CHAPTER

12

Application of Trigonometry

12.1 Introduction

A triangle has six important elements; three angles and three sides. In a triangle *ABC*, the measures of the three angles are usually denoted by α, β, γ and the measures of the three sides opposite to them are denoted by *a, b, c* respectively.

If any three out of these six elements, out of which atleast one side, are given, the remaining three elements can be determined This process of finding the unknown elements is called the **solution of the triangle.**

We have calculated the values of the trigonometric functions of the angles measuring 0°, 30°, 45°, 60° and 90°. But in a triangle, the angles are not necessarily of these few measures. So, in the solution of triangles, we may have to solve problems involving angles of measures other than these. In such cases, we shall have to consult natural sin/cos/tan tables or we may use $\boxed{\sin}, \boxed{\cos}, \boxed{\tan}$ keys on the calculator.

Tables/calculator will also be used for finding the measures of the angles when value of trigonometric ratios are given e.g. to find θ when $\sin\theta = x$.

12.2 Tables of Trigonometric Ratios

Mathematicians have constructed tables giving the values of the trigonometric ratios of large number of angles between 0° and 90°. These are called tables of natural **sines, cosines, tangents** etc. In four-figure tables, the interval is 6 minutes and difference corresponding to 1,2, 3, 4, 5 minutes are given in the **difference columns.**
The following examples will illustrate how to consult these tables.

Example 1: Find the value of

i) sin 38° 24′ ii) sin 38° 28′ iii) tan 65° 30′.

Solution: In the first column on the left hand side headed by degrees (in the Natural Sine table) we read the number 38°. Looking along the row of 38° till the minute column number 24′ is reached, we get the number 0.6211.

$$\therefore \quad \sin 38^\circ 24' = 0.6211$$

ii) To find sin 38° 28′, we first find sin 38° 24′, and then see the right hand column headed by **mean differences.** Running down the column under 4′ till the row of 38° is reached. We find 9 as the difference for 4′. Adding 9 to 6211, we get 6220.

$$\therefore \quad \sin 38^\circ 24' = 0.6220$$

Note: **1.** As sin θ, sec θ and tan θ go on increasing as θ increases from 0° to 90°, so the numbers in the columns of the differences for sin θ, sec θ and tan θ are added.

2. Since cos θ, cosec θ and cot θ decrease as θ increases from 0° to 90°, therefore, for cos θ, cosec θ and cot θ the numbers in the column of the, differences are subtracted.

iii) Turning to the tables of Natural Tangents read the number 65° in the first column on the left hand side headed by degrees. Looking along the row of 65° till the minute column under 30′ is reached, we get the number 1943. The integral part of the figure just next to 65° in the horizontal line is 2.

$$\therefore \quad \tan 65^\circ 30' = 2.1943$$

Example 2: If $\sin x = 0.5100$, find x.

Solution: In the tables of Natural Sines, we get the number (nearest to 5100) 5090 which lies at the intersection of the row beginning with 30° and the column headed by 36′. The difference between 5100 and 5090 is 10 which occurs in the row of 30° under the mean difference column headed by 4′. So, we add 4′ to 30° 36′ and get

$$\sin^{-1}(0.5100) = 30^\circ 40'$$

Hence x = 30° 40

Exercise 12.1

1. Find the values of:

i)	$\sin 53^\circ 40'$	ii)	$\cos 36^\circ 20'$	iii)	$\tan 19^\circ 30'$
iv)	$\cot 33^\circ 50'$	v)	$\cos 42^\circ 38'$	vi)	$\tan 25^\circ 34'$
vii)	$\sin 18^\circ 31'$	viii)	$\cos 52^\circ 13'$	ix)	$\cot 89^\circ 9'$

2. Find θ, if:

i) $\sin\theta = 0.5791$ ii) $\cos\theta = 0.9316$

iii) $\cos\theta = 0.5257$ iv) $\tan\theta = 1.705$

v) $\tan\theta = 21.943$ vi) $\sin\theta = 0.5186$

12.3 Solution of Right Triangles

In order to solve a right triangle, we have to find:

i) the measures of two acute angles

and ii) the lengths of the three sides.

We know that a trigonometric ratio of an acute angle of a right triangle involves 3 quantities "lengths of two sides and measure of an angle". Thus if two out of these three quantities are known, we can find the third quantity.

Let us consider the following two cases in solving a right triangle:

CASE I: When Measures of Two Sides are Given

Example 1: Solve the right triangle ABC, in which b = 30.8, c = 37.2 and γ= 90°.

Solution: From the figure,

$$\cos\alpha = \frac{b}{c} = \frac{30.8}{37.2} = 0.8280$$

$\Rightarrow \quad \alpha = \cos^{-1} 0.8280 = 34^\circ 6'$

$\because \quad \gamma = 90^\circ \quad \Rightarrow \beta = 90^\circ - \alpha = 90^\circ - 34^\circ 6, = 55^\circ 54,$

$\because \quad \frac{a}{c} = \sin\alpha$

$\Rightarrow \quad a = c\sin\alpha = 37.2 \sin 34^\circ 6,$

$= 37.2(0.5606)$

$= 20.855$

$\Rightarrow \quad a = 20.9$

Hence $a = 20.9, \quad \alpha = 34^\circ$ and $\beta = 55^\circ 54$

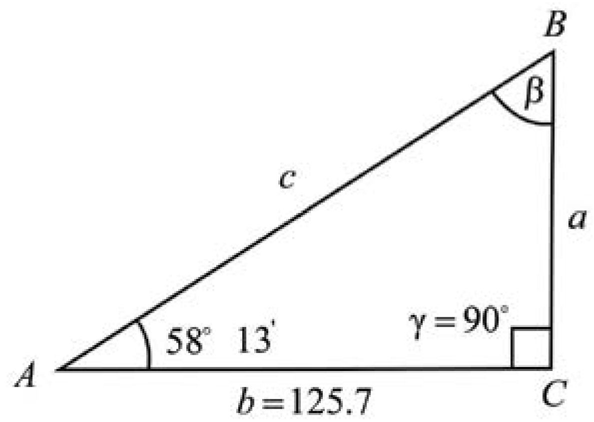
CASE II: When Measures of One Side and One Angle are Given

Example 2: Solve the right triangle, in which

$\alpha = 58^\circ 13'$, $b = 125.7$ and $\gamma = 90^\circ$

Solution: $\because \gamma = 90^\circ, \alpha = 58^\circ 13' \quad \therefore \quad \beta = 90^\circ - 58^\circ 13' = 31^\circ 47'$

From the figure,



$\frac{a}{b} = \tan 58^\circ 13'$

$\Rightarrow \quad a = (125.7)\tan 58^\circ 13'$

$= 125.7(1.6139)$

$= 202.865$

$\therefore \quad a = 202.9$

Again $\quad \frac{a}{c} = \sin 58^\circ 13'$

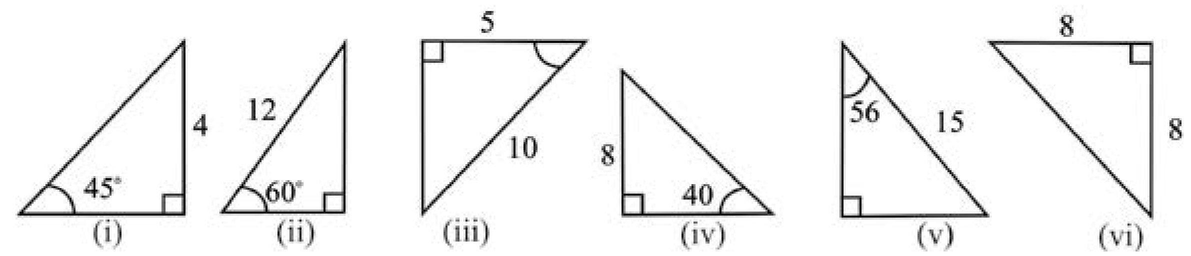
$\Rightarrow \quad c = \frac{202.9}{0.8500}$

$\therefore \quad c = 238.7$

Hence $\quad a = 202.9, \ \beta = 31^\circ 47'$ and $c = 238.7$

Exercise 12.2

1. Find the unknown angles and sides of the following triangles:



Solve the right triangle ABC, in which $\gamma = 90°$

2. $\alpha = 37^\circ 20'$, $a = 243$ **3.** $\alpha = 62^\circ 40'$, $b = 796$

4. $a = 3.28$, $b = 5.74$ **5.** $b = 68.4$, $c = 96.2$

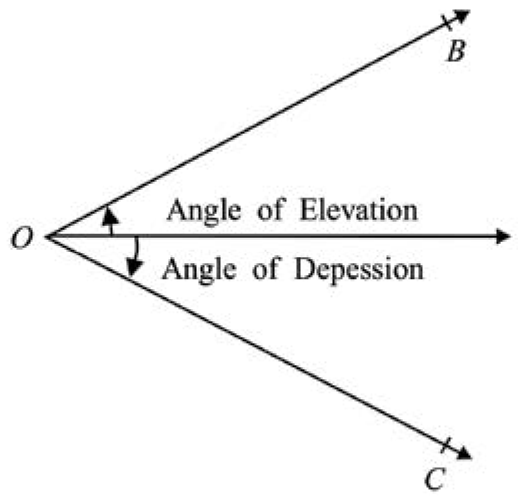
6. $a = 5429$, $c = 6294$ **7.** $\beta = 50^\circ 10'$, $c = 0.832$

12.4 (a) Heights And Distances

One of the chief advantages of trigonometry lies in finding heights and distances of inaccessible objecst:

In order to solve such problems, the following procedure is adopted:

1) Construct a clear labelled diagram, showing the known measurements.
2) Establish the relationships between the quantities in the diagram to form equations containing trigonometric ratios.
3) Use tables or calculator to find the solution.



