CHAPTER

11

Trigonometric Functions and their Graphs

11.1 Introduction

Let us first find domains and ranges of trigonometric functions before drawing their graphs.

11.1.1 Domains and Ranges of Sine and Cosine Functions

We have already defined trigonometric functions $\sin\theta$, $\cos\theta$, $\tan\theta$, $\csc\theta$, $\sec\theta$ and $\cot\theta$. We know that if $P(x, y)$ is any point on unit circle with center at the origin O such that $\angle XOP = \theta$ is standard position, then

$\cos\theta = x$ and $\sin\theta = y$

$\Rightarrow$ for any real number θ there is one and only one value of each x and y .i.e., of each $\cos\theta$ and $\sin\theta$.

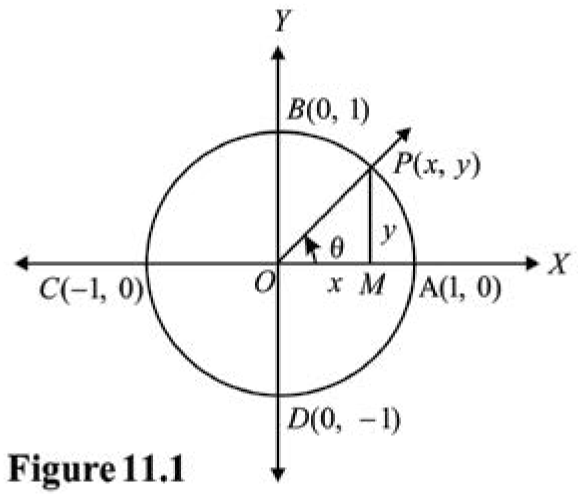


Figure 11.1

Hence $\sin\theta$ and $\cos\theta$ are the functions of θ and their domain is R a set of real numbers. Since $P(x, y)$ is a point on the unit circle with center at the origin O.

$\therefore\quad -1 \le x \le 1 \quad$ and $\quad -1 \le y \le 1$

$\Rightarrow\quad -1 \le \cos\theta \le 1 \quad$ and $\quad -1 \le \sin\theta \le 1$

Thus the range of both the sine and cosine functions is [-1, 1].

11.1.2 Domains and Ranges of Tangent and Cotangent Functions

From figure 11.1

i) $\tan\theta = \dfrac{y}{x} \quad , \quad x \neq 0$

$\Rightarrow$ terminal side $\overrightarrow{OP}$ should not coincide with OY or OY' (i.e., Y–axis)

$$\Rightarrow \quad \theta \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

$$\Rightarrow \quad \theta \neq (2n+1)\frac{\pi}{2}, \quad \text{where} \quad n \in Z$$

$\therefore$ Domain of tangent function $= R - \{x \mid x = (2n+1)\frac{\pi}{2}, \quad n \in Z\}$

and Range of tangent function = R = set of real numbers.

ii) From figure 11.1

$\cot\theta = \dfrac{x}{y} \quad , \quad y \neq 0$

$\Rightarrow$ terminal side $\overrightarrow{OP}$ should not coincide with OX or OX' (i.e., X – axis)

$\Rightarrow \quad \theta \neq 0, \pm\pi, \pm 2\pi, \ldots$

$\Rightarrow \quad \theta \neq n\pi, \quad \text{where} \quad n \in Z$

$\therefore$ Domain of cotangent function $= R - \{x \mid x = n\pi, \quad n \in Z\}$

and Range of cotangent function = R = set of real numbers.

11.1.3 Domain and Range of Secant Function

From figure 11.1

$\sec\theta = \dfrac{1}{x} \quad , \quad x \neq 0$

$\Rightarrow$ terminal side $\overrightarrow{OP}$ should not coincide with OY or OY' (i.e., Y – axis)

$$\Rightarrow \quad \theta \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

$\Rightarrow \quad \theta \neq (2n+1)\frac{\pi}{2}, \quad \text{where} \quad n \in Z$

$\therefore$ Domain of secant function $= R - \{x \mid x = (2n+1)\frac{\pi}{2}, \quad n \in Z\}$

As sec θ attains all real values except those between -1 and 1

$\therefore$ Range of secant function $= R - \{x \mid -1 < x < 1\}$

11.1.4 Domain and Range of Cosecant Function

From figure 11.1

$\csc \theta = \frac{1}{y} \quad , \quad y \neq 0$

$\Rightarrow$ terminal side $\overline{OP}$ should not coincide with OX or OX' (i.e., $X-$ axis)

$\Rightarrow \quad \theta \neq 0, \pm\pi, \pm 2\pi, ...$

$\Rightarrow \quad \theta \neq n\pi, \quad \text{where} \quad n \in Z$

$\therefore$ Domain of cosecant function $= R - \{x \mid x = n\pi, \quad n \in Z\}$

As csc θ attains all values except those between -1 and 1

$\therefore$ Range of cosecant function $= R - \{x \mid -1 < x < 1\}$

The following table summarizes the **domains** and **ranges** of the trigonometric functions:

Function	Domain	Range
$y = \sin x$	$-\infty < x < +\infty$	$-1 \leq y \leq 1$
$y = \cos x$	$-\infty < x < +\infty$	$-1 \leq y \leq 1$

$y = \tan x$	$-\infty < x < +\infty, x \neq \frac{(2n+1)\pi}{2}, n \in Z$	$-\infty < y < +\infty$
$y = \cot x$	$-\infty < x < +\infty, x \neq n\pi, \quad n \in Z$	$-\infty < y < +\infty$
$y = \sec x$	$-\infty < x < +\infty, x \neq \frac{(2n+1)\pi}{2}, n \in Z$	$y \geq 1$ or $y \leq -1$
$y = \text{coses } x$	$-\infty < x < +\infty, x \neq n\pi, \quad n \in Z$	$y \geq 1$ or $y \leq -1$

11.2 Period of Trigonometric Functions

All the six trigonometric functions repeat their values for each increase or decrease of 2π in θ i.e., the values of trigonometric functions for θ and $\theta \pm 2n\pi$, where $\theta \in R$, and $n \in Z$, are the same. This behaviour of trigonometric functions is called **periodicity**.

Period of a trigonometric function is the smallest $+ve$ number which, when added to the original circular measure of the angle, gives the same value of the function.

Let us now discover the periods of the trigonometric functions.

Theorem 11.1: Sine is a periodic function and its period is 2π.

