{24}{25}\right)\left(-\frac{40}{41}\right) - \left(\frac{7}{25}\right)\left(-\frac{9}{41}\right)$$

$$= \frac{960+63}{1025}$$

$$= \frac{1023}{1025}$$

$\because \quad \sin(\alpha+\beta)$ is $-$ ve and $\cos(\alpha+\beta)$ is $+$ ve

$\therefore$ The terminal arm of the angle of measure $(\alpha+\beta)$ is in the IV quadrant.

Example 5: If α, β, γ are the angles of ΔABC prove that:

i) $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

ii) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 1$

Solution: As α, β, γ are the angles of ΔABC

$$\therefore \quad \alpha+\beta+\gamma = 180^\circ$$

i) $\alpha+\beta = 180^\circ - \gamma$

$$\therefore \quad \tan(\alpha+\beta) = \tan(180^\circ-\gamma)$$

$$\Rightarrow \quad \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = -\tan\gamma$$

$$\Rightarrow \tan\alpha+\tan\beta = -\tan\gamma+\tan\alpha\tan\beta\tan\gamma$$

$$\therefore \tan\alpha+\tan\beta+\tan\gamma = \tan\alpha\tan\beta\tan\gamma$$

ii) As $\alpha+\beta+\gamma = 180^\circ \quad \Rightarrow \frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2} = 90^\circ$

So $\frac{\alpha}{2}+\frac{\beta}{2} = 90^\circ-\frac{\gamma}{2}$

$$\therefore \tan\left(\frac{\alpha}{2}+\frac{\beta}{2}\right) = \tan\left(90^\circ-\frac{\gamma}{2}\right)$$

$$\Rightarrow \frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}} = \cot\frac{\gamma}{2} = \frac{1}{\tan\frac{\gamma}{2}}$$

$$\Rightarrow \tan\frac{\alpha}{2}\tan\frac{\gamma}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2} = 1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

$$\therefore \tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 1$$

Example 6: Express $3\sin\theta+4\cos\theta$ in the form $r\sin(\theta+\phi)$, where the terminal side of the angle of measure ϕ is in the I quadrant.

Solution: Let $3 = r\cos\phi$ and $4 = r\sin\phi$

$$\therefore 3^2+4^2 = r^2\cos^2\phi+r^2\sin^2\phi$$

$$\Rightarrow 9+16 = r^2(\cos^2\phi+\sin^2\phi)$$

$$\Rightarrow 25 = r^2$$

$$\Rightarrow 5 = r$$

$$\Rightarrow r = 5$$

$$\frac{4}{3} = \frac{r\sin\phi}{r\cos\phi}$$

$$\Rightarrow \frac{4}{3} = \tan\phi$$

$$\therefore \tan\phi = \frac{4}{3}$$

$$\therefore 3\sin\theta+4\cos\theta = r\cos\phi\sin\theta+r\sin\phi\cos\theta$$

$$= r(\sin\theta\cos\phi+\cos\theta\sin\phi)$$

$$= r\sin(\theta+\phi)$$

Where $r = 5$ and $\phi = \tan^{-1}\frac{4}{3}$

Exercise 10.2

1. Prove that

i) $\sin(180^\circ+\theta) = -\sin\theta$ ii) $\cos(180^\circ+\theta) = -\cos\theta$

iii) $\tan(270^\circ-\theta) = \cot\theta$ iv) $\cos(\theta-180^\circ) = -\cos\theta$

v) $\cos(270^\circ+\theta) = \sin\theta$ vi) $\sin(\theta+270^\circ) = -\cos\theta$

vii) $\tan(180^\circ+\theta) = \tan\theta$ viii) $\cos(360^\circ-\theta) = \cos\theta$

2. Find the values o f the following:

i) $\sin 15^\circ$ ii) $\cos 15^\circ$ iii) $\tan 15^\circ$

iv) $\sin 105^\circ$ v) $\cos 105^\circ$ vi) $\tan 105^\circ$

(**Hint:** $15^\circ = (45^\circ-30^\circ)$ and $105^\circ = (60^\circ+45^\circ)$)

