CHAPTER

2

Sets Functions and Groups

Animation 2.1: Function
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2.1 Introduction

We are familiar with the notion of a **set** since the word is frequently used in everyday speech, for instance, water set, tea set, sofa set. It is a wonder that mathematicians have developed this ordinary word into a mathematical concept as much as it has become a language which is employed in most branches of modern mathematics.

For the purposes of mathematics, a set is generally described as a **well-defined collection of distinct objects.** By a *well-defined collection* is meant a collection, which is such that, given any object, we may be able to decide whether the object belongs to the collection or not. By *distinct objects* we mean objects no two of which are identical (same).

The objects in a set are called its **members** or **elements**. Capital letters A, B, C, X, Y, Z etc., are generally used as names of sets and small letters a, b, c, x, y, z etc., are used as *members* of sets.

There are three different ways of describing a set

i) The Descriptive Method: A set may be described in words. For instance, the set of all vowels of the English alphabets.

ii) The Tabular Method: A set may be described by listing its elements within brackets. If A is the set mentioned above, then we may write:

$$A = \{a, e, i, o, u\}.$$

iii) Set-builder method: It is sometimes more convenient or useful to employ the method of set-builder notation in specifying sets. This is done by using a symbol or letter for an arbitrary member of the set and stating the property common to all the members.

Thus the above set may be written as:

$A = \{x \mid x$ is a vowel of the English alphabet$\}$

This is read as A is the set of all x such that x is a vowel of the English alphabet.

The symbol used for **membership** of a set is $\in$. Thus $a \in A$ means ***a* is an element of *A*** or ***a* belongs to *A***. $c \notin A$ means ***c* does not belong to *A*** or ***c* is not a member of *A***. Elements of a set can be anything: people, countries, rivers, objects of our thought. In algebra we usually deal with sets of numbers. Such sets, alongwith their names are given below:-

N = The set of all natural numbers $= \{1,2,3,...\}$

W = The set of all whole numbers $= \{0,1,2,...\}$

Z = The set of all integers $= \{0,\pm1,+2....\}$.

Z' = The set of all negative integers $= \{-1,-2,-3,...\}$.

O = The set of all odd integers $= \{\pm 1, \pm 3, \pm 5, ...\}$.

E = The set of all even integers $= \{0, \pm2, \pm4, ...\}$.

Q = The set of all rational numbers $= \left\{x \,\middle|\, x = \dfrac{p}{q} \text{ where p,q} \in Z \text{ and q} \neq 0\right\}$

Q' = The set of all irrational numbers $= \left\{x \,\middle|\, x \neq \dfrac{p}{q} \text{ where p,q} \in Z \text{ and q} \neq 0\right\}$

$\mathbb{R}$ = The set of all real numbers $= Q \cup Q'$

Equal Sets: Two sets A and B are equal i.e., $A=B$, if and only if they have the same elements that is, if and only if every element of each set is an element of the other set.

Thus the sets { 1, 2, 3 } and { 2, 1, 3} are equal. From the definition of equality of sets it follows that a mere change in the order of the elements of a set does not alter the set. In other words, while describing a set in the tabular form its elements may be written in any order.

Note: (1) $A = B$ if and only if they have the same elements means if $A = B$ they have the same elements and if A and B have the same elements then $A = B$.

(2) The phrase if and only if is shortly written as "iff ".

Equivalent Sets: If the elements of two sets A and B can be paired in such a way that each element of A is paired with one and only one element of B and vice versa, then such a pairing is called a one-to-one correspondence between A and B e.g., if A = {Billal, Ahsan, Jehanzeb} and B = {Fatima, Ummara, Samina} then six different (1 - 1) correspondences can be established between A and B

Two of these correspondences are shown below; -

i).
$$\begin{array}{ccc} \{\text{Billal}, & \text{Ahsan}, & \text{Jehanzeb}\} \\ \updownarrow & \updownarrow & \updownarrow \\ \{\text{Fatima}, & \text{Ummara}, & \text{Samina}\} \end{array}$$

ii). {Billal, Ahsan, Jehanzeb)

$\updownarrow \quad \updownarrow \quad \updownarrow$

{Fatima, Samina, Ummara)

(Write down the remaining 4 correspondences yourselves)

Two sets are said to be equivalent if *a* (1 – 1) correspondence can be established between them In the above example *A* and *B* are equivalent sets.

Example 1: Consider the sets *N*= {1, 2, 3,... } and *O* = {1, 3, 5,...}

We may establish (1–1) correspondence between them in the following manner:

{1, 2, 3, 4, 5, ...}

$\updownarrow \; \updownarrow \; \updownarrow \; \updownarrow \; \updownarrow$

{1, 3, 5, 7, 9, ...}

Thus the sets *N* and *O* are equivalent. But notice that they are not equal.

Remember that two equal sets are necessarily equivalent, but the converse may not be true i.e., two equivalent sets are not necessarily equal.

Sometimes, the symbol ~ is used to mean **is equivalent to**. Thus *N*~*O*.

Order of a Set: There is no restriction on the number of members of a set. A set may have 0, 1, 2, 3 or any number of elements. Sets with zero or one element deserve special attention. According to the everyday use of the word set or collection it must have at least two elements. But in mathematics it is found convenient and useful to consider sets which have only one element or no element at all.

A set having only one element is called a **singleton set** and a set with no element (zero number of elements) is called the **empty set** or **null set**.The empty set is denoted by the symbol ϕ or { }.The set of odd integers between 2 and 4 is a singleton i.e., the set {3} and the set of even integers between the same numbers is the empty set.

The solution set of the equation $x^2 + 1 = 0$, in the set of real numbers is also the empty set. Clearly the set {0} is a *singleton set* having zero as its only element, and not the empty set.

Finite and Infinite sets: If a set is equivalent to the set {1, 2, 3,...*n*} for some fixed natural number *n*, then the set is said to be *finite* otherwise *infinite*.

Sets of number *N, Z, Z'*etc., mentioned earlier are infinite sets.

The set {1, 3, 5,.......9999} is a finite set but the set { 1, 3, 5, ...}, which is the set of all positive odd natural numbers is an infinite set.

Subset: If every element of a set *A* is an element of set *B*, then *A* is a *subset* of *B*. Symbolically this is written as: $A \subseteq B$ (*A* is subset of *B*)

In such a case we say *B* is a super set of *A*. Symbolically this is written as:

$B \supseteq A$ {*B* is a superset of *A*)

Note: The above definition may also be stated as follows:

$A \subseteq B$ iff $x \in A \Rightarrow x \in B$

Proper Subset: If *A* is a subset of *B* and *B* contains at least one element which is not an element of *A*, then A is said to be a *proper subset* of *B*. In such a case we write: $A \subset B$ (*A* is a proper subset of *B*).