Proof: Suppose p is the period of sine function such that

$\sin(\theta + p) = \sin\theta \quad \text{for all} \quad \theta \in R$ (i)

Now put $\theta = 0$, we have

$\sin(0 + p) = \sin 0$

$\Rightarrow \quad \sin p = 0$

$\Rightarrow \quad p = 0, \pm\pi, \pm 2\pi, \pm 3\pi, ...$

i) if $p = \pi$, then from (i)

$\sin(\theta + \pi) = \sin\theta$ (not true)

$\because \quad \sin(\theta + \pi) = -\sin\theta$

$\therefore \quad \pi$ is not the period of $\sin\theta$.

ii) if $p = 2\pi$, then from (i)

$\sin(\theta + 2\pi) = \sin\theta$, Which is true

As is the smallest +*ve* real number for which

$$\sin(\theta+2\pi) = \sin\theta$$

$\therefore$ 2π is not the period of $\sin\theta$.

Theorem 11.2: Tangent is a periodic function and its period is π.

Proof: Suppose p is the period of tangent function such that

$$\tan(\theta+p) = \tan\theta \quad \text{for all } \theta\in R \qquad \text{(ii)}$$

Now put $\theta=0$, we have

$$\tan(0+p) = \tan 0$$

$$\Rightarrow \tan p = 0$$

$$\therefore p=0, \pi, 2\pi, 3\pi, ...$$

i) if $p=\pi$, then from (i)

$$\tan(\theta+\pi) = \tan\theta,$$ which is true.

As π is the smallest +*ve* number for which

$$\tan(\theta+\pi) = \tan\theta$$

$\therefore$ π is not the period of $\tan\theta$.

Note: By adopting the procedure used in finding the periods of sine and tangent, we can prove that

i) 2π is the period of $\cos\theta$.

ii) 2π is the period of $\csc\theta$.

iii) 2π is the period of $\sec\theta$.

iv) π is the period of $\cot\theta$.

Example 1: Find the periods of: i) $\sin 2x$ ii) $\tan\frac{x}{3}$

Solution: i) We know that the period of sine is 2π

$$\therefore \sin(2x+2\pi) = \sin 2x \qquad \Rightarrow \sin 2(x+\pi) = \sin 2x$$

It means that the value of $\sin 2x$ repeats when x is increased by π.

Hence n is the period of $\sin 2x$.

ii) We know that the period of tangent is π

$$\therefore \tan\left(\frac{x}{3}+\pi\right) = \tan\frac{x}{3} \qquad \Rightarrow \tan\frac{1}{3}(x+3\pi) = \tan\frac{x}{3}$$

It means that the value of $\tan\frac{x}{3}$ repeats when x is increased by 3π.

Hence the period of $\tan\frac{x}{3}$ is 3π.

Exercise 11.1

Find the periods of the following functions:

1. $\sin 3x$ **2.** $\cos 2x$ **3.** $\tan 4x$ **4.** $\cot\frac{x}{2}$

5. $\sin\frac{x}{3}$ **6.** $\text{coses}\frac{x}{4}$ **7.** $\sin\frac{x}{5}$ **8.** $\cos\frac{x}{6}$

9. $\tan\frac{x}{7}$ **10.** $\cot 8x$ **11.** $\sec 9x$ **12.** cosec $10x$.

13. $3\sin x$ **14.** $2\cos x$ **15.** $3\cos\frac{x}{5}$

11.3 Values of Trigonometric Functions

We know the values of trigonometric functions for angles of measure 0°, 30°, 45°, 60°, and 90°. We have also established the following identities:

$$\begin{array}{llllll}
\sin(-\theta) & = -\sin\theta & \cos(-\theta) & = \cos\theta & \tan(-\theta) & = -\tan\theta \\
\sin(\pi-\theta) & = \sin\theta & \cos(\pi-\theta) & = -\cos\theta & \tan(\pi-\theta) & = -\tan\theta \\
\sin(\pi+\theta) & = -\sin\theta & \cos(\pi+\theta) & = -\cos\theta & \tan(\pi+\theta) & = \tan\theta \\
\sin(2\pi-\theta) & = -\sin\theta & \cos(2\pi-\theta) & = \cos\theta & \tan(2\pi-\theta) & = -\tan\theta
\end{array}$$

By using the above identities, we can easily find the values of trigonometric functions of the angles of the following measures:

$-30^\circ, -45^\circ, -60^\circ, -90^\circ$

$\pm 120^\circ, \pm 135^\circ, \pm 150^\circ, \pm 180^\circ$

$\pm 210^\circ, \pm 225^\circ, \pm 240^\circ, \pm 270^\circ$

$\pm 300^\circ, \pm 315^\circ, \pm 330^\circ, \pm 360^\circ$.

11.4 Graphs of Trigonometric Functions

We shall now learn the method of drawing the graphs of all the six trigonometric functions. These graphs are used very often in calculus and social sciences. For graphing the linear equations of the form:

$$a_1x + b_1y + c_1 = 0 \qquad \text{(i)}$$
$$a_2x + b_2y + c_2 = 0 \qquad \text{(ii)}$$

We have been using the following procedure.

i) tables of the ordered pairs are constructed from the given equations,

ii) the points corresponding to these ordered pairs are plotted/located,

and iii) the points, representing them are joined by line segments.

Exactly the same procedure is adopted to draw the graphs of the trigonometric functions except for joining the points by the line segments.

For this purpose,

i) table of ordered pairs (x, y) is constructed, when x is the measure of the angle and y is the value of the trigonometric ratio for the angle of measure x;

ii) The measures of the angles are taken along the X- axis;

iii) The values of the trigonometric functions are taken along the Y-axis;

iv) The points corresponding to the ordered pairs are plotted on the graph paper,

v) These points are joined with the help of **smooth ciurves**.

Note: As we shall see that the graphs of trigonometric functions will be smooth curves and none of them will be line segments or will have sharp corners or breaks **within their domains.** This behaviour of the curve is called **continuity**. It means that the trigonometric functions are continuous, wherever they are defined. Moreover, **as the trigonometric functions are periodic so their curves repeat after a fixed interval.**

11.5 Graph of y = sin x from -2π to 2π

We know that the period of sine function is 2π so, we will first draw the graph for the interval from 0° to 360° i.e., from 0 to 2π.