3. Prove that:

i) $\sin(45^\circ+\alpha) = \frac{1}{\sqrt{2}}(\sin\alpha+\cos\alpha)$

ii) $\cos(\alpha+45^\circ) = \frac{1}{\sqrt{2}}(\cos\alpha-\sin\alpha)$

4. Prove that:

i) $\tan(45^\circ+A)\tan(45^\circ-A) = 1$

ii) $\tan\left(\frac{\pi}{4}-\theta\right)+\tan\left(\frac{3\pi}{4}+\theta\right) = 0$

iii) $\sin\left(\theta+\frac{\pi}{6}\right)+\cos\left(\theta+\frac{\pi}{3}\right) = \cos\theta$

iv) $\frac{\sin\theta-\cos\theta\tan\frac{\theta}{2}}{\cos\theta+\sin\theta\tan\frac{\theta}{2}} = \tan\frac{\theta}{2}$

v) $\dfrac{1-\tan\theta\tan\phi}{1+\tan\theta\tan\phi} = \dfrac{\cos(\theta+\phi)}{\cos(\theta-\phi)}$

5. Show that: $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha$

6. Show that: $\dfrac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)} = \tan\alpha$

7. Show that:

i) $\cot(\alpha+\beta) = \dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$

ii) $\cot(\alpha-\beta) = \dfrac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$

iii) $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta} = \dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$

8. If $\sin\alpha = \dfrac{4}{5}$ and $\cos\beta = \dfrac{40}{41}$, where $0<\alpha<\dfrac{\pi}{2}$ and $0<\beta<\dfrac{\pi}{2}$.

Show that $\sin(\alpha-\beta) = \dfrac{133}{205}$.

9. If $\sin\alpha = \dfrac{4}{5}$ and $\sin\beta = \dfrac{12}{13}$ where $\dfrac{\pi}{2}<\alpha<\pi$ and $\dfrac{\pi}{2}<\beta<\pi$. Find

i) $\sin(\alpha+\beta)$ ii) $\cos(\alpha+\beta)$ iii) $\tan(\alpha+\beta)$

iv) $\sin(\alpha-\beta)$ v) $\cos(\alpha-\beta)$ vi) $\tan(\alpha-\beta)$.

In which quadrants do the terminal sides of the angles of measures $(\alpha+\beta)$ and $(\alpha-\beta)$ lie?

10. Find $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$, given that

i) $\tan\alpha = \dfrac{3}{4}$, $\cos\beta = \dfrac{5}{13}$ and neither the terminal side o f the angle of measure α nor that of β is in the I quadrant.

ii) $\tan\alpha = -\dfrac{15}{8}$ and $\sin\beta = -\dfrac{7}{25}$ and neither the terminal side of the angle of measure α nor that of β is in the IV quadrant.

11. Prove that $\dfrac{\cos 8^\circ \quad \sin 8^\circ}{\cos 8^\circ \quad \sin 8^\circ} \qquad \tan 37$

12. If α,β,γ are the angles of a triangle *ABC*, show that $\cot\dfrac{\alpha}{2}+\cot\dfrac{\beta}{2}+\cot\dfrac{\gamma}{2} = \cot\dfrac{\alpha}{2}\cot\dfrac{\beta}{2}\cot\dfrac{\gamma}{2}$

13. If $\alpha+\beta+\gamma = 180^\circ$, show that

$\cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha = 1$

14. Express the following in the form $r\sin(\theta+\phi)$ or $r\sin(\theta-\phi)$, where terminal sides of the angles of measures θ and ϕ are in the first quadrant:

i) $12\sin\theta+5\cos\theta$ ii) $3\sin\theta-4\cos\theta$ iii) $\sin\theta-\cos\theta$

iv) $5\sin\theta-4\cos\theta$ v) $\sin\theta+\cos\theta$. vi) $3\sin\theta-5\cos\theta$

10.5 Double angle Identities

We have discovered the following results:

$\sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta-\sin\alpha\sin\beta$

and $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

We can use them to obtain the double angle identities as follows:

i) Put $\beta=\alpha$ in $\sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta$

$\therefore \sin(\alpha+\alpha) = \sin\alpha\cos\alpha+\cos\alpha\sin\alpha$

Hence $\sin 2\alpha = 2\sin\alpha\cos\alpha$

ii) Put $\beta=\alpha$ in $\cos(\alpha+\beta) = \cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\cos(\alpha+\alpha) = \cos\alpha\cos\alpha-\sin\alpha\sin\alpha$