Improper Subset: If *A* is subset of *B* and *A* = *B*, then we say that *A* is an improper subset of *B*. From this definition it also follows that every set *A* is an improper subset of itself.

Example 2: Let *A* = { *a*, *b*, *c* }, *B* = { *c*, *a*, *b*} and *C* = { *a*, *b*, *c*, *d*}, then clearly

$A \subset C$, $B \subset C$ but $A = B$ and $B = A$.

Notice that each of *A* and *B* is an improper subset of the other because *A* = *B*

Note: When we do not want to distinguish between proper and improper subsets, we may use the symbol $\subseteq$ for the relationship. It is easy to see that: $N \subset Z \subset Q \subset \mathbb{R}$.

Theorem 1.1: The empty set is a subset of every set.

We can convince ourselves about the fact by rewording the definition of subset as follows: -

A is subset of *B* if it contains no element which is not an element of *B*.

Obviously an empty set does not contain such element, which is not contained by another set.

Power Set: A set may contain elements, which are sets themselves. For example if: *C* = Set of classes of a certain school, then elements of *C* are sets themselves because each class is a set of students. An important set of sets is the *power set* of a given set.

The *power set* of a set *S* denoted by *P* (*S*) is the set containing all the possible subsets of *S*.

Example 3: If $A = \{a, b\}$, then $P(A)=\{\Phi, \{a\}, \{b\}, \{a,b\}\}$
Recall that the empty set is a subset of every set and every set is its own subset.

Example 4: If $B = \{1, 2, 3\}$, then

$$P(B) = \{\Phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Example 5: If $C = \{a, b, c, d\}$, then

Example 6: If $D = \{a\}$, then $P(D)=\{\Phi,\{a\}\}$

Example 7: If $E = \{\ \}$, then $P(E)=\{\Phi\}$

Note: (1) The power set of the empty set is not empty.
(2) Let *n* (*S*) denoted the number of elements of a set *S*, then *n*{*P(S)*} denotes the number of elements of the power set of *S*. From examples 3 to 7 we get the following table of results:

n(s)	0	1	2	3	4	5
n{p(s)}	$1=2^0$	$2=2^1$	$4=2^2$	$8=2^3$	$16=2^4$	$32=2^5$

In general if $n(S) = m$, then, $n\,P(S) = 2^m$

Universal Set: When we are studying any branch of mathematics the sets with which we have to deal, are generally subsets of a bigger set. Such a set is called the **Universal set** or the **Universe of Discourse**. At the elementary level when we are studying arithmetic, we have to deal with *whole numbers* only. At that stage the set of whole numbers can be treated as *Universal Set*. At a later stage, when we have to deal with negative numbers also and fractions, the set of the *rational numbers* can be treated as the *Universal Set*.

For illustrating certain concepts of the Set Theory, we sometimes consider quite

small sets (sets having small number of elements) to be universal. This is only an academic artificiality.

Exercise 2.1

1. Write the following sets in set builder notation:
   i) { 1, 2, 3, ...,1000} ii) { 0, 1,2,........., 100}
   iii) { 0, ± 1, ± 2,............ ±1000} iv) { 0, –1, –2,..........,–500}
   v) {100, 101,102, ...,400} vi) {–100,–101,–102,.., –500}
   vii) {Peshawar, Lahore, Karachi, Quetta}
   viii) { January, June, July }
   xi) The set of all odd natural numbers
   x) The set of all rational numbers
   xi) The set of all real numbers between 1and 2,
   xii) The set of all integers between – 100 and 1000

2. Write each of the following sets in the descriptive and tabular forms:-
   i) $\{x \mid x \in N \wedge x \le 10\}$ ii) $\{x \mid x \in N \wedge 4 < x < 12\}$
   iii) $\{x \mid x \in Z \wedge -5 < x < 5\}$ iv) $\{x \mid x \in E \wedge 2 < x \le 4\}$
   v) $\{x \mid x \in P \wedge x < 12\}$ vi) $\{x \mid x \in O \wedge 3 < x < 12\}$
   vii) $\{x \mid x \in E \wedge 4 \le x \le 10\}$ viii) $\{x \mid x \in E \wedge 4 < x < 6\}$
   ix) $\{x \mid x \in O \wedge 5 \le x \le 7\}$ x) $\{x \mid x \in O \wedge 5 \le x < 7\}$
   xi) $\{x \mid x \in N \wedge x + 4 = 0\}$ xi) $\{x \mid x \in Q \wedge x^2 = 2\}$
   xiii) $\{x \mid x \in \mathbb{R} \wedge x = x\}$ xiv) $\{x \mid x \in Q \wedge x = -x\}$
   xv) $\{x \mid x \in \mathbb{R} \wedge x \ne x\}$ xvi) $\{x \mid x \in \mathbb{R} \wedge x \notin Q\}$

3. Which of the following sets are finite and which of these are infinite?
   i) The set of students of your class.
   ii) The set of all schools in Pakistan.
   iii) The set of natural numbers between 3 and 10.
   iv) The set of rational numbers between 3 and 10.
   v) The set of real numbers between 0 and 1.
   vi) The set of rationales between 0 and 1.
   vii) The set of whole numbers between 0 and 1
   viii) The set of all leaves of trees in Pakistan.

ix) $P(N)$ x) $P\{a, b, c\}$
xi) {1,2,3,4,...} xii) {1,2,3,....,100000000}
xiii) $\{x\, x \mid x \in \mathbb{R} \wedge x \neq x\}$ xiv) $\{x \mid x \in \mathbb{R} \wedge x^2 = -16\}$
xv) $\{x \mid x \in Q \wedge x^2 = 5\}$ xvi) $\{x \mid x \in Q \wedge 0 \leq x \leq 1\}$

4. Write two proper subsets of each of the following sets: -
i) $\{a, b, c\}$ ii) { 0, 1} iii) N iv) Z
v) Q vi) $\mathbb{R}$ vii) W viii) $\{x \mid x \in Q \wedge 0 < x \leq 2\}$

5. Is there any set which has no proper sub set? If so name that set.

6. What is the difference between $\{a, b\}$ and $\{\{a, b\}\}$?

7. Which of the following sentences are true and which of them are false?
i) { 1,2} = { 2,1} ii) $\Phi \subseteq \{\{a\}\}$ iii) $\{a\} \subseteq \{\{a\}\}$
v) $\{a\} \in \{\{a\}\}$ vi) $a \in \{\{a\}\}$ vii) $\Phi \in \{\{a\}\}$