(b) Angles of Elevation and Depression

If $\overrightarrow{OA}$ is the horizontal ray through the eye of the observer at point *O*, and there are two objects *B* and *C* such that *B* is above and *C* is below the horizontal ray $\overrightarrow{OA}$, then,

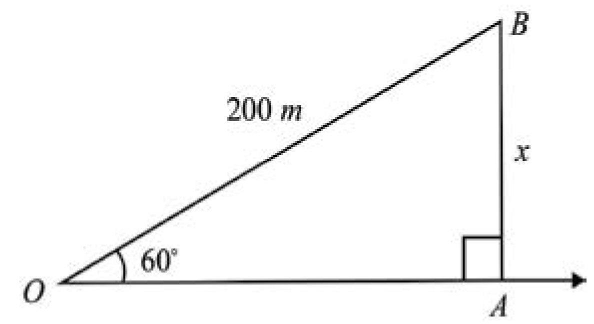
i) for looking at *B* above the horizontal ray, we have to raise our eye , and $\angle AOB$ is called the **Angle of Elevation** and

ii) for looking at *C* below the horizontal ray we have to lower our eye , and $\angle AOC$ is called the **Angle of Depression.**

Example 1: A string of a flying kite is 200 meters long, and its angle of elevation is 60°. Find the height of the kite above the ground taking the string to be fully stretched.

Solution: Let *O* be the position of the observer, *B* be the position of the kite and $\overrightarrow{OA}$ be the horizontal ray through *O*.

Draw $\overline{BA} \perp \overrightarrow{OA}$



Now $m\angle O = 60^\circ$ and $OB = 200m$

Suppose $AB = x$ meters

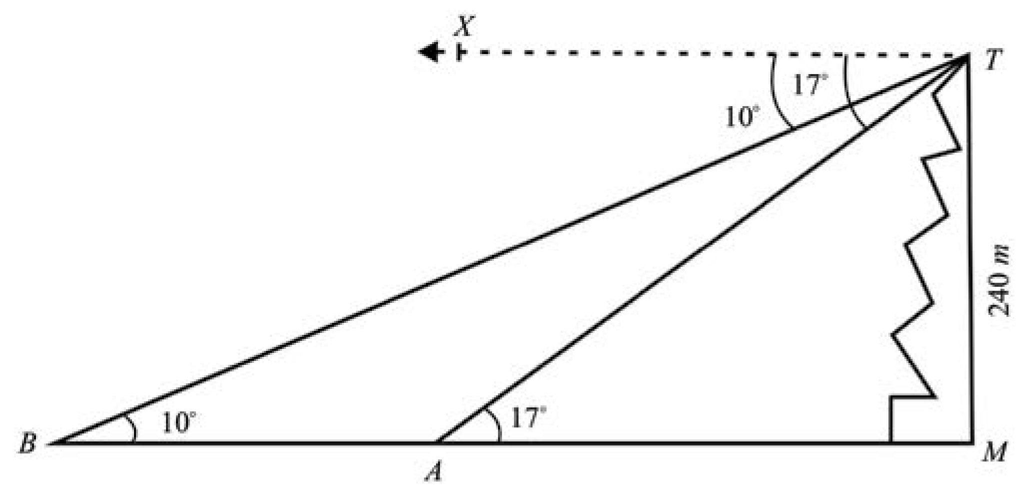
In right ΔOAB,

$$\frac{x}{200} = \sin 60^\circ = \frac{\sqrt{3}}{2} = \frac{1.732}{2}$$

$$\Rightarrow \quad x = 200\left(\frac{1.732}{2}\right) = 100(1.732) = 173.2$$

Hence the height of the kite above the ground = 173.2 m.

Example 2: A surveyor stands on the top of 240 m high hill by the side of a lake. He observes two boats at the angles of depression of measures 17° and 10°. If the boats are in the same straight line with the foot of the hill just below the observer, find the distance between the two boats, if they are on the same side of the hill.



Solution: Let *T* be the top of the hill $\overline{TM}$, where the observer is stationed, *A* and *B* be the positions of the two boats so that $m\angle XTB$ = 10° and $m\angle XTA$ = 17° and *TM* = 240*m* :

Now, $m\angle MAT = m\angle XTA = 17^\circ \quad (\because \overline{TX} \parallel \overline{MA})$

and $m\angle MBT = m\angle XTB = 10^\circ \quad (\because \overline{TX} \parallel \overline{MA})$

From the figure, $\dfrac{\overline{TM}}{\overline{AM}} = \tan 17^\circ$

$$\Rightarrow \quad \overline{AM} = \frac{\overline{TM}}{\tan 17^\circ} = \frac{240}{0.3057}$$

$$\Rightarrow \quad \overline{AM} = 785m$$

and $\dfrac{\overline{TM}}{\overline{BM}} = \tan 10^\circ$

$$\Rightarrow \overline{BM} = \frac{\overline{TM}}{\tan 10^\circ} = \frac{240}{0.1763} = 1361m$$

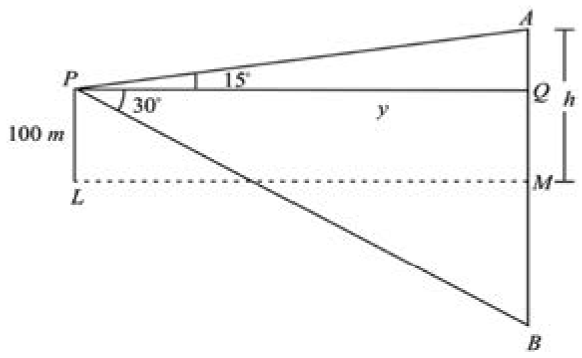
$$\therefore \overline{AB} = \overline{BM} - \overline{AM} = 1361 - 785 = 576m$$

Hence the distance between the boats = 576m.

Example 3: From a point 100 m above the surface of a lake, the angle of elevation of a peak of a cliff is found to be 15° and the angle of depression of the image of the peak is 30°. Find the height of the peak.

Solution:

Let A be the top of,the peak $\overline{AM}$ and $\overline{MB}$ be its image. Let P be the point of observation and L be the point just below P (on the surface of the lake).



such that $\overline{PL}$ = 100m

From P, draw $\overline{PQ} \perp \overline{AM}$.

Let $\overline{PQ}$ = y metres and $\overline{AM}$ = h metres.

$$\therefore \overline{AQ} = h - \overline{QM} = h - \overline{PL} = h - 100$$

From the figure,

$$\tan 15^\circ = \frac{\overline{AQ}}{\overline{PQ}} = \frac{h-100}{y} \quad \text{and} \quad \tan 30^\circ = \frac{\overline{BQ}}{\overline{PQ}} = \frac{100+h}{y}$$

By division, we get

$$\frac{\tan 15^\circ}{\tan 30^\circ} = \frac{h-100}{h+100}$$

By Componendo and Dividendo, we have

$$\frac{\tan 15^\circ + \tan 30^\circ}{\tan 15^\circ - \tan 30} = \frac{h-100+h+100}{h-100-h-100} = \frac{2h}{-200} = \frac{h}{-100}$$

$$\therefore \quad h = \frac{\tan 30^\circ + \tan 15^\circ}{\tan 30^\circ - \tan 15} \times 100 = \left[\frac{0.5774 + 0.2679}{0.5774 - 0.2679}\right] \times 100$$

$$\Rightarrow \quad h = 273.1179.$$

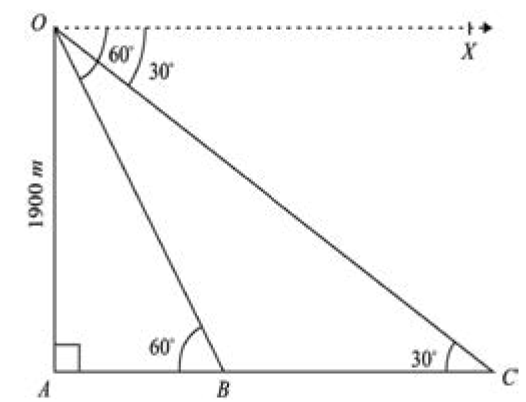
Hence height of the peak = 273 m. (Approximately)

12.5 Engineering and Heights and Distances

Engineers have to design the construction of roads and tunnels for which the knowledge of heights and distance is very useful to them. Moreover, they are also required to find the heights and distances of the out of reach objects.

Example 4: An O.P., sitting on a cliff 1900 meters high, finds himself in the same vertical plane with an anti-air-craft gun and an ammunition depot of the enemy. He observes that the angles of depression of the gun and the depot are 60° and 30° respectively. He passes this information on to the headquarters. Calculate the distance between the gun and the depot.