To graph the *sine* function, first, recall that $-1 \le \sin x \le 1$ for all $x \in R$

i.e., the range of the sine function is [-1, 1], so the graph will be between the horizontal lines y = +1 and y = – 1

The table of the ordered pairs satisfying y = sin x is as follows:

x	0 or 0°	$\frac{\pi}{6}$ or 30°	$\frac{\pi}{3}$ or 60°	$\frac{\pi}{2}$ or 90°	$\frac{2\pi}{3}$ or 120°	$\frac{5\pi}{6}$ or 150°	π or 180°	$\frac{7\pi}{6}$ or 210°	$\frac{4\pi}{3}$ or 240°	$\frac{3\pi}{2}$ or 270°	$\frac{5\pi}{3}$ or 300°	$\frac{11\pi}{6}$ or 330°	2π or 360°
Sin x	0	0.5	0.87	1	0.87	0.5	0	–0.5	–0.87	–1	–0.87	–0.5	0

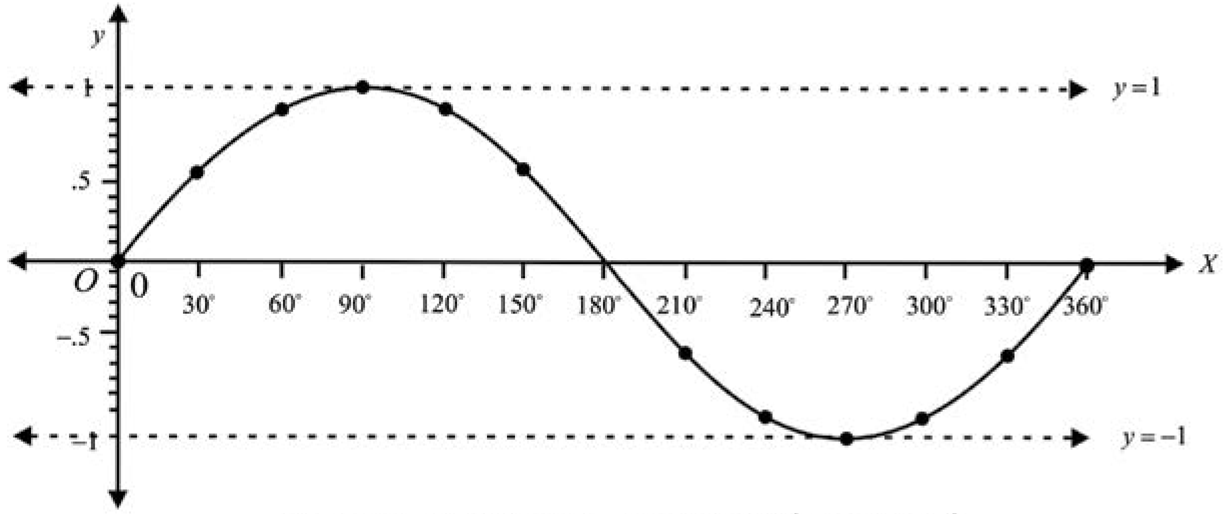
To draw the graph

i) Take a convenient scale $\begin{cases} 1 \text{ side of small square on the } x-\text{axis} = 10^\circ \\ 1 \text{ side of big square on the } y-\text{axis} = 1 \text{unit} \end{cases}$

ii) Draw the coordinate axes.

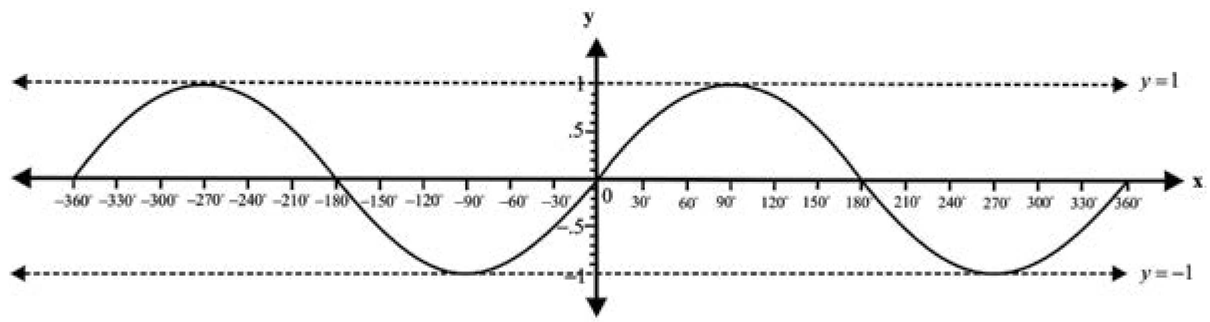
iii) Plot the points corresponding to the ordered pairs in the table above i.e., (0, 0), (30°, 0.5), (60°, 0.87) and so on,

(iv) Join the points with the help of a smooth curve as shown so we get the graph of y = sin x from 0 to 360° i.e., from 0 to 2π.



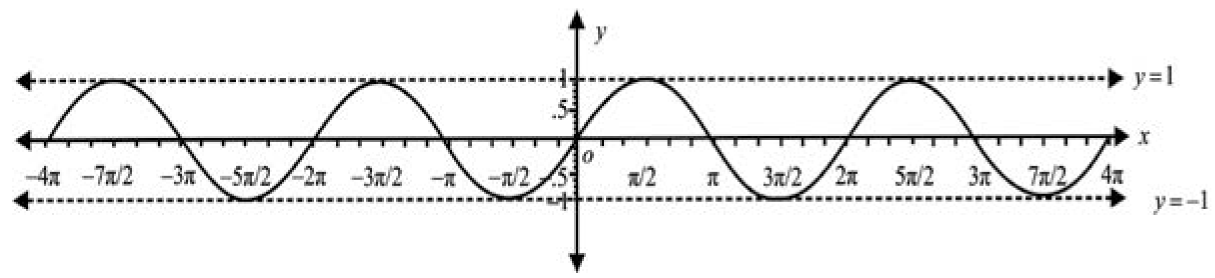
Graph of $y = \sin x$ from 0° to 360°

In a similar way, we can draw the graph for the interval from 0° to –360°. This will complete the graph of y = sin x from –360° to 360° i.e. from – 2π to 2π, which is given below:



Graph of $y = \sin x$ from −360° to 360°

The graph in the interval [0, 2π] is called a cycle. Since the period of sine function is 2π, so the sine graph can be extended on both sides of x-axis through every interval of 2π (360°) as shown below:



11.6 Graph of y = cos x from -2π to 2π

We know that the period of cosine function is 2π so, we will first draw the graph for the interval from 0° to 360° i.e., from 0 to 2π