8. What is the number of elements of the power set of each of the following sets?
i) { } ii) {0,1} iii) {1,2,3,4,5,6,7}
v) {0,1,2,3,4,5,6,7} vi) $\{a, \{b, c\}\}$ vii) $\{\{a,b\},\{b,c\},\{d,e\}\}$

9. Write down the power set of each of the following sets: -
i) {9,11 } ii) $\{+, -, \times, \div\}$ iii) $\{\Phi\}$ iv) $\{a, \{b,c\}\}$

10. Which pairs of sets are equivalent? Which of them are also equal?
i) $\{a, b, c\}$, {1, 2, 3}
ii) The set of the first 10 whole members, {0, 1, 2, 3,....,9}
iii) Set of angles of a quadrilateral *ABCD*,
set of the sides of the same quadrilateral.
iv) Set of the sides of a hexagon *ABCDEF*,
set of the angles of the same hexagon;
v) {1,2,3,4,.....}, {2,4,6,8,.....} vi) {1,2,3,4,.....}, $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$

viii) {5, 10, 15,.....,55555}, {5, 10, 15, 20,....... }

2.2 Operations on Sets

Just as operations of addition, subtraction etc., are performed on numbers, the operations of unions, intersection etc., are performed on sets. We are already familiar with them. A review of the main rules is given below: -

Union of two sets: The Union of two sets *A* and *B*, denoted by $A \cup B$, is the set of all elements, which belong to *A* or *B*. Symbolically;

$$A \cup B \; \{x \mid x \in A \vee x \in B\}$$

Thus if A = {1, 2, 3}, B = { 2, 3, 4, 5}, then $A \cup B$ ={1,2,3,4,5}

Notice that the elements common to *A* and *B*, namely the elements 2, 3 have been written only once in $A \cup B$ because repetition of an element of a set is not allowed to keep the elements distinct.

Intersection of two sets: The intersection of two sets *A* and *B*, denoted by $A \cap B$, is the set of all elements, which belong to both *A* and *B*. Symbolically;

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Thus for the above sets *A* and *B*, $A \cap B$ ={2,3}

Disjoint Sets: If the intersection of two sets is the empty set then the sets are said to be disjoint sets. For example; if

S_1 = The set of odd natural numbers and S_2 = The set of even natural numbers, then S_1 and S_2 are disjoint sets.

The set of arts students and the set of science students of a college are disjoint sets.

Overlapping sets: If the intersection of two sets is non-empty but neither is a subset of the other, the sets are called overlapping sets, e.g., if

L = {2,3,4,5,6} and M= {5,6,7,8,9,10}, then L and M are two overlapping sets.

Complement of a set: The complement of a set A, denoted by A' or A^c relative to the universal set U is the set of all elements of U, which do not belong to A.

Symbolically: $A' = \{x | x \in U \wedge x \notin A\}$

For example, if $U=N$, then $E' = O$ and $O'=E$

Example 1: If U = set of alphabets of English language,

C = set of consonants,

W = set of vowels, then $C' = W$ and $W' = C$.

Difference of two Sets: The Difference set of two sets A and B denoted by $A–B$ consists of all the elements which belong to A but do not belong to B.

The Difference set of two sets B and A denoted by B-A consists of all the elements, which belong to B but do not belong to A.

Symbolically, $A–B = \{x | x \in A \wedge x \notin B\}$ and $B–A = \{x | x \in B \wedge x \notin A\}$

Example 2: If $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8,9,10\}$, then

$A–B = \{1,2,3\}$ and $B–A = \{6,7,8,9,10\}$.

Notice that $A–B \neq B–A$.

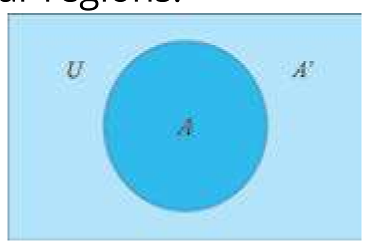
Note: In view of the definition of complement and difference set it is evident that for any set A, $A' = U – A$

2.3 Venn Diagrams

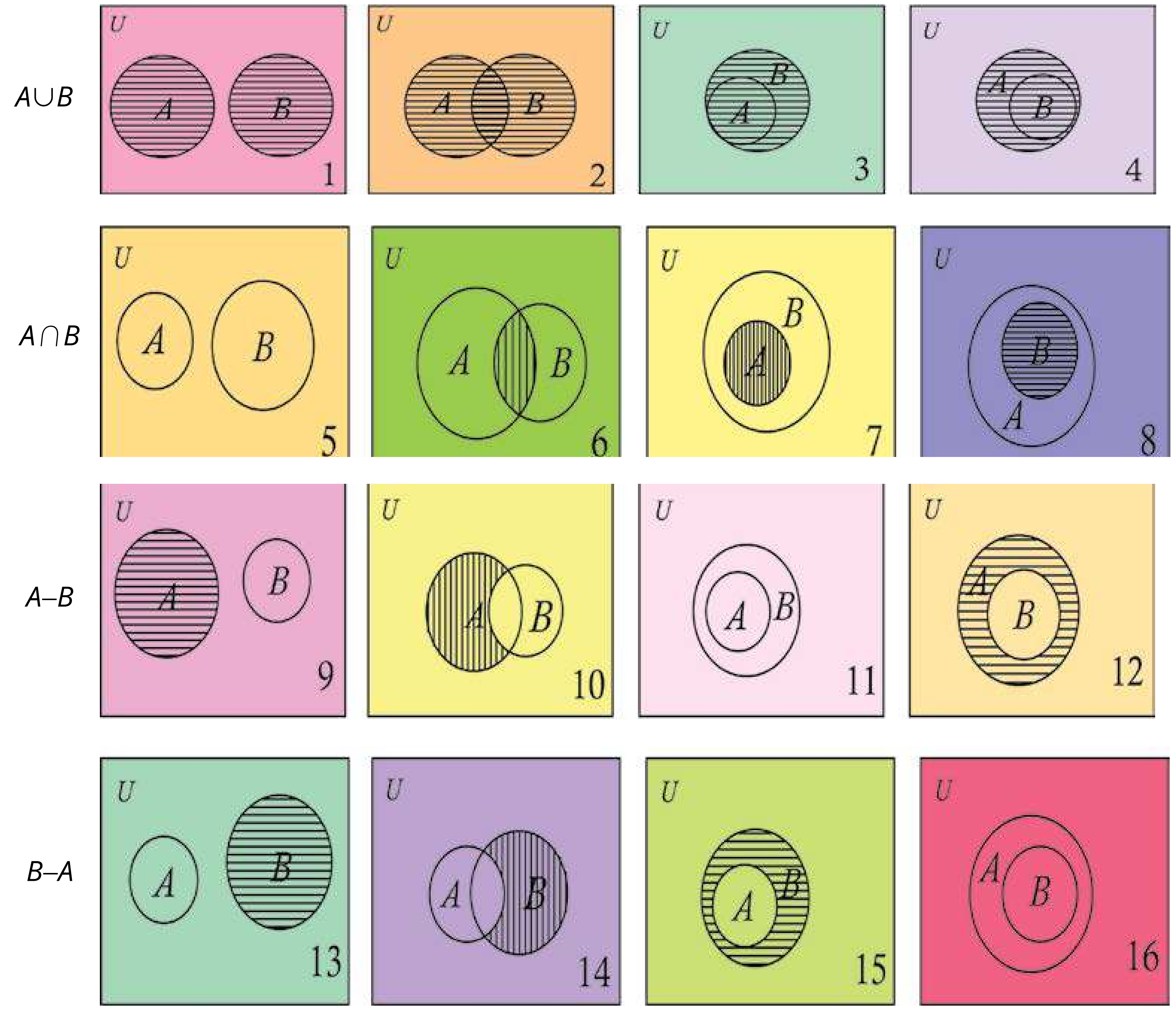
Venn diagrams are very useful in depicting visually the basic concepts of sets and relationships between sets. They were first used by an English logician and mathematician John Venn (1834 to 1883 A.D).