Solution: Let O be the position of the O.P., A be the point on the ground just below him and B and C be the positions of the gun and the depot respectively.



$$\overline{OA} = 1900m$$

$$m\angle BOX = 60^\circ$$

and $m\angle COX = 30^\circ$

$$\Rightarrow m\angle ABO = m\angle BOX = 60^\circ, \quad m\angle ACO = 30^\circ$$

In right ΔBAO, In right ΔCAO,

$$\frac{1900}{\overline{AB}} = \tan 60^\circ \qquad \frac{1900}{\overline{AC}} = \tan 30^\circ$$

$$\Rightarrow \overline{AB} = \frac{1900}{\tan 60^\circ} = \frac{1900}{\sqrt{3}} \qquad \overline{AC} = \frac{1900}{\tan 30^\circ}$$

Now $\overline{BC} = \overline{AC} - \overline{AB}$ $\Rightarrow$ $\overline{AC} = 1900\sqrt{3}$

$\Rightarrow$ $\overline{BC} = 1900\sqrt{3} - \dfrac{1900}{\sqrt{3}} = 2193.93$

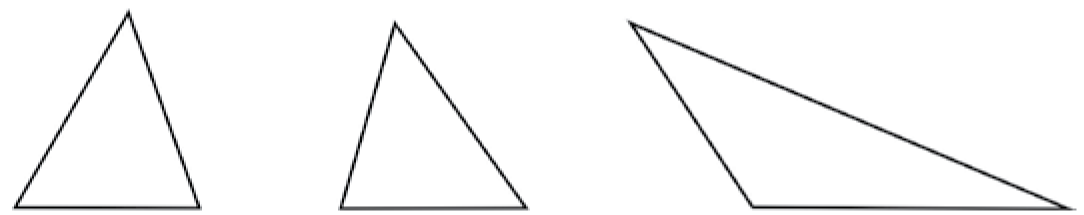
$\therefore$ Required distance = 2194 meters.

Exercise 12.3

1. A vertical pole is 8 m high and the length of its shadow is 6 m. What is the angle of elevation of the sun at that moment?
2. A man 18 dm tall observes that the angle of elevation of the top of a tree at a distance of 12 m from him is 32 . What is the height of the tree?
3. At the top of a cliff 80 m high, the angle of depression of a boat is 12°. How far is the boat from the cliff?
4. A ladder leaning against a vertical wall makes an angle of 24° with the wall. Its foot is 5m from the wall. Find its length.
5. A kite flying at a height of 67.2 m is attached to a fully stretched string inclined at an angle of 55° to the horizontal. Find the length of the string.
6. When the angle between the ground and the suri is 30°, flag pole casts a shadow of 40m long. Find the height of the top of the flag.
7. A plane flying directly above a post 6000 m away from an anti-aircraft gun observes the gun at an angle of depression of 27°.Find the height of the plane.
8. A man on the top of a 100 m high light-house is in line with two ships on the same side of it, whose angles of depression from the man are 17° and 19° respecting. Find the distance between the ships.
9. *P* and *Q* are two points in line with a tree. If the distance between *P* and *Q* be 30 m and the angles of elevation of the top of the tree at *P* and *Q* be 12° and l5° respectively, find the height of the tree.
10 Two men are on the opposite sides of a 100 m high tower. If the measures of the angles of elevation of the top of the tower are 18° and 22°respectively find the distance between them.
11. A man standing 60 m away from a tower notices that the angles of elevation of the top and the bottom of a flag staff on the top of the tower are 64° and 62° respectively. Find the length of the flag staff.
12. The angle of elevation of the top of a 60 m high tower from a point *A*, on the same level as the foot of the tower, is 25°. Find the angle of elevation of the top of the tower from a point *B*, 20 m nearer to *A* from the foot of the tower.
13. Two buildings *A* and *B* are 100 m apart. The angle of elevation from the top of the building *A* to the top of the building *B* is 20°. The angle of elevation from the base of the building *B* to the top of the building *A* is 50°. Find the height of the building *B*.
14. A window washer is working in a hotel building. An observer at a distance of 20 m from the building finds the angle of elevation of the worker to be of 30°. The worker climbs up 12 m and the observer moves 4 m farther away from the building. Find the new angle of elevatiqn of the worker.
15 A man standing on the bank of a canal observes that the measure of the angle of elevation of a tree on the other side of the canal, is 60. On retreating 40 meters from the bank, he finds the measure of the angle of elevation of the tree as 30 . Find the height of the tree and the width of the canal.

12.6 Oblique Triangles

A triangle, which is not right, is called an oblique triangle. Following triangles are not right, and so each one of them is oblique:



We have learnt the methods of solving right triangles. However, in solving oblique triangles, we have to make use of the relations between the sides *a, b, c* and the angle α,β,γ of such triangles, which are called **law of cosine, law of sines** and **law of tangents**.

Let us discover these laws one by one before solving oblique triangles.

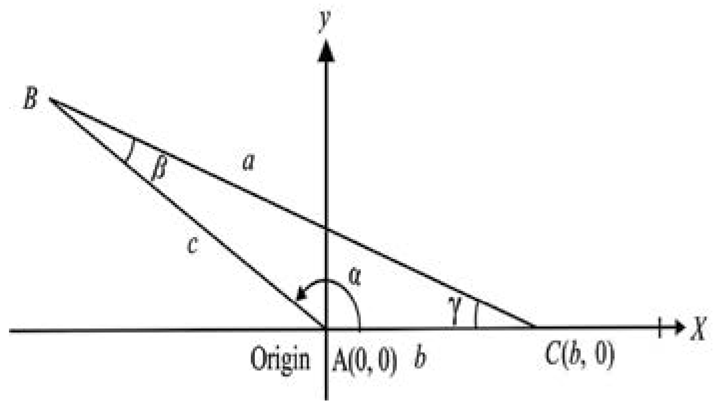
12.6.1 The Law of Cosine

In any triangle *ABC*, with usual notations, prove that:

i) $a^2 = b^2 + c^2 - 2bc\cos\alpha$

ii) $b^2 = c^2 + a^2 - 2ca\cos\beta$

iii) $c^2 = a^2 + b^2 - 2ab\cos\gamma$



Proof: Let side $\overline{AC}$ of triangle *ABC* be along the positive direction of the x-axis with vertex *A* at origin, then $\angle BAC$ will be in the standard position.

$\because\ \overline{AB} = c$ and $m\angle BAC = \alpha$

$\therefore$ coodinates of B are $(c\cos\alpha, c\sin\alpha)$

$\because\ AC = b$ and point *C* is on the x-axis

$\therefore$ Coordinates of *C* are (*b*, 0)

By distance formula,

$$\left|\overline{BC}\right|^2 = (c\cos\alpha - b)^2 + (c\sin\alpha - 0)^2$$

$$\Rightarrow\ a^2 = c^2\cos^2\alpha + b^2 - 2bc\cos\alpha + c^2\sin^2\alpha \quad (\because\ \overline{BC} = a)$$

$$\Rightarrow\ a^2 = c^2(\cos^2\alpha + \sin^2\alpha) + b^2 - 2bc\cos\alpha$$

$$\Rightarrow\ a^2 = b^2 + c^2 - 2bc\cos\alpha \qquad \text{(i)}$$

In a similar way, we can prove that

$$b^2 = c^2 + a^2 - 2ca\cos\beta \qquad \text{(ii)}$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \qquad \text{(iii)}$$

(i), (ii) and (iii) are called law of cosine. They can also be expressed as:

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\beta = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

Note: If ΔABC is right, then

Law of cosine reduces to Pythagorous Theorem i.e.,

if $\alpha = 90^\circ$ then $b^2 + c^2 = a^2$

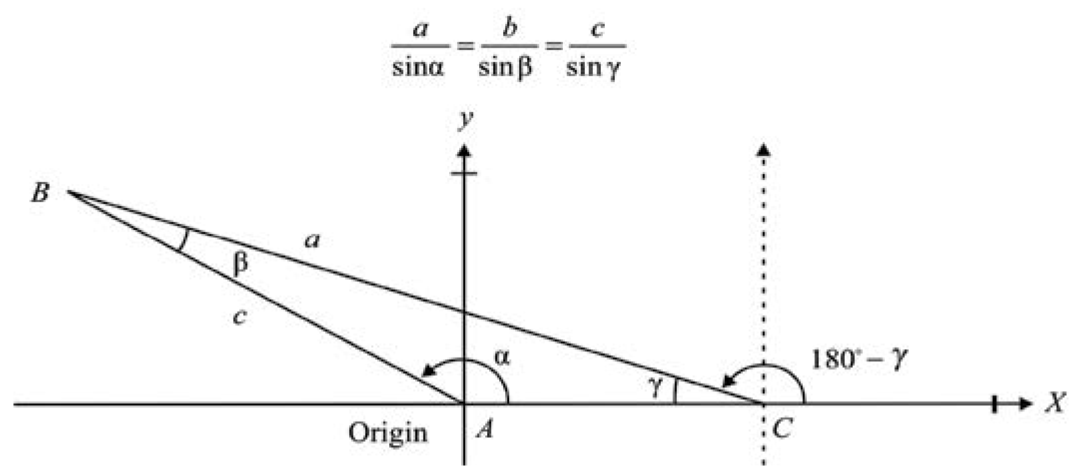
or if $\beta = 90^\circ$ then $c^2 + a^2 = b^2$

or if $\gamma = 90^\circ$ then $a^2 + b^2 = c^2$

12.6.2 The Law of Sines

In any triangle *ABC*, with usual notations, prove that:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$



Proof: Let side $\overline{AC}$ of.triangle *ABC* be along the positive direction of the x-axis with vertex *A* at origin, then $\angle BAC$ will be in the standard position.