Hence $\cos 2\alpha = \cos^2\alpha-\sin^2\alpha$

$\because\ \cos 2\alpha = \cos^2\alpha-\sin^2\alpha$

$\therefore\ \cos 2\alpha = \cos^2\alpha-(1-\cos^2\alpha) \qquad (\because\ \sin^2\alpha = 1-\cos^2\alpha)$

$= \cos^2\alpha-1+\cos^2\alpha$

$\therefore\ \cos 2\alpha = 2\cos^2\alpha-1$

$\because\ \cos 2\alpha = \cos^2\alpha-\sin^2\alpha$

$\therefore\ \cos 2\alpha = (1-\sin^2\alpha)-\sin^2\alpha \qquad (\because\ \cos^2\alpha = 1-\sin^2\alpha)$

$\therefore\ \cos 2\alpha = 1-2\sin^2\alpha$

iii) Put $\beta=\alpha$ in $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$\tan(\alpha+\alpha) = \dfrac{\tan\alpha+\tan\alpha}{1-\tan\alpha\tan\alpha}$

$\therefore\ \tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

10.6 Half angle Identities

The formulas proved above can also be written in the form of half angle identities, in the following way:

i) $\because\ \cos\alpha = 2\cos^2\dfrac{\alpha}{2}-1 \quad \Rightarrow \cos^2\dfrac{\alpha}{2} = \dfrac{1+\cos\alpha}{2}$

$\Rightarrow\ \cos\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$

ii) $\because\ \cos\alpha = 1-2\sin^2\dfrac{\alpha}{2} \quad \Rightarrow\ \sin^2\dfrac{\alpha}{2} = \dfrac{1-\cos\alpha}{2}$

$\Rightarrow\ \sin\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$

iii) $\tan\dfrac{\alpha}{2} = \dfrac{\sin\dfrac{\alpha}{2}}{\cos\dfrac{\alpha}{2}} = \pm\dfrac{\sqrt{\dfrac{1-\cos\alpha}{2}}}{\sqrt{\dfrac{1+\cos\alpha}{2}}}$

$\Rightarrow\ \tan\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}$

10.7 Triple angle Identities

i) $\sin 3\alpha = 3\sin\alpha-4\sin^3\alpha$

ii) $\cos 3\alpha = 4\cos^3\alpha-3\cos\alpha$

iii) $\tan 3\alpha = \dfrac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}$

Proof:

i) $\sin 3\alpha = \sin(2\alpha+\alpha)$

$= \sin 2\alpha\cos\alpha+\cos 2\alpha\sin\alpha$

$= 2\sin\alpha\cos\alpha\cos\alpha+(1-2\sin^2\alpha)\sin\alpha$

$= 2\sin\alpha\cos^2\alpha+\sin\alpha-2\sin^3\alpha$

$= 2\sin\alpha(1-\sin^2\alpha)+\sin\alpha-2\sin^3\alpha$

$$= 2\sin\alpha - 2\sin^3\alpha + \sin\alpha - 2\sin^3\alpha$$

$\therefore$ $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

ii) $\cos 3\alpha = \cos(2\alpha + \alpha)$

$$= \cos 2\alpha\cos\alpha - \sin 2\alpha\sin\alpha$$
$$= (2\cos^2\alpha - 1)\cos\alpha - 2\sin\alpha\cos\alpha\sin\alpha$$
$$= 2\cos^3\alpha - \cos\alpha - 2\sin^2\alpha\cos\alpha$$
$$= 2\cos^3\alpha - \cos\alpha - 2(1-\cos^2\alpha)\cos\alpha$$
$$= 2\cos^3\alpha - \cos\alpha - 2\cos\alpha + 2\cos^3\alpha$$

$\therefore$ $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

iii) $\tan 3\alpha = \tan(2\alpha + \alpha)$

$$= \frac{\tan 2\alpha + \tan\alpha}{1 - \tan 2\alpha\tan\alpha}$$

$$= \frac{\frac{2\tan\alpha}{1-\tan^2\alpha} + \tan\alpha}{1 - \frac{2\tan\alpha}{1-\tan^2\alpha}.\tan\alpha} = \frac{2\tan\alpha + \tan\alpha - \tan^3\alpha}{1 - \tan^2\alpha - 2\tan^2\alpha}$$