In a Venn diagram, a rectangular region represents the universal set and regions bounded by simple closed curves represent other sets, which are subsets of the universal set. For the sake of beauty these regions are generally shown as circular regions.

In the adjoining figures, the shaded circular region represents a set A and the remaining portion of rectangle representing the universal set U represents A' or $U – A$.



Below are given some more diagrams illustrating basic operations on two sets in different cases (lined region represents the result of the relevant operation in each case given below).



The above diagram suggests the following results: -

Fig No.	Relation between A and B	Result Suggested
1.	A and B disjoint sets $A \cap B = \Phi$	$A \cup B$ consists of all the elements of A and all the elements of B. Also $n(A \cup B) = n(A) + n(B)$
2.	A and B are overlapping $A \cap B \neq \Phi$	$A \cup B$ contains elements which are i) in A and not in B ii) in B and not in A iii) in both A and B. Also $n(A \cup B) = n(A) + n(B) - (A \cap B)$
3.	$A \subseteq B$	$A \cup B = B$; $n(A \cup B) = n(B)$
4.	$B \subseteq A$	$A \cup B = A$; $n(A \cup B) = n(A)$
5.	$A \cap B = \Phi$	$A \cap B = \Phi$; $n(A \cap B) = 0$
6.	$A \cap B \neq \Phi$	$A \cap B$ contains the elements which are in *A and B*
7.	$A \subseteq B$	$A \cap B = A$; $n(A \cap B) = n(A)$
8.	$B \subseteq A$	$A \cap B = B$; $n(A \cap B) = n(B)$
9.	A and B are disjoint sets.	$A - B = A$; $n(A - B) = n(A)$
10.	A and B are overlapping	$n(A - B) = n(A) - n\ A \cap B$
11.	$A \subseteq B$	$A - B = \Phi$; $n(A - B) = 0$
12.	$B \subseteq A$	$A - B \neq \Phi$; $n(A - B) = n(A) - n(B)$
13.	A and B are disjoint	$B - A = B$; $n(B - A) = n(B)$
14.	A and B are overlapping	$n(B - A) = n(B) - n(A \cap B)$
15.	$A \subseteq B$	$B - A \neq \Phi$; $n(B - A) = n(B) - n(A)$
16.	$B \subseteq A$	$B - A = \Phi$; $n(B - A) = 0$

Note (1) Since the empty set contains no elements, therefore, no portion of U represents it.

(2) If in the diagrams given on preceding page we replace B by the empty set (by imagining the region representing B to vanish).

$A \cup \Phi = A$ (FromFig. 1 o r 4)

$A \cap \Phi = \Phi$ (From Fig. 5 or 8)

$A - \Phi = A$ (From Fig. 9 or 12)

$\Phi - A = \Phi$ (FromFig. 13 or 16)

Also by replacing B by A (by imagining the regions represented by A and B to coincide), we obtain the following results:

$A \cup A = A$ (From fig. 3 or 4)

$A \cap A = A$ (From fig. 7 or 8)

$A - A = \Phi$ (From fig. 12)

Again by replacing B by U, we obtain the results: -

$A \cup U = U$ (From fig. 3); $A \cap U = A$ **(From fig. 7)**

$A - U = \Phi$ (From fig. 11); $U - A = A'$ **(From fig. 15)**

(3) Venn diagrams are useful only in case of abstract sets whose elements are not specified. It is not desirable to use them for concrete sets (Although this is erroneously done even in some foreign books).

Exercise 2.2

1. Exhibit $A \cup B$ and $A \cap B$ by Venn diagrams in the following cases: -

i) $A \subseteq B$ ii) $B \subseteq A$ iii) $A \cup A'$

iv) A and B are disjoint sets. v) A and B are overlapping sets

2. Show $A - B$ and $B - A$ by Venn diagrams when: -

i) A and B are overlapping sets ii) $A \subseteq B$ iii) $B \subseteq A$

3. Under what conditions on A and B are the following statements true?

i) $A \cup B = A$ ii) $A \cup B = B$ iii) $A - B = A$

iv) $A \cap B = B$ v) $n(A \cup B) = n(A) + n(B)$ vi) $n(A \cap B) = n(A)$

vii) $A - B = A$ vii) $n(A \cap B) = 0$ ix) $A \cup B = U$
x) $A \cup B = B \cup A$ xi) $n(A \cap B) = n(B)$. xii) $U - A = \Phi$

4. Let $U = \{1,2,3,4,5,6,7,8,9,10\}$, $A = \{2,4,6,8,10\}$, $B = \{1,2,3,4,5\}$ and $C = \{1,3,5,7,9\}$
List the members of each of the following sets: -
i) A^c ii) B^c iii) $A \cup B$ iv) $A - B$
v) $A \cap C$ vi) $A^c \cup C^c$ vii) $A^c \cup C$. viii) U^c

5. Using the Venn diagrams, if necessary, find the single sets equal to the following: -
i) A^c ii) $A \cap U$ iii) $A \cup U$ iv) $A \cup \Phi$ v) $\Phi \cap \Phi$