$\therefore \quad \overline{AB} = c \quad$ and $\quad m\angle BAC = \alpha$

$\therefore$ The coodinates of the point B are $(c \cos\alpha, c \sin\alpha)$

If the origin A is shifted to C, then $\angle BCX$ will be in the standard position,

$\because \quad \overline{BC} = a \quad$ and $\quad m\angle BCX = 180^\circ - \gamma$

$\therefore$ The coodinates of B are $[a \cos(180^\circ - \gamma),\ a \sin(180^\circ - \gamma)]$

In both the cases, the y-coordinate of B remains the same

$\Rightarrow \quad a \sin(180 - \gamma) = c \sin\alpha$

$a \sin\gamma = c \sin\alpha$

$$\Rightarrow \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \qquad \text{(i)}$$

In a similar way, with side $\overline{AB}$ along +ve x-axis, we can prove that:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \qquad \text{(ii)}$$

From (i) and (ii), we have $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$

This is called the law of sines.

12.6.3 The Law of Tangents

In any triangle ABC, with usual notations, prove that:

i) $\dfrac{a-b}{a+b} = \dfrac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}}$ ii) $\dfrac{b-c}{b+c} = \dfrac{\tan\frac{\beta-\gamma}{2}}{\tan\frac{\beta+\gamma}{2}}$

iii) $\dfrac{c-a}{c+a} = \dfrac{\tan\frac{\gamma-\alpha}{2}}{\tan\frac{\gamma+\alpha}{2}}$

Proof: We know that by the law of sines:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{\sin\alpha}{\sin\beta}$$

By componendo and dividendo,

$$\frac{a-b}{a+b} = \frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} = \frac{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}$$

$$\Rightarrow \quad \frac{a-b}{a+b} = \frac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}} \qquad \text{(i)}$$

Similarly, we can prove that:

$$\frac{b-c}{b+c} = \frac{\tan\frac{\beta-\gamma}{2}}{\tan\frac{\beta+\gamma}{2}} \quad \text{(ii)} \qquad \text{and} \qquad \frac{c-a}{c+a} = \frac{\tan\frac{\gamma-\alpha}{2}}{\tan\frac{\gamma+\alpha}{2}} \quad \text{(iii)}$$

$\Rightarrow$ (i), (ii) and (iii) are called **Law of Tangents.**

12.6.4 Half Angle Formulas

We shall now prove some more formulas with the help of the law of cosine, which are called half-angle formulas:

a) The Sine of Half the Angle in Terms of the Sides

In any triangle *ABC*, prove that :

$$\left.\begin{aligned}&\text{(i)}\quad \sin\frac{\alpha}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}\\&\text{(ii)}\quad \sin\frac{\beta}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}\\&\text{(iii)}\quad \sin\frac{\gamma}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}\end{aligned}\right\}\quad \text{where } 2s=a+b+c$$

Proof: We know that

$$2\sin^2\frac{\alpha}{2}=1-\cos\alpha$$

$$\therefore\ 2\sin^2\frac{\alpha}{2}=1-\frac{b^2+c^2-a^2}{2bc}\qquad \left\{\because\ \cos\alpha=\frac{b^2+c^2-a^2}{2bc}\right.$$

$$=\frac{2bc-b^2-c^2+a^2}{2bc}$$

$$\therefore\ 2\sin^2\frac{\alpha}{2}=\frac{a^2-(b^2+c^2-2bc)}{2bc}=\frac{a^2-(b-a)^2}{2bc}$$

$$\therefore\ \sin^2\frac{\alpha}{2}=\frac{(a+b-c)(a-b+c)}{4bc}$$

$$\therefore\ \sin^2\frac{\alpha}{2}=\frac{2(s-c).2(s-b)}{4bc}\qquad \{\because\ a+b+c=2s\}$$

Hence: $\sin\frac{\alpha}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$ $\qquad\left\{\begin{array}{l}\text{is the measure of}\\ \text{an angle of } ABC\\ \therefore\ \frac{\alpha}{2}< 90\ \Rightarrow \sin\frac{\alpha}{2}=+\text{ve}\end{array}\right.$

In a similar way, we can prove that

$$\sin\frac{\beta}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}\quad\text{and}\quad \sin\frac{\gamma}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$$

b) The Cosine of Half the Angle in Term of the Sides

In any triangle *ABC*, with usual notation, prove that:

$$\left.\begin{aligned}&\text{i)}\quad \cos\frac{\alpha}{2}=\sqrt{\frac{s(s-a)}{bc}}\\&\text{ii)}\quad \cos\frac{\beta}{2}=\sqrt{\frac{s(s-b)}{ac}}\\&\text{iii)}\quad \cos\frac{\gamma}{2}=\sqrt{\frac{s(s-c)}{ab}}\end{aligned}\right\}\quad \text{where } 2s=a+b+c$$

Proof: We know that

$$2\cos^2\frac{\alpha}{2}=1+\cos\alpha=1+\frac{b^2+c^2-a^2}{2bc}\quad\left[\because\ \cos\alpha=\frac{b^2+c^2-a^2}{2bc}\right]$$

$$=\frac{2bc+b^2+c^2-a^2}{2bc}=\frac{(b+c)^2-a^2}{2bc}$$

$$=\frac{(b+c+a)(b+c-a)}{2bc}$$

$$\therefore\ \cos^2\frac{\alpha}{2}=\frac{(a+b+c)(b+c-a)}{4bc}$$

$$\therefore\ \cos^2\frac{\alpha}{2}=\frac{2s.2(s-a)}{4bc}\qquad (\therefore\ 2s=a+b+c)$$

$$\Rightarrow\ \cos\frac{\alpha}{2}=\sqrt{\frac{s(s-a)}{bc}}\qquad\left\{\begin{array}{l}\because\ \alpha\text{ is measure of}\\ \text{an angle of }\Delta ABC\\ \therefore\ \frac{\alpha}{2}\text{ is acute }\Rightarrow \cos=\frac{\alpha}{2}=+\text{ve}\end{array}\right.$$

In a similar way, we can prove that

$$\cos\frac{\beta}{2}=\sqrt{\frac{s(s-b)}{ca}}\quad\text{and}\quad\cos\frac{\gamma}{2}=\sqrt{\frac{s(s-c)}{ab}}$$

c) The Tangent of Half the Angle in Terms of the Sides

In any triangle *ABC*, with usual notation, prove that:

$$\left.\begin{array}{ll}\text{(i)} & \tan\frac{\alpha}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\\ \text{(ii)} & \tan\frac{\beta}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}\\ \text{(iii)} & \tan\frac{\gamma}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}\end{array}\right\}\quad \text{where } 2s=a+b+c$$

Proof: We know that:

$$\sin\frac{\alpha}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}} \quad \text{and} \quad \cos\frac{\alpha}{2}=\sqrt{\frac{s(s-a)}{bc}}$$

$$\Rightarrow \quad \tan\frac{\alpha}{2}=\frac{\sin\frac{\alpha}{2}}{\cos\frac{\beta}{2}}=\frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}}$$

$$\therefore \quad \tan\frac{\alpha}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

In a similar way, we can prove that:

$$\tan\frac{\beta}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \quad \text{and} \quad \tan\frac{\gamma}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

12.7 Solution of Oblique Triangles

We know that a triangle can be constructed if:

i) one side and two angles are given,

or ii) two sides and their included angle are given

or iii) three sides are given.

In the same way, we can solve an oblique triangle if

i) one side and two angles are known,

or ii) two sides and their included angle are known

or iii) three sides are known.

Now we shall discover the methods of solving an oblique triangle in each of the above cases:

12.7.1 Case I: When measures of one side and two angles are given

In this case, the law of sines can be applied.

Example 1: Solve the triangle *ABC*, given that

$$\alpha = 35^\circ 17', \quad \beta = 45^\circ 13', \quad b = 421$$

Solution: $\because \alpha+\beta+\gamma = 180^\circ$

$$\therefore \quad \gamma = 180^\circ-(\alpha+\beta)=180^\circ-(35^\circ 17'+45^\circ 13')=99^\circ 30'$$

By Law of sines, we have

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$\Rightarrow \quad a = b\frac{\sin\alpha}{\sin\beta} = \frac{421\times\sin 35^\circ 17'}{\sin 45^\circ 13'} = \frac{421(0.5776)}{0.7098}$$

$\therefore$ a = 342.58 = 343 approximately.

Again $$\frac{c}{\sin\gamma} = \frac{b}{\sin\beta}$$

$$\therefore \quad c = b\frac{\sin\gamma}{\sin\beta} = \frac{421\times\sin 99^\circ 30'}{\sin 45^\circ 13'} = \frac{421(0.9863)}{0.7098}$$

= 584.99 = 585 approximately.

Hence $\gamma = 99^\circ 30'$, $a = 343$, $c = 585$.