$\therefore$ $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$

Example 1: Prove that $\dfrac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \tan A$

Solution: L.H.S $= \dfrac{\sin A + 2\sin A\cos A}{1 + \cos A + 2\cos^2 A - 1} = \dfrac{\sin A(1 + 2\cos A)}{\cos A(1 + 2\cos A)}$

$$= \frac{\sin A}{\cos A} = \tan A = \text{R.H.S}$$

Hence $\dfrac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \tan A.$

Example 2: Show that

i) $\sin 2\theta = \dfrac{2\tan\theta}{1 + \tan^2\theta}$ ii) $\cos 2\theta = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$

Solution: i) $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\sin\theta\cos\theta}{1} = \dfrac{2\sin\theta\cos\theta}{\cos^2\theta + \sin^2\theta}$

$$= \frac{\frac{2\sin\theta\cos\theta}{\cos^2\theta}}{\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}} = \frac{2\frac{\sin\theta}{\cos\theta}}{\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}}$$

$\therefore$ $\sin 2\theta = \dfrac{2\tan\theta}{1 + \tan^2\theta}$

ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta = \dfrac{\cos^2\theta - \sin^2\theta}{1} = \dfrac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta}$

$$= \frac{\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}}{\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}} = \frac{\frac{\cos^2\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}}{\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}}$$

$\therefore$ $\cos 2\theta = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$

Example 3: Reduce $\cos^4\theta$ to an expression involving only function of multiples of θ, raised to the first power.

Solution: We know that:

$$2\cos^2\theta = 1 + \cos 2\theta \Rightarrow \cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\therefore \cos^4\theta = (\cos^2\theta)^2 = \left[\frac{1 + \cos 2\theta}{2}\right]^2$$

$$= \frac{1+2\cos 2\theta+\cos^2 2\theta}{4}$$

$$= \frac{1}{4}[1+2\cos 2\theta+\cos^2 2\theta]$$

$$= \frac{1}{4}\left[1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right]$$

$$= \frac{1}{4\times 2}[2+4\cos 2\theta+1+\cos 4\theta]$$

$$= \frac{1}{8}[3+4\cos 2\theta+\cos 4\theta]$$

Exercise 10.3

1. Find the values of $\sin 2\alpha$, $\cos 2\alpha$ and $\tan 2\alpha$, when:

i) $\sin\alpha = \frac{12}{13}$ ii) $\cos\alpha = \frac{3}{5}$, where $0<\alpha<\frac{\pi}{2}$

Prove the following identities:

2. $\cot\alpha - \tan\alpha = 2\cot 2\alpha$ **3.** $\frac{\sin 2\alpha}{1+\cos 2\alpha} = \tan\alpha$

4. $\frac{1-\cos\alpha}{\sin\alpha} = \tan\frac{\alpha}{2}$ **5.** $\frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha} = \sec 2\alpha - \tan 2\alpha$

6. $\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}} = \frac{\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}}$ **7.** $\frac{\text{coses}\,\theta+\text{coses}\,2\theta}{\sec\theta} = \cot\frac{\theta}{2}$

8. $1+\tan\alpha\tan 2\alpha = \sec 2\alpha$ **9.** $\frac{2\sin\theta\sin 2\theta}{\cos\theta+\cos 3\theta} = \tan 2\theta\tan\theta$

10. $\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 2$ **11.** $\frac{\cos 3\theta}{\cos\theta} + \frac{\sin 3\theta}{\sin\theta} = 4\cos 2\theta$

12. $\frac{\tan\frac{\theta}{2}+\cot\frac{\theta}{2}}{\cot\frac{\theta}{2}-\tan\frac{\theta}{2}} = \sec\theta$ **13.** $\frac{\sin 3\theta}{\cos\theta} + \frac{\cos 3\theta}{\sin\theta} = 2\cot 2\theta$

14. Reduce $\sin^4\theta$ to an expression involving only function of multiples of θ, raised to the first power.

15. Find the values of $\sin\theta$ and $\cos\theta$ without using table or calculator, when θ is

i) 18° ii) 36° iii) 54° iv) 72°

Hence prove that: $\cos 36^\circ\cos 72^\circ\cos 108^\circ\cos 144^\circ = \frac{1}{16}$

Hint :

Let $\theta = 18^\circ$	Let $\theta = 36^\circ$
$5\theta = 90^\circ$	$5\theta = 180^\circ$
$(3\theta+2\theta) = 90^\circ$	$3\theta+2\theta = 180^\circ$
$3\theta = 90^\circ-2\theta$	$3\theta = 180^\circ-2\theta$
$\sin 3\theta = \sin(90^\circ-2\theta)$	$\sin 3\theta = \sin(180^\circ-2\theta)$
etc.	etc.