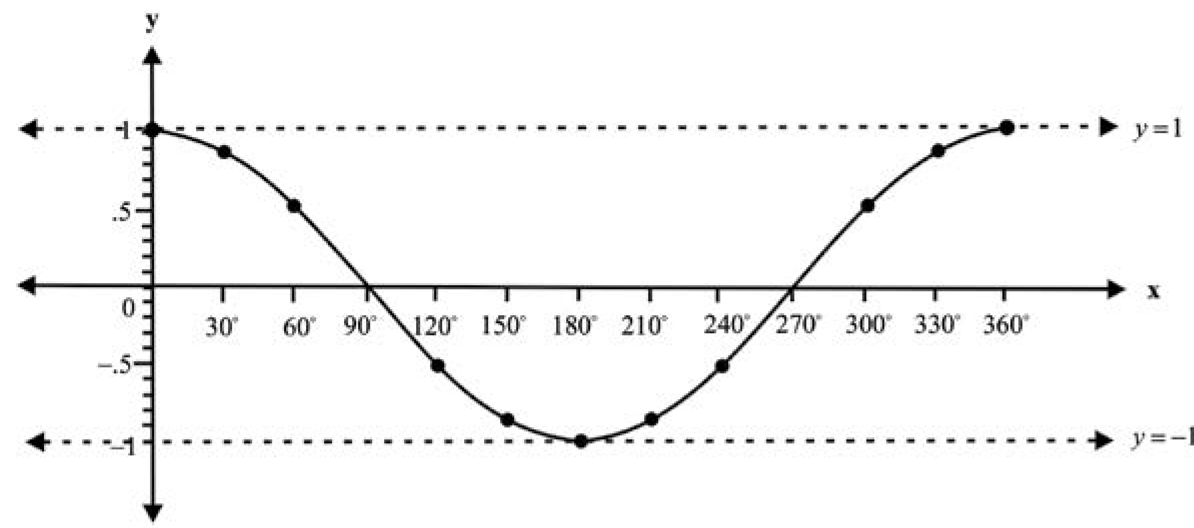
To graph the cosine function, first, recall that $-1 \le \sin x \le 1$ for all $x \in R$

i.e., the range of the cosine function is [-1, 1], so the graph will be between the horizontal lines y = +1 and y = –1

The table of the ordered pairs satisfying y = cos x is as follows:

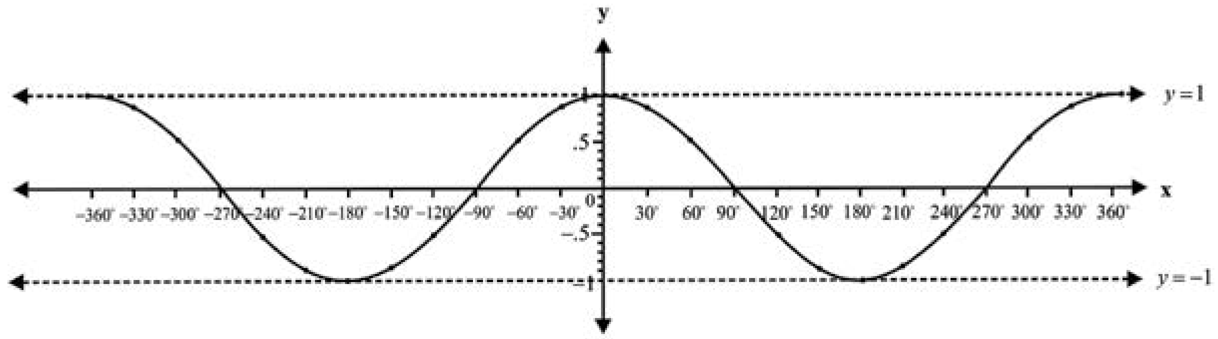
x	0 or 0°	$\frac{\pi}{6}$ or 30°	$\frac{\pi}{3}$ or 60°	$\frac{\pi}{2}$ or 90°	$\frac{2\pi}{3}$ or 120°	$\frac{5\pi}{6}$ or 150°	π or 180°	$\frac{7\pi}{6}$ or 210°	$\frac{4\pi}{3}$ or 240°	$\frac{3\pi}{2}$ or 270°	$\frac{5\pi}{3}$ or 300°	$\frac{11\pi}{6}$ or 330°	2π or 360°
cos x	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	0.5	0.87	1

The graph of y = cos x from 0° to 360° is given below:



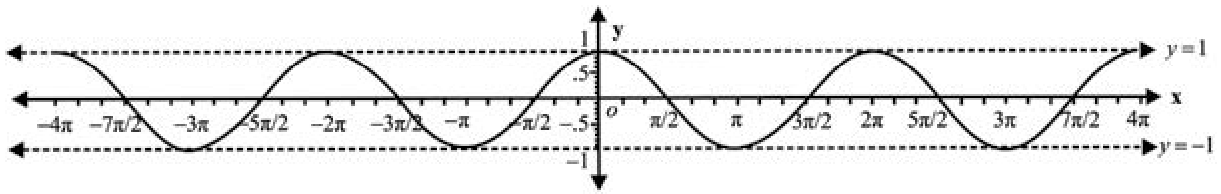
Graph of y = cos x from 0° to 360°

In a similar way, we can draw the graph for the interval from 0° to –360°. This will complete the graph of y = cos x from –360° to 360° i.e. from -2π to 2π, which is given below:



Graph of $y = \cos x$ from -360° to 360°

As in the case of *sine* graph, the *cosine* graph is also extended on both sides of x-axis through an interval of 2π as shown above:



Graph of $y = \sin x$ from -4 π to 4 π

11.7 Graph of y = tan x from $-\pi$ To π

We know that tan $(-x)$ = – tan x and tan $(\pi - x)$ = – tan x, so the values of tan x for x = 0°, 30°, 45°, 60° can help us in making the table.

Also we know that tan x is undefined at x = ± 90°, when

i) x approaches $\frac{\pi}{2}$ from left i.e., $x \to \frac{\pi}{2} - 0$, tan x increases indefinitely in I Quard.

ii) x approaches $\frac{\pi}{2}$ from right i.e., $x \to \frac{\pi}{2} + 0$, tan x increases indefinitely in IV Quard.

iii) x approaches $-\frac{\pi}{2}$ from left i.e., $x \to -\frac{\pi}{2} - 0$ tan x increases indefinitely in II Quard.