Exercise 12.4

Solve the triangle *ABC*, if

1. $\beta = 60^\circ$, $\gamma = 15^\circ$, $b = \sqrt{6}$

2. $\beta = 52^\circ$, $\gamma = 89^\circ 35'$, $a = 89.35$

3. $b = 125$, $\gamma = 53^\circ$, $\alpha = 47^\circ$
4. $c = 16.1$, $\alpha = 42^\circ 45'$, $\gamma = 74^\circ 32'$
5. $a = 53$, $\beta = 88^\circ 36'$, $\gamma = 31^\circ 54'$

12.7.2 Case II: When measures of two sides and their included angle are given

In this case, we can use any one of the following methods:
i) First law of cosine and then law of sines,
or ii) First law of tangents and then law of sines.

Example 1: Solve the triangle *ABC*, by using the cosine and sine laws, given that *b* = 3, *c* = 5 and *a* = 120°.

Solution: By cosine laws,

$$a^2 = b^2 + c^2 - 2bc\cos\alpha = 9 + 25 - 2(3)(5)\cos 120^\circ$$

$$= 9 + 25 - 2(3)(5)\left(-\frac{1}{2}\right) = 9 + 25 + 15 = 49$$

$\therefore$ a = 7

NOW $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta}$

$$\Rightarrow \sin\beta = \frac{b\sin\alpha}{a} = \frac{3\times\sin 120^\circ}{7} = \frac{3\times 0.866}{7} = 0.3712$$

$\therefore \beta = 21^\circ 47'$

$\therefore \gamma = 180^\circ - (\alpha+\beta) = 180^\circ - (120^\circ + 21^\circ 47')$

$\gamma = 88^\circ 13'$

Hence $a = 7, \beta = 21^\circ 47'$ and $\gamma = 38^\circ 13'$

Example 2: Solve the triangle ABC, in which:

$$a = 36.21,\ c = 30.14,\quad \beta = 78^\circ 10'$$

Solution: Here $a > c \quad \therefore \alpha > \gamma$

$\because \alpha + \beta + \gamma = 180^\circ$

$\because \alpha + \gamma = 180^\circ - \beta = 180^\circ - 78^\circ 10'$

$\Rightarrow \alpha + \gamma = 101^\circ 50'$

$\Rightarrow \dfrac{\alpha+\gamma}{2} = 50^\circ 55'$

$\therefore$ By the law of tangents,

$$\frac{\tan\frac{\alpha-\gamma}{2}}{\tan\frac{\alpha+\gamma}{2}} = \frac{a-c}{a+c} \Rightarrow \tan\frac{\alpha-\gamma}{2} = \frac{a-c}{a+c}\tan\frac{\alpha+\gamma}{2}$$

so $\tan\dfrac{\alpha-\gamma}{2} = \dfrac{36.21-30.14}{36.21+30.14}\cdot\tan 50^\circ 55'$

$\tan\dfrac{\alpha-\gamma}{2} = \dfrac{6.07}{66.35}\times 1.2312$

$\Rightarrow \tan\dfrac{\alpha-\gamma}{2} = 0.1126$

$\Rightarrow \tan\dfrac{\alpha-\gamma}{2} = 6^\circ 26'$

$\alpha - \gamma = 12^\circ 52'$ (ii)

Solving (i) and (ii) we have

$$\alpha = 57^\circ 21 \text{ and } \gamma = 44^\circ 29'$$

To find side *b*, we use law of sines

$$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha} \Rightarrow b = \frac{a\sin\beta}{\sin\alpha}$$

$$b = \frac{36.21\times\sin 78^\circ 10'}{\sin 57^\circ 21'} = \frac{(36.21)(0.9788)}{(0.8420)} = 420.09$$

Hence $b = 42.09,\ \gamma = 44^\circ 29'$ and $\alpha = 57^\circ 21'$

Example 3: Two forces of 20 Newtons and 15 Newtons, inclined at an angle of 45°, are applied at a point on a body. If these forces are represented by two adjacent sides of a parallelogram then, their resultant is represented by its diagonal. Find the resultan force and also the angle which the resultant makes with the force of 20 Newtons.

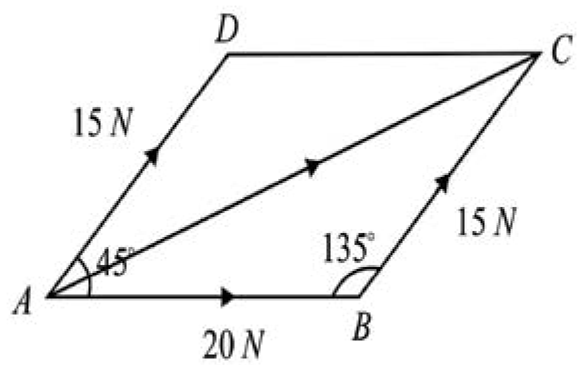
Solution:

Let *ABCD* be a ||^m, such that

$|\overrightarrow{AB}|$ represent 20 Newtons

$|\overrightarrow{AD}|$ represents 15 Newtons

and $m\angle BAD = 45^\circ$



$\because \quad ABCD$ is a $||^m$

$$\therefore \begin{cases} |\overrightarrow{BC}| = |\overrightarrow{AD}| = 15\ N \\ m\angle ABC = 180^\circ - m\angle BAD = 180^\circ - 45^\circ = 135^\circ \end{cases}$$

By the law of cosine,

$$\left(|\overrightarrow{AC}|\right)^2 = \left(|\overrightarrow{AB}|\right)^2 + \left(\overrightarrow{BC}\right)^2 - 2|\overrightarrow{AB}|\times|\overrightarrow{BC}|\times \cos 135^\circ$$

$$= (20)^2 + (15)^2 - 2\times 20\times 15\times\frac{-1}{\sqrt{2}}$$

$$= 400 + 225 + 424.2$$

$$= 1049.2$$

$$\therefore \quad |\overrightarrow{AC}| = \sqrt{1049.2} = 32.4\ N$$

By the law of sines,

$$\frac{|\overrightarrow{BC}|}{\sin m\angle BAC} = \frac{|\overrightarrow{AC}|}{\sin 135^\circ}$$

Make $|\overrightarrow{AB}|, |\overrightarrow{BC}|, |\overrightarrow{AD}|$ and $|\overrightarrow{AC}|$

$$\therefore \quad \sin m\angle BAC = \frac{|\overrightarrow{BC}|\times \sin 135^\circ}{|\overrightarrow{AC}|} = \frac{15\times 0.707}{32.4} = 0.3274$$

$$\therefore \quad m\angle BAC = 19^\circ 6'$$

Exercise 12.5

Solve the triangle *ABC* in which:

1. $b = 95$ $c = 34$ and $\alpha = 52^\circ$

2. $b = 12.5$ $c = 23$ and $\alpha = 38^\circ 20'$

3. $a = \sqrt{3} - 1$ $b = \sqrt{3} + 1$ and $\gamma = 60^\circ$

4. $a = 3$ $c = 6$ and $\beta = 36^\circ 20'$

5. $a = 7$ $b = 3$ and $\gamma = 38^\circ 13'$

Solve the following triangles, using first Law of tangents and then Law of sines:

6. $a = 36.21$ $b = 42.09$ and $\gamma = 44^\circ 29'$

7. $a = 93$ $b = 101$ and $\beta = 80^\circ$

8. $a = 14.8$ $c = 16.1$ and $\alpha = 42^\circ 45'$

9. $a = 319$ $b = 168$ and $\gamma = 110^\circ 22$

10. $a = 61$ $a = 32$ and $\alpha = 59^\circ 30$

11. Measures of two sides of a triangle are in the ratio 3 : 2 and they include an angle of measure 57°. Find the remaining two angles.

12. Two forces of 40 *N* and 30 *N* are represented by $\overrightarrow{AB}$ and $\overrightarrow{BC}$ which are inclined at an angle of 147° 25″. Find $\overrightarrow{AC}$, the resultant of $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

12.7.3 Case. lll: When Measures of Three Sides are Given

In this case, we can take help of the following formulas:

i) the law of cosine;

or ii) the half angle formulas:

Example 1: Solve the triangle *ABC*, by using the law of cosine when
$a = 7$, $b = 3$, $c = 5$

Solution: We know that

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$$

$$\therefore \quad \cos\alpha = \frac{9+25-49}{30} = -\frac{15}{30} = -\frac{1}{2}$$

$$\therefore \quad \alpha = 120^\circ$$

$$\cos\beta = \frac{c^2+a^2-b^2}{2ca} = \frac{25+49-9}{70} = \frac{65}{70} = 0.9286$$

$$\therefore \quad \beta = 21^\circ 17'$$

and $\gamma = 180^\circ - (\alpha+\beta) = 180^\circ - (120^\circ + 21^\circ 47') = 38^\circ 13'$

Example 2: Solve the triangle *ABC*, by half angle formula, when
$a = 283$, $b = 317$, $c = 428$

Solution: $2s = a + b + c = 283 + 317 + 428 = 1028$

$s = 514$

$s - a = 514 - 283 = 231$

$s - b = 514 - 317 = 197$

$s - c = 514 - 428 = 96$

Now, $$\tan\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{197\times 86}{514\times 231}} = 0.3777$$

$$\frac{\alpha}{2} = 20^\circ 53' \Rightarrow \alpha = 41^\circ 24'$$ and

$$\tan\frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \sqrt{\frac{86\times 231}{514\times 197}} = 0.4429$$

$$\therefore \quad \frac{\beta}{2} = 23^\circ 53' \Rightarrow \beta = 47^\circ 46'$$

$$\therefore \quad \gamma = 180^\circ - (\alpha+\beta) = 180^\circ - (41^\circ 24' + 47^\circ 46') = 90^\circ 50'$$

Exercise 12.6

Solve the following triangles, in which

1. $a = 7$, $b = 7$, $c = 9$
2. $a = 32$, $b = 40$, $c = 66$
3. $a = 28.3$, $b = 31.7$, $c = 42.8$
4. $a = 31.9$, $b = 56.31$, $c = 40.27$
5. $a = 4584$, $b = 5140$, $c = 3624$
6. Find the smallest angle of the triangle *ABC*, when $a = 37.34$, $b = 3.24$, $c = 35.06$.
7. Find the measure of the greatest angle, if sides of the triangle are 16, 20, 33.
8. The sides of a triangle are x^2+x+1, $2x+1$ and x^2-1. Prove that the greatest angle of the triangle is 120°.
9. The measures of side of a triangular plot are 413, 214 and 375 meters. Find the measures of the comer angles of the plot.
10. Three villages *A*, *B* and *C* are connected by straight roads 6 km. 9 km and 13 km. What angles these roads make with each other?