6. Use Venn diagrams to verify the following: -
i) $A - B = A \cap B^c$ ii) $(A - B)^c \cap B = B$

2.4 Operations on Three Sets

If A, B and C are three given sets, operations of union and intersection can be performed on them in the following ways: -
i) $A \cup (B \cup C)$ ii) $(A \cup B) \cup C$ iii) $A \cap (B \cup C)$
iv) $(A \cap B) \cap C$ v) $A \cup (B \cap C)$ vi) $(A \cap C) \cup (B \cap C)$
vii) $(A \cup B) \cap C$ viii) $(A \cap B) \cup C$. ix) $(A \cup C) \cap (B \cup C)$
Let $A = \{1, 2, 3\}$, $B = \{2,3,4,5\}$ and $C = \{3,4,5,6,7,8\}$
We find sets (i) to (iii) for the three sets (Find the remaining sets yourselves).
i) $B \cup C = \{2,3,4,5,6,7,8\}$, $A \cup (B \cup C) = \{1,2,3,4,5,6,7,8\}$
ii) $A \cup B = \{1,2,3,4,5\}$, $(A \cup B) \cup C = \{1,2,3,4,5,6,7,8\}$
iii) $B \cap C = \{3,4,5\}$, $A \cap (B \cap C) = \{3\}$

2.5 Properties of Union and Intersection

We now state the fundamental properties of union and intersection of two or three sets. Formal proofs of the last four are also being given.

Properties:
i) $A \cup B = B \cup A$ (Commutative property of Union)
ii) $A \cap B = B \cap A$ (Commutative property of Intersection)
iii) $A \cup (B \cup C) = (A \cup B) \cup C$ (Associative property of Union)
iv) $A \cap (B \cap C) = (A \cap B) \cap C$ (Associative property of Intersection).
v) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributivity of Union over intersection)
vi) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributivity of intersection over Union)

vii) $(A \cup B)' = A' \cap B'$
viii) $(A \cap B)' = A' \cup B'$ De Morgan's Laws

Proofs of De Morgan's laws and distributive laws:

i) $(A \cup B)' = A' \cap B'$
Let $x \in (A \cup B)'$
$\Rightarrow$ $x \notin A \cup B$
$\Rightarrow$ $x \notin A$ and $x \notin B$
$\Rightarrow$ $x \in A'$ and $x \in B'$
$\Rightarrow$ $x \in A' \cap B'$
But x is an arbitrary member of $(A \cup B)'$ (1)
Therefore, (1) means that $(A \cup B)' \subseteq A' \cap B'$ (2)
Now suppose that $y \in A' \cap B'$
$\Rightarrow$ $y \in A'$ and $y \in B'$
$\Rightarrow$ $y \notin A$ and $y \notin B$
$\Rightarrow$ $y \notin A \cup B$
$\Rightarrow$ $y \in (A \cup B)'$
Thus $A' \cap B' \subseteq (A \cup B)'$
From (2) and (3) we conclude that (3)
$$(A \cup B)' = A' \cap B'$$

ii) $(A \cap B)' = A' \cup B'$
It may be proved similarly or deducted from (i) by complementation

iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Let $x \in A \cup (B \cap C)$
$\Rightarrow$ $x \in A$ or $x \in B \cap C$
$\Rightarrow$ If $x \in A$ it must belong to $A \cup B$ and $x \in A \cup C$

$\Rightarrow \quad x \in (A \cup B) \cap (A \cup C)$

Also if $x \in B \cap C$, then $x \in B$ and $x \in C$. (1)

$\Rightarrow \quad x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow \quad x \in (A \cup B) \cap (A \cup C)$

Thus $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ (2)

Conversely, suppose that

$y \in (A \cup B) \cap (A \cup C)$

There are two cases to consider: -

$y \in A,\ y \notin A$

In the first case y $\in A \cup (B \cap C)$

If $y \notin A$, it must belong to B as well as C

i.e., y $\in (B \cap C)$

$\therefore y \in A \cup (B \cap C)$

So in either case

y $\in (A \cup B) \cap (A \cup C) \Rightarrow$ y $\in A \cup (B \cap C)$

thus $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ (3)

From (2) and (3) it follows that

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

It may be proved similarly or deducted from (iii) by complementation

Verification of the properties:

Example 1: Let $A = \{1,2,3\}$, $B = \{2,3,4,5\}$ and $C = \{3,4,5,6,7,8\}$

i) $A \cup B = \{1,2\ 3\} \cup \{2,3,4,5\}$ $\quad B \cup A = \{2,3,4,5\} \cup \{1,2,3\}$

$= \{1,2,3,4,5\}$ $\quad = \{2,3,4,5,1\}$

$\therefore A \cup B = B \cup A$

ii) $A \cap B = \{1,2,3\} \cap \{2,3,4,5\}$ $\quad B \cap A = \{2,3,4,5\} \cap \{1,2,3\}$

$= \{2,3\}$ $\quad = \{2,3\}$

$\therefore A \cap B = B \cap A$

(iii) and (iv) Verify yourselves.

(v) $A \cup (B \cap C) = \{1,2,3\} \cup (\{2,3,4,5\} \cap \{3,4,5,6,7,8)$

$= \{1,2,3\} \cup \{3,4,5\}$

$= \{1,2,3,4,5\}$ (1)

$(A \cup B) \cap (A \cup C) = (\{1,2,3\} \cup \{2,3,4,5\}) \cap (\{1,2,3\} \cup \{3,4,5,6,7,8\})$

$= \{1,2,3,4,5\} \cap \{1,2,3,4,5,6,7,8\}$

$= \{1,2,3,4,5\}$ (2)

From (1) and (2),

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

vi) Verify yourselves.

vii) Let the universal set be $U = \{1,2,3,4,5,6,7,8,9,10\}$

$A \cup B = \{1,2,3\} \cup \{2,3,4,5\} = \{1,2,3,4,5\}$

$(A \cup B)' = \{6,7,8,9,10\}$ (1)

$A' = U - A = (4,5,6,7,8,9,10)$

$B' = U - B = \{1,6,7,8,9,10\}$

$A' \cap B' = (4,5,6,7,8,9,10) \cap \{1,6,7,8,9,10\}$

$= \{6,7,8,9,10\}$ (2)

From (1) and (2),

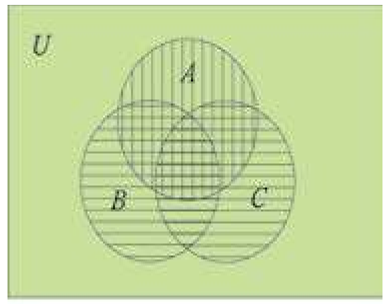
$(A \cup B)' = A' \cap B'$

viii) Verify yourselves.