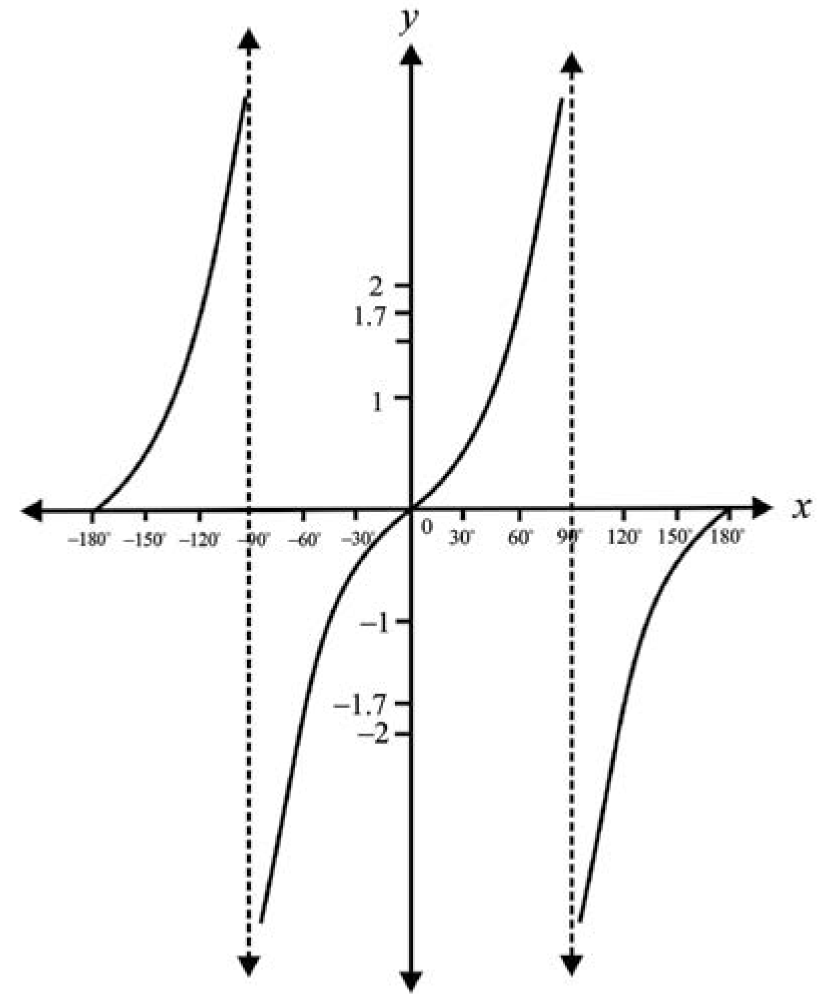
iv) x approaches $-\frac{\pi}{2}$ from right i.e., $x \to -\frac{\pi}{2} + 0$, tan x increases indefinitely in III Quard.

We know that the period of tangent is π, so we shall first draw the graph for the interval from $-\pi$ to π i.e., from –180° to 180°

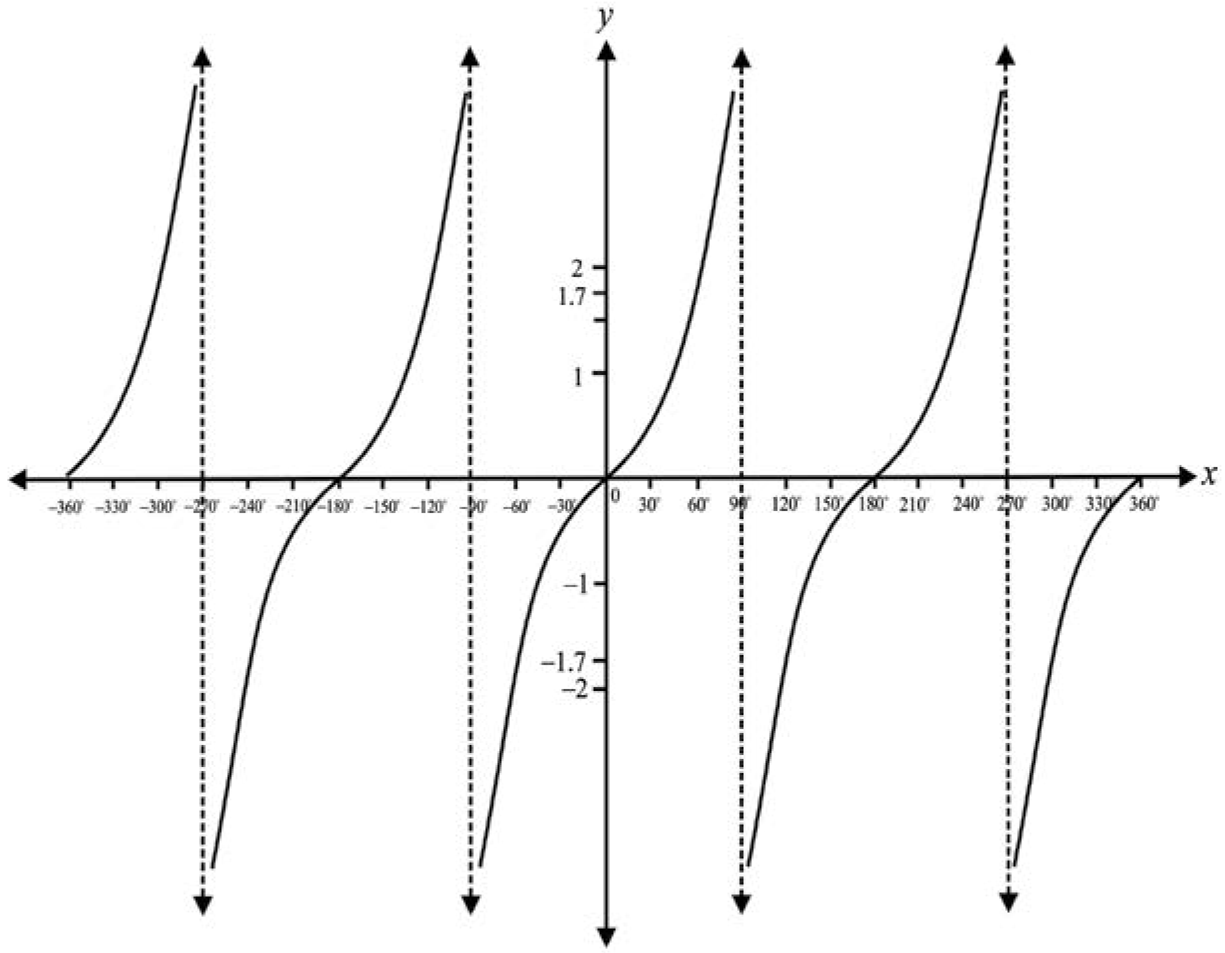
$\therefore$ The table of ordered pairs satisfying y = tan x is given below:

x	$-\pi$ or -180°	$-\frac{5\pi}{6}$ or -150°	$-\frac{2\pi}{3}$ or -120°	$-\frac{\pi}{2}-0$ or $-90^\circ-0$	$-\frac{\pi}{2}+0$ or $-90+0$	$-\frac{\pi}{3}$ or -60°	$-\frac{\pi}{6}$ or -30°	0 or 0	$\frac{\pi}{6}$ or 30°	$\frac{\pi}{3}$ or 60°	$\frac{\pi}{2}-0$ or $90^\circ-0$	$\frac{\pi}{2}+0$ or $90^\circ+0$	$\frac{2\pi}{3}$ or 120°	$\frac{5\pi}{6}$ or 150°	π or 180°
Tan x	0	0.58	1.73	$+\infty$	$-\infty$	–1.73	–0.58	0	0.58	1.73	$+\infty$	$-\infty$	–1.73	–0.58	0