12.8 Area of Triangle

We have learnt the methods of solving different types of triangle. Now we shall find the methods of finding the area of these triangles.

case 1 Area of Triangle in Terms of the Measures of Two Sides and Their Included Angle

With usual notations, prove that:

$$\text{Area of triangle } ABC = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ca\sin\beta = \frac{1}{2}ab\sin\gamma$$

Proof: Consider three different kinds of triangle *ABC* with $m\angle C = \gamma$ as

i) acute ii) obtuse and iii) right

From *A*, draw $\overline{AD} \perp \overline{BC}$ or $\overline{BC}$ produced.

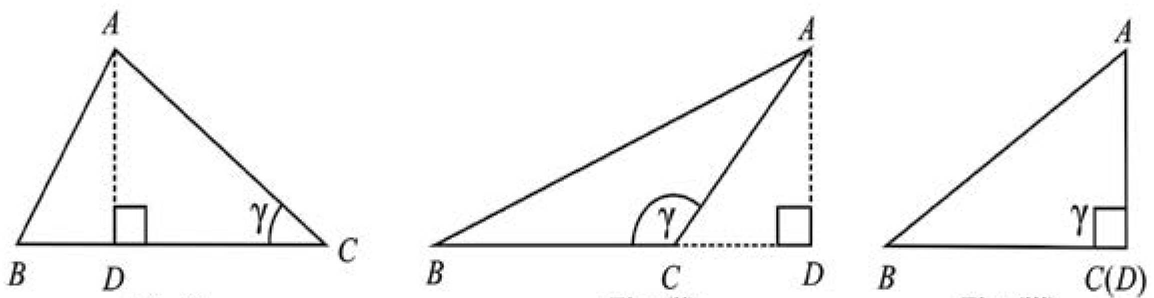


Fig: (i) Fig: (ii) Fig: (iii)

In figure. (i), $\frac{\overline{AD}}{\overline{AC}} = \sin\gamma$

In figure. (ii), $\frac{\overline{AD}}{\overline{AC}} = \sin(180^\circ - \gamma) = \sin\gamma$

In figure. (iii), $\frac{\overline{AD}}{\overline{AC}} = 1 = \sin 90^\circ = \sin\gamma$

In all the three cases, we have

$$\overline{AD} = \overline{AC}\sin\gamma = b\sin\gamma$$

Let Δ denote the area of triangle *ABC*.

By elementary geometry we know that

$$\Delta = \frac{1}{2}\text{(base)(altitude)}$$

$$\therefore \quad \Delta = \frac{1}{2}\overline{BC}\,.\,\overline{AD}$$

$$\therefore \quad \Delta = \frac{1}{2}ab\sin\gamma$$

Similarly, we can prove that:

$$\Delta = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ca\sin$$

Case II. Area of Triangle in Terms of the Measures of One Side and two Angles

In a triangle ΔABC, with usual notations, prove that:

$$\text{Area of triangle} = \frac{a^2\sin\beta\sin\gamma}{2\sin\alpha} = \frac{b^2\sin\gamma\sin\alpha}{2\sin\beta} = \frac{c^2\sin\alpha\sin\beta}{2\sin\gamma}$$

Proof: By the law of sines, we know that:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

$$\Rightarrow \quad a = c\frac{\sin\alpha}{\sin\gamma} \quad \text{and} \quad b = c\frac{\sin\beta}{\sin\gamma}$$

We know that area of triangle *ABC* is

$$\Delta = \frac{1}{2}ab\sin\gamma$$

$$\Rightarrow \quad \Delta = \frac{1}{2}\left(c\frac{\sin\alpha}{\sin\gamma}\right)\left(\frac{c\sin\beta}{\sin\gamma}\right)\sin\gamma$$

$$\therefore \quad \Delta = \frac{c^2\sin\alpha\sin\beta}{2\sin\gamma}$$

In a similar way, we can prove that:

$$\Delta ABC = \frac{a^2\sin\beta\sin\gamma}{2\sin\alpha} \quad \text{and} \quad \Delta ABC = \frac{b^2\sin\gamma\sin\alpha}{2\sin\beta}$$

Case III. Area of Triangle in Terms of the Measures of its Sides

In a triangle *ABC*, with usual notation, prove that:

$$\text{Area of triangle} = \sqrt{s(s-a)(s-b)(s-c}$$

Proof: We know that area of triangle *ABC* is

$$\Delta = \frac{1}{2}bc\sin\alpha$$

$$= \frac{1}{2}bc.\ 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \qquad \left(\because\ \sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)$$

$$= bc\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}$$ (by half angle formulas)

$$= bc\frac{\sqrt{s(s-a)(s-b)(s-c)}}{bc}$$

$$\therefore\quad \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

Which is also called **Hero's formula**

Example 1: Find the area of the triangle ABC, in which
$b = 21.6$, $c = 30.2$ and $\alpha = 52°\ 40'$

Solution: We know that:

$$\Delta ABC = \frac{1}{2}bc\sin\alpha = \frac{1}{2}(21.6)(30.2)\ \sin 52^\circ\ 40'$$

$$= \frac{1}{2}(21.6)(30.2)(0.7951)$$

$$\therefore\quad \Delta ABC = 259.3 \text{ sq.units}$$

Example 2: Find the area of the triangle *ABC*, when
$\alpha = 35^\circ\ 17'$, $\gamma = 45^\circ 13'$ and $b = 42.1$

Solution: $\because\ \alpha + \beta + \gamma = 180^\circ$

$$\therefore\quad \beta = 180^\circ - (\alpha + \gamma) = 180^\circ - (35^\circ 17' + 45^\circ 13') = 99^\circ 30'$$

Also $b = 42.1$ $\alpha = 35^\circ 17'$, $\gamma = 45^\circ 13'$, $\beta = 99^\circ\ 30'$

We know that the area of triangle *ABC* is

$$\Delta = \frac{1}{2}\frac{b^2\sin\gamma\sin\alpha}{\sin\beta}$$

$$\therefore\quad = \frac{1}{2}\frac{(42.1)^2\sin 45^\circ 13'\sin 35^\circ\ 17'}{\sin 99^\circ\ 30'}$$

$$= \frac{1}{2}\frac{(42.1)^2(0.7097)(0.5776)}{(0.9863)}$$

$\therefore\quad \Delta = 368.3$ square units.

Example 3: Find the area of the triangle *ABC* in which
$a = 275.4$, $b = 303.7$, $c = 342.5$

Solution: $\because\ a = 275.4, b = 303.7,\ c = 342.5$

$\therefore\quad 2s = a + b + c$
$= 275.4 + 303.7 + 342.5 = 921.6$

$\therefore\quad s = 460.8$

Now $s - a = 460.8 - 275.4 = 185.4$
$s - b = 460.8 - 303.7 = 157.1$
$s - c = 460.8 - 342.5 = 118.3$

Now $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$
$= \sqrt{460.8 \times 185.4 \times 157.1 \times 118.3}$

$\therefore\quad \Delta = 39847$ sq. units

Exercise 12.7

1. Find the area of the triangle *ABC*, given two sides and their included angle:

i) $a = 200$, $b = 120$, $\gamma = 150^\circ$
ii) $b = 37$, $c = 45$, $\alpha = 30^\circ 50'$
iii) $b = 4.33$, $b = 9.25$, $\gamma = 56^\circ 44'$

2. Find the area of the triangle *ABC*, given one side and two angles:

i) $b = 25.4$, $\gamma = 36^\circ 41'$, $\alpha = 45^\circ 17'$
ii) $c = 32$, $\alpha = 47^\circ 24'$, $\beta = 70^\circ 16'$
iii) $a = 8.2$, $\alpha = 83^\circ 42'$, $\gamma = 37^\circ 12'$

3. Find the area of the triangle *ABC*, given three sides:

i) $a = 18$, $b = 24$, $c = 30$

ii) $a = 524$, $b = 276$, $c = 315$

iii) $a = 32.65$, $b = 42.81$, $c = 64.92$

4. The area of triangle is 2437. If $a = 79$, and $c = 97$, then find angle β.

5. The area of triangle is 121.34. If $\alpha = 32° 15$ $\beta = 65° 37$ then find c and angle γ.

6. One side of a triangular garden is 30 m. If its two corner angles are $22^\circ \frac{1}{2}$ and $112^\circ \frac{1}{2}$, find the cost of planting the grass at the rate of Rs. 5 per square meter.