Verification of the properties with the help of Venn diagrams.

i) and (ii): Verification is very simple, therefore, do it yourselves,

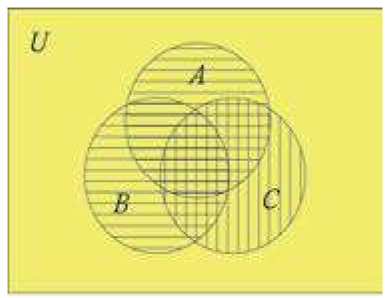
iii): In fig. (1) set A is represented by vertically lined region and $B \cup C$ is represented by horizontally lined region. The set $A \cup (B \cup C)$ is represented by the region which is lined either in one or both ways.



A |||
B ∪ C
≡ ∨ ⩩
A∪(B ∪ C)
|||,≡ ∨ ⩩

Fig (1)

In figure(2) $A \cup B$ is represented by horizontally lined region and C by vertically lined region. $(A \cup B) \cup C$ is represented by the region which is lined in either one or both ways.



A ∪B
C |||
(A∪B)∪C
≡ ||| ∨ ⩩

Fig (2)

From fig (1) and (2) we can see that
$A \cup (B \cup C) = (A \cup B) \cup C$
(iv) In fig (3) doubly lined region represents.
$A \cap (B \cap C)$

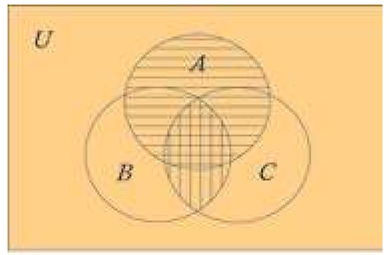


Fig (3)

$A \equiv$
$B \cap C \,|||$
$A \cap (B \cap C$

In fig (4) doubly lined region represents
$(A \cap B) \cap C$.
Since in fig (3) and (4) these regions are the same therefore,
$A \cap (B \cap C) = (A \cap B) \cap C$.

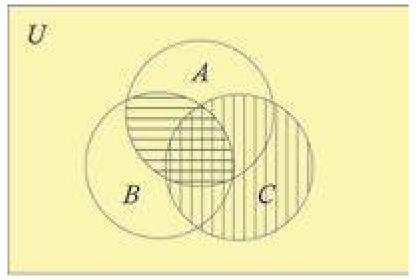


Fig (4)

$A \cap B \equiv$
$C \,|||$
$(A \cap B) \cap C$
#

(v) in fig. (5) $A \cup (B \cap C)$ is represented by the region which is lined horizontally or vertically or both ways.

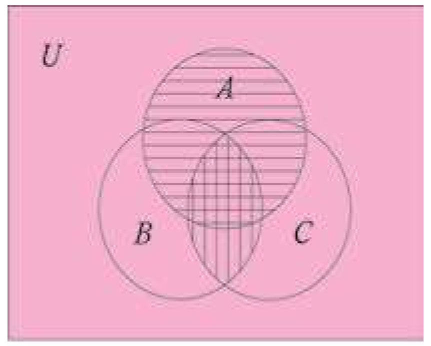


Fig (5)

$A \equiv$
$B \cap C \,|||$
$A \cup (B \cap C)$
$\equiv, ||| \vee$ #

In fig. (6) $(A \cup B) \cap (A \cup C)$ is represented by the doubly lined region. Since the two region in fig (5) and (6) are the same, therefore

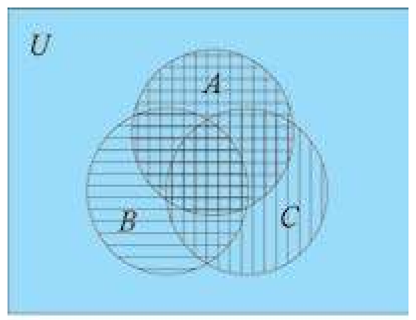


Fig (6)

$A \cup B \equiv$
$A \cup C \,|||$
$(A \cup B) \cap (A \cup C)$
#

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(vi) Verify yourselves.
(vii) In fig (7) $(A \cup B)'$ is represented by vertically lined region.

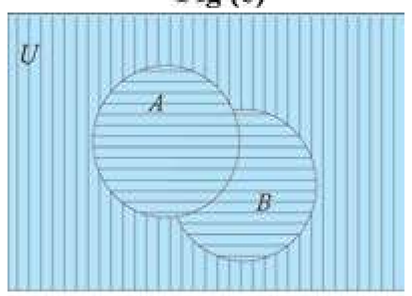


Fig (7)

$A \cup B \equiv$
$(A \cup B)' \,|||$

In fig. (8) doubly lined region represents.
$A' \cap B'$.
The two regions in fig (7). And (8) are the same, therefore, $(A \cup B)' = A' \cap B'$

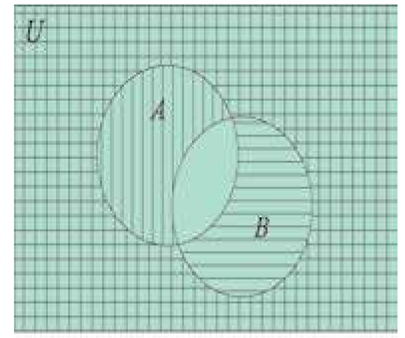


Fig (8)

$A' \equiv$
B'
$A' \cap B'$
#

(viii) Verify yourselves.

Note: In all the above Venn diagrams only overlapping sets have been considered. Verification in other cases can also be effected similarly. Detail of verification may be written by yourselves.

Exercise 2.3

1. Verify the commutative properties of union and intersection for the following pairs of sets: -
i) $A = \{1,2,3,4,5\}$, $B = \{4,6,8,10\}$ ii) N, Z
iii) $A = \{x \mid x \in \mathbb{R} \wedge x \geq 0\}$, $B = \mathbb{R}$.

2. Verify the properties for the sets A, B and C given below: -
i) Associativity of Union ii) Associativity of intersection.
iii) Distributivity of Union over intersection.
iv) Distributivity of intersection over union.
a) $A = \{1,2,3,4\}$, $B = \{3,4,5,6,7,8\}$, $C = \{5,6,7,9,10\}$
b) $A = \Phi$, $B = \{0\}$, $C = \{0,1,2\}$
c) N, Z, Q

3. Verify De Morgan's Laws for the following sets:
$U = \{1,2,3, \ldots, 20\}$, $A = \{2,4,6, \ldots, 20\}$ and $B = \{1,3,5, \ldots, 19\}$.