Graph of $y = \tan x$ from -180° to 180°



We know that the period of the tangent function is π. The graph is extended on both sides of x-axis through an interval of π in the same pattern and so we obtain the graph of y = tan x from –360° to 360° as shown below:



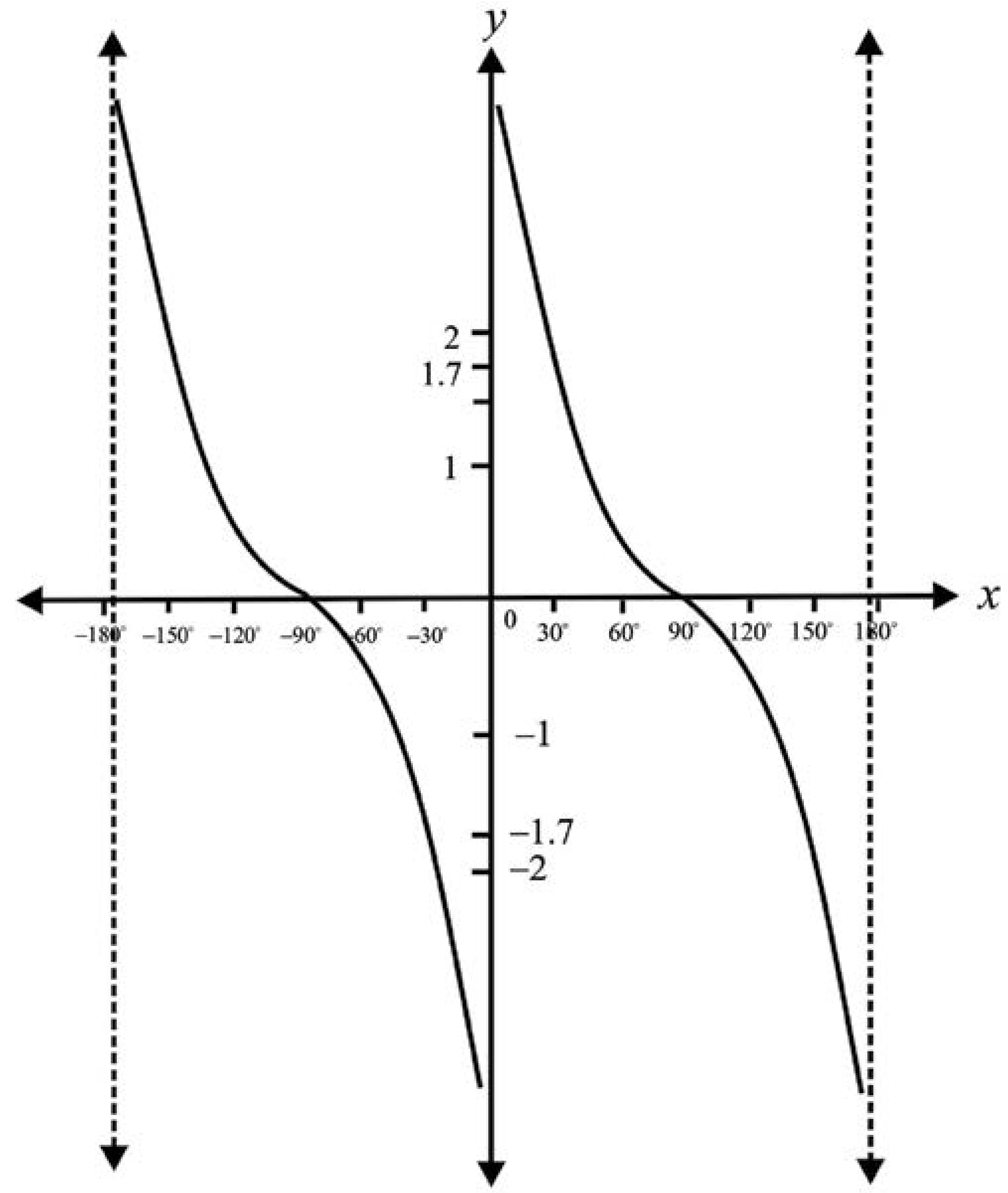
Graph of y = tan x from – 360° to 360°

11.8 Graph of y = cot x From -2π to π

We know that cot ($-x$) = – cot x and cot (π – x) = – cot x, so the values of cot x for x = 0°, 30°, 45°, 60°, 90° can help us in making the table.

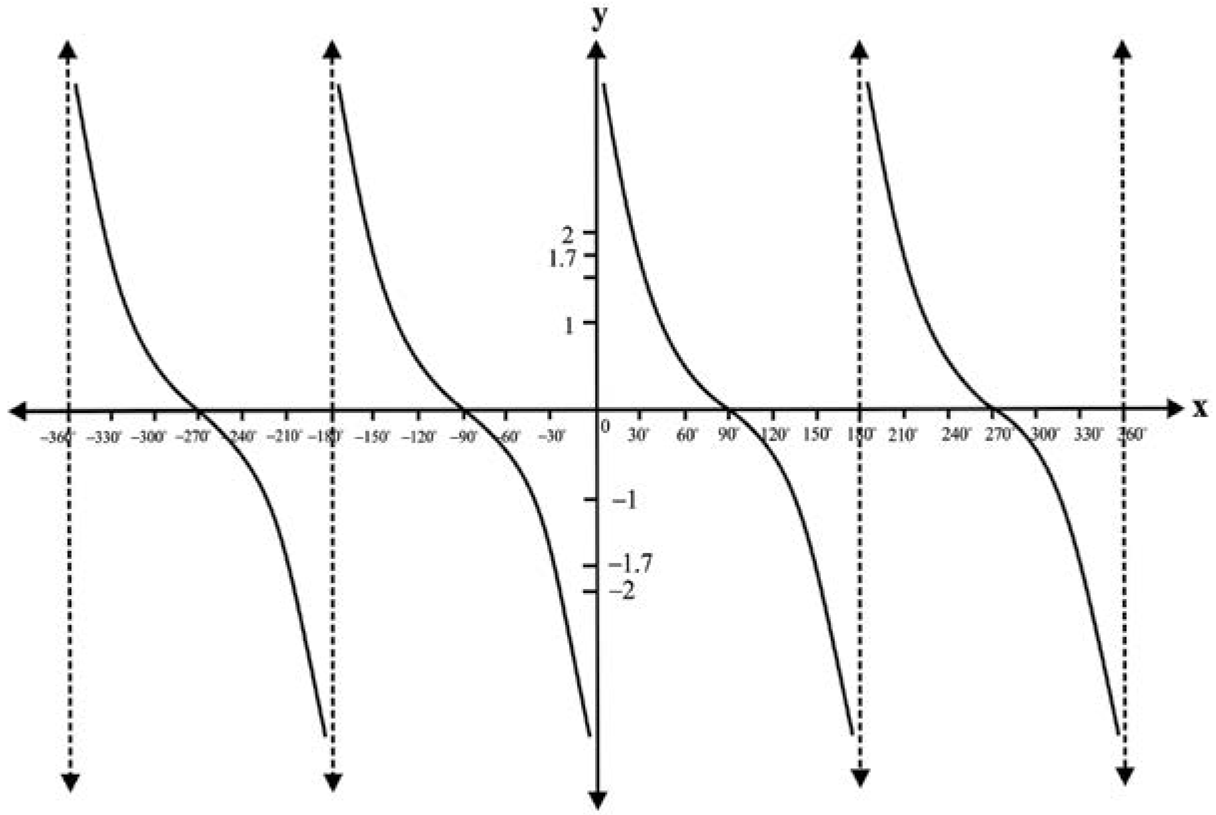
The period of the *cotangent function* is also π. So its graph is drawn in a similar way of *tangent* graph using the table given below for the interval from –180° to 180°.