12.9 Circles Connected with Triangle

In our previous classes, we have learnt the methods of drawing the following three kinds of circles related to a triangle:

i) Circum-Circle ii) In-Circle iii) Ex-Circle.

12.9.1 Circum-Circle:

The circle passing through the three vertices of a triangle is called a **Circum- Circle**. Its centre is called the **circum-centre**, which is the point of intersection of the right bisectors of the sides of the triangle. Its radius is called the **circum-radius** and is denoted by *R*.

a) Prove that: $R = \dfrac{a}{2\sin\alpha} = \dfrac{b}{2\sin\beta} = \dfrac{c}{2\sin\gamma}$ with usual notations.

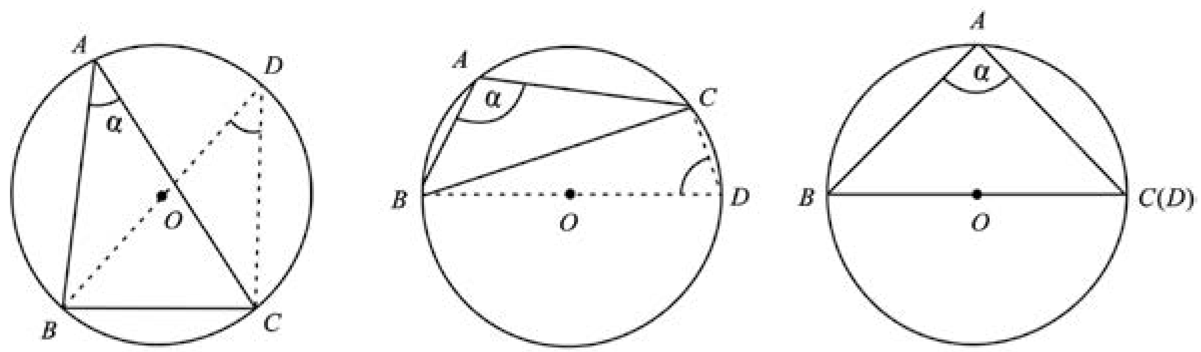


Fig. (i) ($\angle BAC$ is acute) Fig. (ii) ($\angle BAC$ is obtuse) Fig. (iii) ($\angle BAC$ is right)

Proof: Consider three different kinds of triangle *ABC* with $m\angle A = \alpha$

i) acute ii) obtuse iii) right.

Let *O* be the circum-centre of ΔABC. Join *B* to *O* and produce $\overline{BO}$ to -meet the circle again at *D*. Join *C* to *D*. Thus we have the measure of diameter $m\overline{BD} = 2R$ and $m\overline{BC} = a$

I. In fig. (i), $m\angle BDC = m\angle A = \alpha$ (Angles in the same segment)

In right triangle *BCD*,

$$\frac{m\ \overline{BC}}{m\ \overline{BD}} = \sin m\angle BDC = \sin\alpha$$

II. In fig. (ii),

$$m\angle BDC + m\angle A = 180^\circ \quad \text{(Sum of opposite angles of a cyclic quadrilateral} = 180^\circ)$$

$$\Rightarrow \quad m\angle BDC + \alpha = 180^\circ$$

$$\Rightarrow \quad m\angle BDC = 180^\circ - \alpha$$

In right triangle *BCD*,

$$\frac{m\ \overline{BC}}{m\ \overline{BD}} = \sin m\angle BDC = \sin(180^\circ - \alpha) = \sin\alpha$$

III. In fig. (iii), $m\angle A = \alpha = 90^\circ$

$$\therefore \quad \frac{m\ \overline{BC}}{m\ \overline{BD}} = 1 = \sin 90^\circ = \sin\ \alpha$$

In all the three figures, we have proved that

$$\frac{m\ \overline{BC}}{m\ \overline{BD}} = \sin\alpha$$

$$\Rightarrow \quad \frac{a}{2R} = \sin\ \alpha \Rightarrow 2R\sin\alpha = a$$

$$\therefore \quad R = \frac{a}{2\ \sin\ \alpha}$$

Similarly, we can prove that

$$R = \frac{b}{2\sin\beta} \quad \text{and} \quad R = \frac{R.c}{2\sin\gamma}$$

Hence $\quad R = \dfrac{a}{2\sin\alpha} = \dfrac{b}{2\sin\beta} = \dfrac{c}{2\sin\gamma}$

a) Deduction of Law of Sines:

We know that $R = \dfrac{a}{2\sin\alpha} = \dfrac{b}{2\sin\beta} = \dfrac{c}{2\sin\gamma}$

$$\Rightarrow \quad \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$$

$\therefore \quad \dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$, which is the law of sines.

b) Prove that: $R = \dfrac{abc}{4\Delta}$

Proof: We know that: $\quad R = \dfrac{a}{2\sin\alpha}$

$$\Rightarrow \quad R = \frac{a}{2.2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} \qquad \left(\because \sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)$$

$$= \frac{a}{4\sqrt{\frac{s(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}} \quad \text{(by half angle formulas)}$$

$$= \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

$$\therefore \quad R = \frac{abc}{4\Delta} \qquad \left(\because \Delta = \sqrt{s(s-a)(s-b)(s-c)}\right)$$

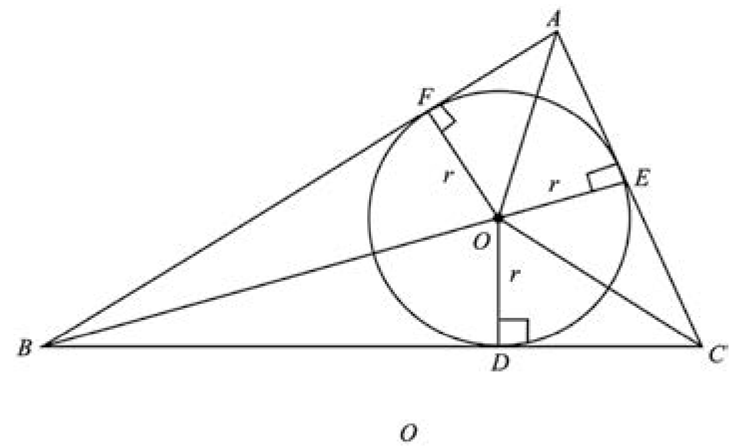
12.9.2 In-Circle

The circle drawn inside a triangle touching its three sides is called its **inscribed circle** or **in-circle**. Its centre is known as the **in-centre**, it is the point of intersection of the bisectors of angles of the triangle. Its radius is called **in-radius** and is denoted by *r*.

a) **Prove that:** $r = \dfrac{\Delta}{s}$ with usual notations.

Proof: Let the internal bisectors of angles of triangle *ABC* meet at *O*, the in-centre

Draw $\overline{OD} \perp \overline{BC}$, $\overline{OE} \perp \overline{AC}$ and $\overline{OF} \perp \overline{AB}$



Let, $m\overline{OD} = m\overline{OE} = m\overline{OF} = r$

From the figure Area ΔABC = AreaΔOBC + AreaΔOCA + AreaΔOAB

$$\therefore \quad \Delta = \frac{1}{2}\overline{BC}\times\overline{OD} + \frac{1}{2}\overline{CA}\times\overline{OE} + \frac{1}{2}\overline{AB}\times\overline{OF}$$

$$= \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$$

$$= \frac{1}{2}r(a+b+c)$$

$$\Delta = \frac{1}{2}r\,.\,2s \qquad (\because 2s = a+b+c)$$

$$\Rightarrow \boxed{r = \frac{\Delta}{s}}$$

12.9.3 Escribed Circles

A circle, which touches one side of the triangle externally and the other two produced sides, is called an **escribed circle** or **ex-circle** or **e-circle**. Obviously, there could be only three such circles of a triangle, one opposite to each angle of the triangle.

The centres of these circles, which are called **ex-centres** are the points where the internal bisector of one and the external bisectors of the other two angles of the triangle meet.

In ΔABC , centre of the ex-circle opposite to the **vertex A** is usually taken as I_1 and its raidus is denoted by r_1. Similarly, centres of ex-circles opposite to the vertices *B* and *C* are taken as I_2 and I_3 and their radii are denoted by r_2 and r_3 respectively.

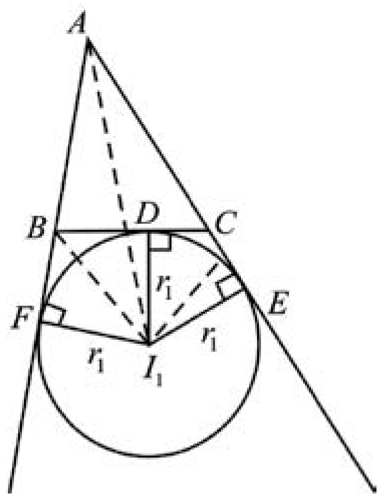
a) With usual notation, prove that:

$$r_1 = \frac{\Delta}{s-a}, \quad r_2 = \frac{\Delta}{s-b}, \quad \text{and } r_3 = \frac{\Delta}{s-c}$$

Proof: Let I_1 be the centre of the escribed circle opposite to the vertex *A* of ΔABC ,

From I_1 draw $\overline{I_1D} \perp \overline{BC}$, $\overline{I_1E} \perp \overline{AC}$ produced and $\overline{I_1F} \perp \overline{AB}$ produced.