4. Let U = The set of the English alphabet
$A = \{x \mid x$ is a vowel$\}$, $B = \{y \mid y$ is a consonant$\}$,
Verify De Morgan's Laws for these sets.

5. With the help of Venn diagrams, verify the two distributive properties in the following cases w.r.t union and intersection.

i) $A \subseteq B$, $A \cap C = \Phi$ and B and C are overlapping.

ii) A and B are overlapping, B and C are overlapping but A and C are disjoint.

6. Taking any set, say $A = \{1,2,3,4,5\}$ verify the following: -

i) $A \cup \Phi = A$ ii) $A \cup A = A$ iii) $A \cap A = A$

7. If $U = \{1,2,3,4,5,...., 20\}$ and $A = \{1,3,5, ..., 19\}$, verify the following:-

i) $A \cup A' = U$ ii) $A \cap U = A$ iii) $A \cap A' = \Phi$

8. From suitable properties of union and intersection deduce the following results:

i) $A \cap (A \cup B) = A \cup (A \cap B)$ ii) $A \cup (A \cap B) = A \cap (A \cup B)$.

9. Using venn diagrams, verify the following results.

i) $A \cap B' = A$ iff $A \cap B = \Phi$ ii) $(A - B) \cup B = A \cup B$.

iii) $(A - B) \cap B = \Phi$ iv) $A \cup B = A \cup (A' \cap B)$.

2.6 Inductive and Deductive Logic

In daily life we often draw general conclusions from a limited number of observations or experiences. A person gets penicillin injection once or twice and experiences reaction soon afterwards. He generalises that he is allergic to penicillin. We generally form opinions about others on the basis of a few contacts only. This way of drawing conclusions is called **induction**.

Inductive reasoning is useful in natural sciences where we have to depend upon repeated experiments or observations. In fact greater part of our knowledge is based on induction.

On many occasions we have to adopt the opposite course. We have to draw conclusions from accepted or well-known facts. We often consult lawyers or doctors on the basis of their good reputation. This way of reasoning i.e., drawing conclusions from premises believed to be true, is called **deduction**. One usual example of deduction is: All men are mortal. We are men. Therefore, we are also mortal.

Deduction is much used in higher mathematics. In teaching elementary mathematics we generally resort to the inductive method. For instance the following sequences can be continued, inductively, to as many terms as we like:

i) 2,4,6,... ii) 1,4,9,... iii) 1,-1,2,-2,3,-3,...

iv) 1,4,7,... v) $\frac{1}{3}, \frac{1}{12}, \frac{1}{36}, ...$ vi) $\frac{1}{10}, \frac{2}{100}, \frac{4}{1000}, ...$

As already remarked, in higher mathematics we use the deductive method. To start with we accept a few statements (called postulates) as true without proof and draw as many conclusions from them as possible.

Basic principles of deductive logic were laid down by Greek philosopher, Aristotle. The illustrious mathematician Euclid used the deductive method while writing his 13 books of geometry, called Elements. Toward the end of the 17th century the eminent German mathematician, Leibniz, symbolized deduction. Due to this device deductive method became far more useful and easier to apply.

2.6.1 Aristotelian and non-Aristotelian logics

For reasoning we have to use *propositions*. A daclarative statement which may be true or false but not both is called a proposition. According to Aristotle there could be only two possibilities - a proposition could be either true or false and there could not be any third possibility. This is correct so far as mathematics and other exact sciences are concerned. For instance, the statement $a = b$ can be either true or false. Similarly, any physical or chemical theory can be either true or false. However, in statistical or social sciences it is sometimes not possible to divide all statements into two mutually exclusive classes. Some statements may be, for instance, undecided.

Deductive logic in which every statement is regarded as true or false and there is no other possibility, is called Aristotlian Logic. Logic in which there is scope for a third or fourth possibility is called non-Aristotelian. we shall be concerned at this stage with Aristotelian logic only.

2.6.2 Symbolic logic

For the sake of brevity propositions will be denoted by the letters p , q etc. We give a brief list of the other symbols which will be used.

Symbol	How to be read	Symbolic expression	How to be read
$\sim$	not	$\sim p$	Not p, negation of p
$\wedge$	and	$p \wedge q$	p and q
$\vee$	or	$p \vee q$	p or q
$\rightarrow$	If... then, implies	$p \rightarrow q$	If p then q p implies q
$\leftrightarrow$	Is equivalent to, if and only if	$p \leftrightarrow q$	p if and only if q p is equivalent to q

Explanation of the use of the Symbols:

1) **Negation:** If p is any proposition its negation is denoted by $\sim p$, read 'not p'. It follows from this definition that if p is true, $\sim p$ is false and if p is false, $\sim p$ is true. The adjoining table, called truth table, gives the possible truth- values of p and $\sim p$.

p	$\sim p$
T	F
F	T

Table (1)

2) **Conjunction** of two statements p and q is denoted symbolically as $p \wedge q$ (p and q). A conjunction is considered to be true only if both its components are true. So the truth table of $p \wedge q$ is table (2).

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table (2)

Example 1:

i) Lahore is the capital of the Punjab and Quetta is the capital of Balochistan.

ii) $4 < 5 \wedge 8 < 10$

iii) $4 < 5 \wedge 8 > 10$

iv) $2 + 2 = 3 \wedge 6 + 6 = 10$

Clearly conjunctions (i) and (ii) are true whereas (iii) and (iv) are false.

3) **Disjunction** of p and q is p or q. It is symbolically written $p \vee q$. The disjunction $p \vee q$ is considered to be true when at least one of the components p and q is true. It is false when both of them are false. Table (3) is the truth table.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table (3)

Example 2:

i) 10 is a positive integer or π is a rational number. Find truth value of this disjunction.

Solution: Since the first component is true, the disjunction is true.

ii) A triangle can have two right angles or Lahore is the capital of Sind.

Solution: Both the components being false, the composite proposition is false.

2.7 Implication or conditional

A compound statement of the form if p then q, also written ***p* implies *q***, is called a **conditional** or an **implication**, p is called the **antecedent** or **hypothesis** and q is called the **consequent** or the **conclusion**.

A conditional is regarded as false only when the antecedent is true and consequent is false. In all other cases it is considered to be true. Its truth table is, therefore, of the adjoining form.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table (4)

Entries in the first two rows are quite in consonance with common sense but the entries of the last two rows seem to be against common sense. According to the third row the conditional

If p then q

is true when p is false and q is true and the compound proposition is true (according to the fourth row of the table) even when both its components are false. We attempt to clear the position with the help of an example. Consider the conditional

If a person A lives at Lahore, then he lives in Pakistan.