10.8. Sum, Difference and Product of Sines and Cosines

We know that:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta \quad \text{(i)}$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta-\cos\alpha\sin\beta \quad \text{(ii)}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta-\sin\alpha\sin\beta \quad \text{(iii)}$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta \quad \text{(iv)}$$

Adding (i) and (ii) we get

$$\sin(\alpha+\beta)+\sin(\alpha-\beta) = 2\sin\alpha\cos\beta \quad \text{(v)}$$

Subtracting (ii) from (i) we get

$$\sin(\alpha+\beta)-\sin(\alpha-\beta) = 2\cos\alpha\sin\beta \qquad \text{(vi)}$$

Adding (iii) and (iv) we get

$$\cos(\alpha+\beta)+\cos(\alpha-\beta) = 2\cos\alpha\cos\beta \qquad \text{(vii)}$$

Subtracting (iv) from (iii) we get

$$\cos(\alpha+\beta)-\cos(\alpha-\beta) = -2\sin\alpha\sin\beta \qquad \text{(viii)}$$

So we get four identities as:

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta)+\sin(\alpha-\beta)$$
$$2\cos\alpha\sin\beta = \sin(\alpha+\beta)-\sin(\alpha-\beta)$$
$$2\cos\alpha\cos\beta = \cos(\alpha+\beta)+\cos(\alpha-\beta)$$
$$-2\sin\alpha\sin\beta = \cos(\alpha+\beta)-\cos(\alpha-\beta)$$

Now putting $\alpha+\beta=P$ and $\alpha-\beta=Q$, we get

$$\alpha = \frac{P+Q}{2} \quad \text{and} \quad \beta = \frac{P-Q}{2}$$

$$\sin P+\sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

$$\sin P-\sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$$
$$\cos P+\cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$$
$$\cos P-\cos Q = -2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

Example 1: Express $2\sin 7\theta\cos 3\theta$ as a sum or difference.

Solution: $2\sin 7\theta\cos 3\theta = \sin(7\theta+3\theta)+\sin(7\theta-3\theta)$

$= \sin 10\theta+\sin 4\theta$

Example 2: Prove without using tables / calculator, that

$$\sin 19^\circ\cos 11^\circ+\sin 71^\circ\sin 11^\circ = \frac{1}{2}$$

Solution: L.H.S. $= \sin 19^\circ\cos 11^\circ+\sin 71^\circ\sin 11^\circ$

$$= \frac{1}{2}[2\sin 19^\circ\cos 11^\circ+2\sin 71^\circ\sin 11^\circ]$$

$$= \frac{1}{2}[\{\sin(19^\circ+11^\circ)+\sin(19^\circ-11^\circ)\}-\{\cos(71^\circ+11^\circ)-\cos(71^\circ-11^\circ)\}]$$

$$= \frac{1}{2}[\sin 30^\circ+\sin 8^\circ-\cos 82^\circ+\cos 60^\circ]$$

$$= \frac{1}{2}\left[\frac{1}{2}+\sin 8^\circ-\cos(90^\circ-8^\circ)+\frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}+\sin 8^\circ-\sin 8^\circ+\frac{1}{2}\right] \quad (\because \cos 82^\circ=\cos(90^\circ-8^\circ)=\sin 8^\circ)$$

$$= \frac{1}{2}\left[\frac{1}{2}+\frac{1}{2}\right]$$

$$= \frac{1}{2}$$

= R.H.S.

Hence $\sin 19^\circ\cos 11^\circ+\sin 71^\circ+\sin 11^\circ = \frac{1}{2}$

Example 3: Express $\sin 5x+\sin 7x$ as a product.

Solution: $\sin 5x+\sin 7x = 2\sin\frac{5x+7x}{2}\cos\frac{5x-7x}{2} = 2\sin 6x\cos(-x)$

$= 2\sin 6x\cos x \qquad (\because \cos(-\theta)=\cos\theta)$

Example 4: Express $\cos A+\cos 3A+\cos 5A+\cos 7A$ as a product.