Join I_1 to *A, B* and *C*.



Let $m\overline{I_1D} = m\overline{I_1E} = m\overline{I_1F} = r_1$

From the figure

$$\Delta ABC = \Delta I_1AB + \Delta I_1AC - \Delta I_1BC$$

$$\Rightarrow \quad \Delta = \frac{1}{2}\overline{AB}\times\overline{I_1F} + \frac{1}{2}\overline{AC}\times\overline{I_1E} - \frac{1}{2}\overline{BC}\times\overline{I_1D}$$

$$= \frac{1}{2}c\,r_1 + \frac{1}{2}b\,r_1 - \frac{1}{2}a\,r_1$$

$$\Delta = \frac{1}{2}r_1(c+b-a)$$

$$= \frac{1}{2}r_1 \,.\, 2(s-a) \qquad (2s = a+b+c)$$

$$= (s-a)\ r_1$$

Hence $\boxed{r_1 = \dfrac{\Delta}{s-a}}$

In a similar way, we can prove that:

$$\boxed{r_2 = \frac{\Delta}{s-b}} \quad \text{and} \quad \boxed{r_3 = \frac{\Delta}{s-c}}$$

Example 1: Show that:

$$r = (s-a)\tan\frac{\alpha}{2} = (s-b)\tan\frac{\beta}{2} = (s-c)\tan\frac{\gamma}{2}$$

Solution: To prove $r = (s-a)\tan\dfrac{\alpha}{2}$

We know that: $\tan\dfrac{\alpha}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$

$$\text{R.H.S} = (s-a)\tan\frac{\alpha}{2} = (s-a)\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2}} = \frac{\Delta}{s} = r$$

$$\therefore \ (s-a)\tan\frac{\alpha}{2} = r$$

In a similar way, we can prove that:

$$r = (s-b)\tan\frac{\beta}{2} \quad \text{and} \quad r = (s-c)\tan\frac{\gamma}{2}$$

Example 2: Show that $r_1 = 4R\sin\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}\cos\dfrac{\gamma}{2}$.

Solution: R.H.S. $= 4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$

$$= 4\,.\,\frac{abc}{4\Delta}\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-b)}{ca}}\sqrt{\frac{s(s-c)}{ab}}$$

$$= \frac{s(s-b)(s-c)}{\Delta}$$

$$= \frac{s(s-a)(s-b)(s-c)}{\Delta\,.\,(s-a)}$$

$$= \frac{\Delta^2}{\Delta(s-a)}$$

$$= \frac{\Delta}{s-a} = r_1 = \text{L.H.S}$$

Hence $r_1 = 4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$

Example 3 : Prove that $\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=\frac{a^2+b^2+c^2}{\Delta^2}$

Solution: L.H.S. $= \frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}$

$$= \frac{s^2}{\Delta^2}+\frac{(s-a)^2}{\Delta^2}+\frac{(s-b)^2}{\Delta^2}+\frac{(s-c)^2}{\Delta^2}$$

$$= \frac{s^2+(s-a)^2+(s-b)^2+(s-c)^2}{\Delta^2}$$

$$= \frac{4s^2-2s(a+b+c)+a^2+b^2+c^2}{\Delta^2}$$

$$= \frac{4s^2-2s.2s+a^2+b^2+c^2}{\Delta^2}$$

$$= \frac{a^2+b^2+c^2}{\Delta^2}$$

= R.H.S.

Hence the result.

Example 4: If the measures of the sides of a triangle *ABC* are 17, 10, 21. Find R, r, r_1, r_2 and r_3.

Solution: Let $a = 17,\quad b = 10,\quad c = 21$

$\therefore\quad 2s = a+b+c = 17+10+21 = 48$

$\Rightarrow\quad S = 24$

$\therefore\quad s-a = 24-17=7,\ s-b=24-10=14$ and $s-c = 24-21=3$

Now $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\Rightarrow\quad \Delta = \sqrt{24(7)(14)(3)} = 84$

Now $R = \frac{abc}{4\Delta} = \frac{17\,.\,10\,.\,21}{4\,.\,84} = \frac{85}{8}$

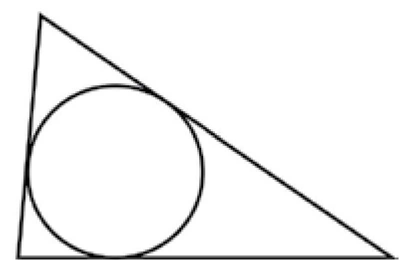
$r = \frac{\Delta}{s} = \frac{84}{24} = \frac{7}{2},\quad r_1 = \frac{\Delta}{s-a} = \frac{84}{7} = 12,$

$r_2 = \frac{\Delta}{s-b} = \frac{84}{14} = 6,\quad r_3 = \frac{\Delta}{s-c} = \frac{84}{3} = 28$

12.10 Engineering and Circles Connected With Triangles

We know that frames of all rectilinear shapes with the exception of triangular ones, change their shapes when pressed from two corners. But a triangular frame does not change its shape, when it is pressed from any two vertices. It means that a triangle is the only **rigid rectilinear figure.** It is on this account that the engineers make frequent use of triangles for the strength of material in all sorts of construction work.

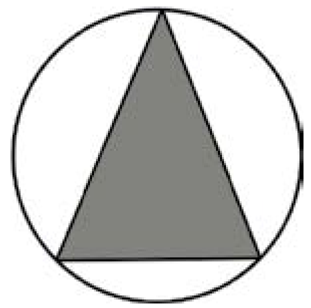
Besides triangular frames etc., circular rings can stand greater pressure when pressed from any two points on them. That is why the wells are always made cylindrical whose circular surfaces can stand the pressure of water from all around their bottoms. Moreover, the arches below the bridges are constructed in the shape of arcs of circles so that they can bear the burden of the traffic passing over the bridge.



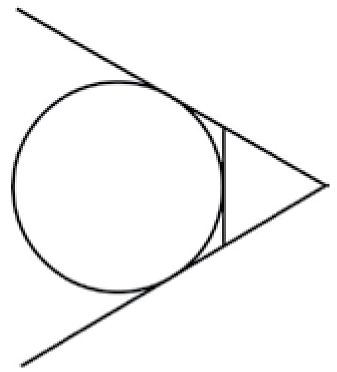
a) We know that triangular frames change their rectilinear nature when they are pressed from the sides. From the strength of material point of view, the engineers have to fix circular rings touching the sides of the triangular frames.

For making these rings, they have to find the **in-radii** of the triangles.

***b*)** In order to protect the triangular discs from any kind of damage, the engineers fit circular rings enclosing the discs. For making rings of proper size, the engineers are bound to calculate the **circum-radii** of the triangles.



***c*)** In certain triangular frames, the engineers have to extend two sides of the frames. In order to strengthen these loose wings, the engineer feels the necessity of fixing circular rings touching the extended sides andthe third side of the frames.



For making appropriate rings, the engineers have to find **ex-radii** of the triangles.

The above discussion shows that the methods of calculations of the radii of **incircle, circum-circle and ex-circles** of traingles must be known to an engineer for performing his professional duty efficiently.

Exercise 12.8

1. Show that: $r = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$

ii) $s = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

2. Show that: $r = a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2} = b\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\sec\frac{\beta}{2}$

$= c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}$

3. Show that: i) $r_1 = 4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

ii) $r_2 = 4R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}$

iii) $r_3 = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}$

4. Show that:

i) $r_1 = s\tan\frac{\alpha}{2}$ ii) $r_2 = s\tan\frac{\beta}{2}$ iii) $r_3 = s\tan\frac{\gamma}{2}$

5. Prove that:

i) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$ ii) $rr_1r_2r_3 = \Delta^2$

iii) $r_1 + r_2 + r_3 - r = 4R$ iv) $r_1r_2r_3 = rs^2$

6. Find R, r, r_1, r_2 and r_3, if measures of the sides of triangle ABC are

i) $a = 13$, $b = 14$, $c = 15$

ii) $a = 34$, $b = 20$, $c = 42$

7. Prove that in an equilateral triangle,

i) $r : R : r_1 = 1 : 2 : 3$

ii) $r : R : r_1 : r_2 : r_3 = 1 : 2 : 3 : 3 : 3$

8. Prove that:

i) $\Delta = r^2\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$

ii) $r = s\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

iii) $\Delta = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

9. Show that: i) $\frac{1}{2rR} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$

ii) $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

10. Prove that:

$$r = \frac{a\ \sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}} = \frac{b\sin\frac{\alpha}{2}.\sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}} = \frac{c\ \sin\frac{\alpha}{2}.\sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}$$

11. Prove that: $abc\,(\sin\alpha+\sin\beta+\sin\gamma) = 4\Delta s$

12. Prove that: i) $(r_1+r_2)\ \tan\frac{\gamma}{2} = c.$

ii) $(r_3-r)\ \cot\frac{\gamma}{2} = c$