x	$-\pi$ or -180°	$-\frac{5\pi}{6}$ or -150°	$-\frac{2\pi}{3}$ or -120°	$-\frac{\pi}{2}-0$ or $-90^\circ-0$	$-\frac{\pi}{2}+0$ or $-90^\circ+0$	$-\frac{\pi}{3}$ or -60°	$-\frac{\pi}{6}$ or -30°	0	$\frac{\pi}{6}$ or 30°	$\frac{\pi}{3}$ or 60°	$\frac{\pi}{2}-0$ or $90^\circ-0$	$\frac{\pi}{2}+0$ or $90^\circ+0$	$\frac{2\pi}{3}$ or 120°	$\frac{5\pi}{6}$ or 150°	π or 180°
cot x	$\pm\infty$	1.73	0.58	$+\infty$	$-\infty$	–0.58	–1.73	$\pm\infty$	1.73	0.58	$+\infty$	$-\infty$	–0.58	–1.73	$\pm\infty$



Graph of y = cot x from – 180° to 180°

We know that the period of the cotangent function is π. The graph is extended on both sides of x – axis through an interval of π in the same pattern and so we obtain the graph of y = cot$x$ from from –360° to 360° as shown below:



Graph of y = cot x from – 360° to 360°

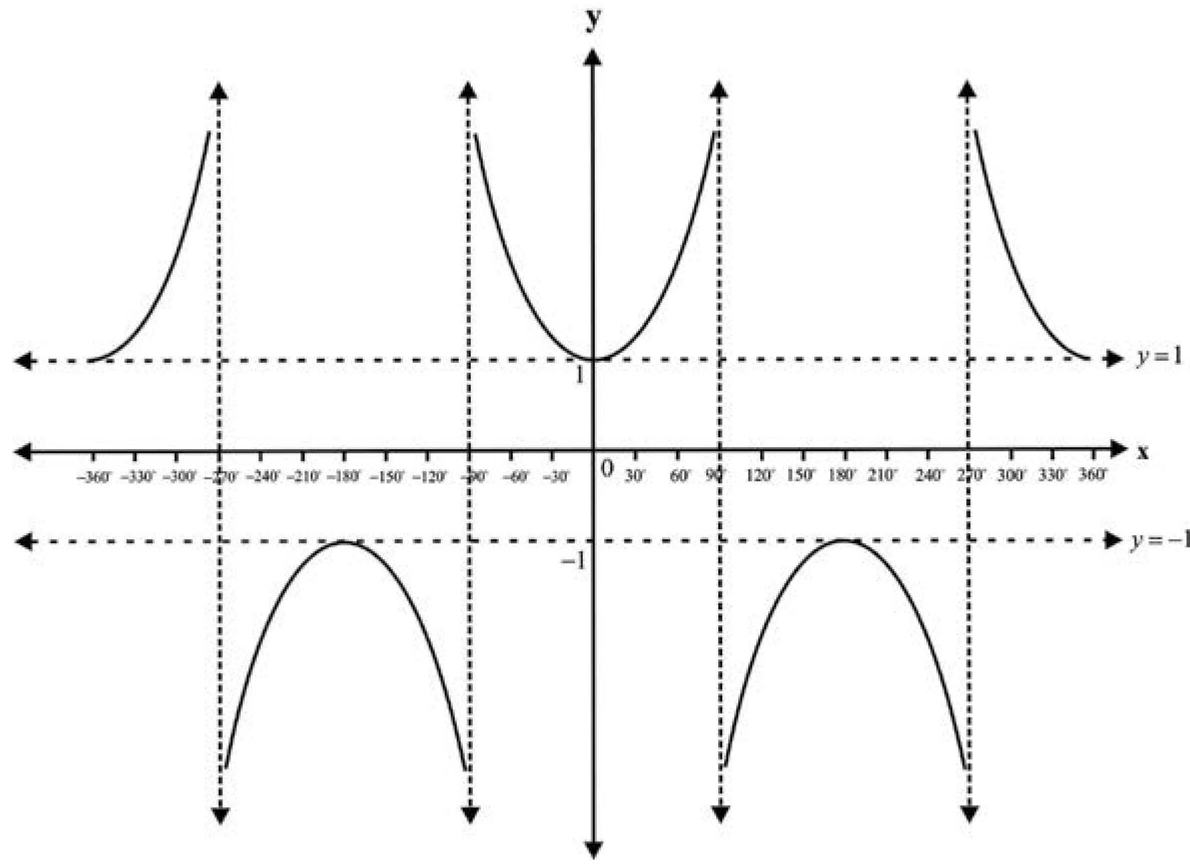
11.9 Graph of y = sec x from -2π to 2π

We know that $\sec(-x) = \sec x$ and $\sec(\pi - x) = -\sec x$,

So the values of sec x for x = 0°, 30°, 45°, 60°, can help us in making the following table of the ordered pairs for drawing the graph of y = sec x for the interval 0° to 360°:

x	0 or 0	$\frac{\pi}{6}$ or 30°	$\frac{\pi}{3}$ or 60°	$\frac{\pi}{2}-0$ or 90−0	$\frac{\pi}{2}+0$ or 90+0	$\frac{2\pi}{3}$ or 120°	$\frac{5\pi}{6}$ or 150°	π or 180°	$\frac{7\pi}{6}$ or 210°	$\frac{4\pi}{3}$ or 240°	$\frac{3\pi}{2}-0$ or 270−0	$\frac{3\pi}{2}+0$ or 270+0	$\frac{5\pi}{3}$ or 300°	$\frac{5\pi}{6}$ or 330°	2π or 360°
Sec x	1	1.15	2	∞	$-\infty$	−2	−1.15	−1	−1.15	−2	$-\infty$	$+\infty$	2	1.15	1