Solution: $\cos A+\cos 3A+\cos 5A+\cos 7A$

$=(\cos 3A+\cos A)+(\cos 7A+\cos 5A)$

$=2\cos\frac{3A+A}{2}\cos\frac{3A-A}{2}+2\cos\frac{7A+A}{2}\cos\frac{7A-5A}{2}$

$=2\cos 2A\cos A+2\cos 6A\cos A$

$=2\cos A(\cos 6A+\cos 2A)$

$=2\cos A\left[2\cos\frac{6A+2A}{2}\cos\frac{6A-2A}{2}\right]$

$=2\cos A(2\cos 4A\cos 2A) = 4\cos A\cos 2A\cos 4A.$

Example 4: Show that $\cos 20^\circ\cos 40^\circ\cos 80^\circ=\frac{1}{8}$

Solution: L.H.S. $=\cos 20^\circ\cos 40^\circ\cos 80^\circ$

$=\frac{1}{4}(4\cos 20^\circ\cos 40^\circ\cos 80^\circ)$

$=\frac{1}{4}[(2\cos 40^\circ\cos 20^\circ).2\cos 80^\circ]$

$=\frac{1}{4}[(\cos 60^\circ+\cos 20^\circ).2\cos 80^\circ]$

$=\frac{1}{4}\left[\left(\frac{1}{2}+\cos 20^\circ\right).2\cos 80^\circ\right]$

$=\frac{1}{4}(\cos 80^\circ+2\cos 80^\circ\cos 20^\circ)$

$=\frac{1}{4}(\cos 80^\circ+\cos 100^\circ+\cos 60^\circ)$

$=\frac{1}{4}[\cos 80^\circ+\cos(180^\circ-80^\circ)+\cos 60^\circ]$

$=\frac{1}{4}\left(\cos 80^\circ-\cos 80^\circ+\frac{1}{2}\right)$ $[\because \cos(180-\theta)=-\cos\theta]$

$=\frac{1}{4}\left(\frac{1}{2}\right)=\frac{1}{8}$ R.H.S.

Hence $\cos 20^\circ\cos 40^\circ\cos 80^\circ=\frac{1}{8}.$

Exercise 10.4

1. Express the following products as sums or differences:

i) $2\sin 3\theta\cos\theta$ ii) $2\cos 5\theta\sin 3\theta$

iii) $\sin 5\theta\cos 2\theta$ iv) $2\sin 7\theta\sin 2\theta$

v) $\cos(x+y)\sin(x-y)$ vi) $\cos(2x+30^\circ)\cos(2x-30^\circ)$

vii) $\sin 12^\circ\sin 46^\circ$ viii) $\sin(x+45^\circ)\sin(x-45^\circ)$

2. Express the following sums or differences as products:

i) $\sin 5\theta+\sin 3\theta$ ii) $\sin 8\theta-\sin 4\theta$

iii) $\cos 6\theta+\cos 3\theta$ iv) $\cos 7\theta-\cos\theta$

v) $\cos 12^\circ+\cos 48^\circ$ vi) $\sin(x+30^\circ)+\sin(x-30^\circ)$

3. Prove the following identities:

i) $\frac{\sin 3x-\sin x}{\cos x-\cos 3x}=\cot 2x$ ii) $\frac{\sin 8x+\sin 2x}{\cos 8x+\cos 2x}=\tan 5x$

iii) $\frac{\sin\alpha-\sin\beta}{\sin\alpha+\sin\beta}=\tan\frac{\alpha-\beta}{2}\cot\frac{\alpha+\beta}{2}$

4. Prove that:

i) $\cos 20^\circ+\cos 100^\circ+\cos 140^\circ=0$

ii) $\sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}+\theta\right)=\frac{1}{2}\cos 2\theta$

iii) $\dfrac{\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta}{\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta} = \tan 4\theta$

5. Prove that:

i) $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \dfrac{1}{16}$

ii) $\sin\dfrac{\pi}{9}\sin\dfrac{2\pi}{9}\sin\dfrac{\pi}{3}\sin\dfrac{4\pi}{9} = \dfrac{3}{16}$

iii) $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \dfrac{1}{16}$