Since the period of sec x is also 2π, so we have the following graph of y = sec x from – 360° to 360° i.e., from -2π to 2π:



Graph of y = sec x from – 360° to 360°

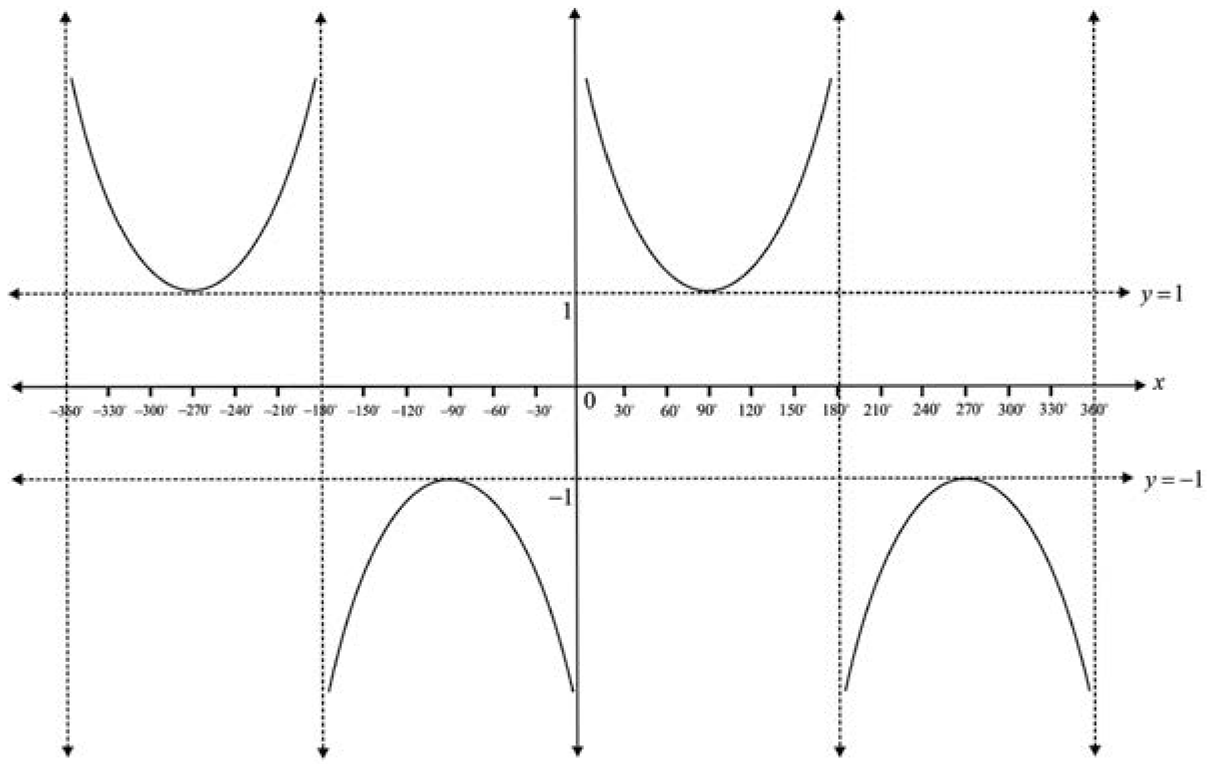
11.10 Graph of y = csc x from -2π to 2π

We know that: csc $(-x)$ = – csc x and csc $(\pi - x)$ = csc x

So the values of csc x for x = 0°, 30°, 45°, 60°, can help us in making the following table of the ordered pairs for drawing the graph of y = csc x for the interval 0° to 360°:

x	0+0 or 0+0	$\frac{\pi}{6}$ or 30°	$\frac{\pi}{6}$ or 60°	$\frac{\pi}{2}$ or 90°	$\frac{2\pi}{3}$ or 120°	$\frac{3\pi}{6}$ or 150°	$\pi-0$ or 180−0	$\pi+0$ or 180+0	$\frac{7\pi}{6}$ or 210°	$\frac{4\pi}{3}$ or 240°	$\frac{3\pi}{2}$ or 270°	$\frac{5\pi}{2}$ or 300°	$\frac{4\pi}{3}$ or 330°	$2\pi-0$ or 360°
csc x	∞	2	1.15	1	1.15	2	∞	$-\infty$	−2	−1.15	−1	−1.15	−2	$-\infty$

Since the period of csc x is also 2π, so we have the following graph of



$y = \csc x$ from -360° to 360° i.e., from -2π to 2π:

Graph of $y = \csc x$ from -360° to 360°

Note 1: From the graphs of trigonometric functions we can check their *domains* and *ranges*.

Note 2: By making use of the periodic property, each one of these graphs can be extended on the left as well as on the right side of x-axis depending upon the *period* of the functions.

Note 3: The dashes lines are vertical asymptotes in the graphs of tan x, cot x, sec x and csc x.

Exercise 11.2

1. Draw the graph of each of the following function for the intervals mentioned against each :

i) $y = -\sin x, \quad x \in [-2\pi, 2\pi]$

ii) $y = 2\cos x, \quad x \in [0, 2\pi]$

iii) $y = \tan 2x, \quad x \in [-\pi, \pi]$

iv) $y = \tan x, \quad x \in [-2\pi, 2\pi]$

v) $y = \sin \dfrac{x}{2}, \quad x \in [0, 2\pi]$

vi) $y = \cos \dfrac{x}{2}, \quad x \in [-\pi, \pi]$

2. On the same axes and to the same scale, draw the graphs of the following function for their complete period:

i) $y = \sin x$ and $y = \sin 2x$

ii) $y = \cos x$ and $y = \cos 2x$

3. Solve graphically:

i) $\sin x = \cos x, \quad x \in [0, \pi]$

ii) $\sin x = x, \quad x \in [0, \pi]$