If the antecedent is false i.e., A does not live in Lahore, all the same he may be living in Pakistan. We have no reason to say that he does not live in Pakistan.

We cannot, therefore, say that the conditional is false. So we must regard it as true. It must be remembered that we are discussing a problem of Aristotlian logic in which every proposition must be either true or false and there is no third possibility. In the case under discussion there being no reason to regard the proposition as false, it has to be regarded as true. Similarly, when both the antecedent and consequent of the conditional under consideration are false, there is no justification for quarrelling with the proposition. Consider another example.

A certain player, Z, claims that if he is appointed captain, the team will win the tournament. There are four possibilities: -

i) Z is appointed captain and the team wins the tournament. Z's claim is true.

ii) Z is appointed captain but the team loses the tournament. Z's claim is falsified.

iii) Z is not appointed captain but the team all the same wins the tournament. There is no reason to falsify Z's claim.

iv) Z is not appointed captain and the team loses the tournament. Evidently, blame cannot be put on Z.

It is worth noticing that emphasis is on the conjunction if occurring in the beginning of the ancedent of the conditional. If condition stated in the antecedent is not satisfied we should regard the proposition as true without caring whether the consequent is true or false.

For another view of the matter we revert to the example about a Lahorite:

'If a person A lives at Lahore, then he lives in Pakistan'.

p: A person A lives at Lahore.

q: He lives in Pakistan

When we say that this proposition is true we mean that in this case it is not possible that 'A lives at Lahore' is true and that 'A does not live in Pakistan' is also true, that is $p \to q$ and $\sim(p \wedge \sim q)$ are both simultaneously true. Now the truth table of $\sim(p \wedge \sim q)$ is shown below:

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

Table (5)

Looking at the last column of this table we find that truth values of the compound proposition $\sim(p \wedge \sim q)$ are the same as those adopted by us for the conditional $p \to q$. This shows that the two propositions $p \to q$ and $\sim(p \wedge \sim q)$ are logically equivalent. Therefore, the truth values adopted by us for the conditional are correct.

2.7.1 Biconditional : $p \leftrightarrow q$

The proposition $p \to q \wedge q \to p$ i s shortly written $p \leftrightarrow q$ and is called the **biconditional** or **equivalence**. It is read ***p*** iff ***q*** (iff stands for "if and only if ')

We draw up its truth table.

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Table (6)

From the table it appears that $p \leftrightarrow q$ is true only when both p and q are true or both p and q are false.

2.7.2 Conditionals related with a given conditional.

Let $p \to q$ be a given conditional. Then

i) $q \to p$ is called the **converse** of $p \to q$;

ii) $\sim p \to \sim q$ is called the **inverse** of $p \to q$;

iii) $\sim q \to \sim p$ is called the **contrapositive** of $p \to q$.

To compare the truth values of these new conditionals with those of $p \to q$ we draw up their joint table.

				Given conditional	Converse	Inverse	Contrapositive
p	q	$\sim p$	$\sim q$	$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	F	T	T	T	T

Table (7)

From the table it appears that

i) Any conditional and its contrapositive are equivalent therefore any theorem may be proved by proving its contrapositive.

ii) The converse and inverse are equivalent to each other.

Example 3: Prove that in any universe the empty set ϕ is a subset of any set *A*.

First Proof: Let U be the universal set consider the conditional:

$$\forall x \in U,\ x \in \phi \rightarrow x \in A \qquad (1)$$

The antecedent of this conditional is false because no $x \in U$, is a member of ϕ .

Hence the conditional is true.

Second proof: (By contrapositive)

The contrapositive of conditional (1) is

$$\forall x \in U,\ x \notin A \rightarrow x \notin \phi \qquad (2)$$

The consequent of this conditional is true. Therefore, the conditional is true.

Hence the result.

Example 4: Construct the truth table ot $[(p \rightarrow q) \wedge p \rightarrow q]$

Solution : Desired truth table is given below: -

p	p	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge p \rightarrow q]$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Table (8)

2.7.3 Tautologies

i) A statement which is true for all the possible values of the variables involved in it is

called a tautology, for example, $p \rightarrow q \leftrightarrow (\sim q \rightarrow \sim p)$ is a *tautology*.(are already verified by a truth table).

ii) A statement which is always false is called an **absurdity** or a contradiction e.g., $p \rightarrow \sim p$

iii) A statement which can be true or false depending upon the truth values of the variables involved in it is called a **contingency** e.g., $(p \rightarrow q) \wedge (p \vee q)$ is a contingency.

(You can verify it by constructing its truth table).

2.7.4 Quantifiers

The words or symbols which convey the idea of quantity or number are called quantifiers.

In mathematics two types of quantifiers are generally used.

i) **Universal quantifier** meaning for all

Symbol used : $\forall$

ii) **Existential quantifier:** There exist (some or few, at least one) symbol used: $\exists$

Example 5:

i) $\forall x \in A$, $p(x)$ is true.

(To be read : For all x belonging to *A* the statement $p(x)$ is true).

ii) $\exists x \in A \ni p(x)$ is true.

(To be read : There exists x belonging to *A* such that statement $p(x)$ is true).

The symbol $\ni$ stands for such that

Exercise 2.4

1. Write the converse, inverse and contrapositive of the following conditionals: -

i) $\sim p \rightarrow q$ ii) $q \rightarrow p$ iii) $\sim p \rightarrow \sim q$ iv) $\sim q \rightarrow \sim p$

2. Construct truth tables for the following statements: -

i) $(p \rightarrow \sim p) \vee (p \rightarrow q)$ ii) $(p \wedge \sim p) \rightarrow q$

iii) $\sim(p \rightarrow q) \leftrightarrow (p \wedge \sim q)$

3. Show that each of the following statements is a tautology: -
   i) $(p \wedge q) \rightarrow p$ ii) $p \rightarrow (p \vee q)$
   iii) $\sim(p \rightarrow q) \rightarrow p$ iv) $\sim q \wedge (p \rightarrow q) \rightarrow \sim p$

4. Determine whether each of the following is a tautology, a contingency or an absurdity: -
   i) $p \wedge \sim p$ ii) $p \rightarrow (q \rightarrow p)$ iii) $q \vee (\sim q \vee p)$
5. Prove that $p \vee (\sim p \wedge \sim q) \vee (p \wedge q) = p \vee (\sim p \wedge \sim q)$

2.8 Truth Sets, A link between Set Theory and Logic.

Logical propositions p, q etc., are formulae expressed in terms of some variables. For the sake of simplicity and convenience we may assume that they are all expressed in terms of a single variable x where x is a real variable. Thus $p = p(x)$ where, $x \in \mathbb{R}$. All those values of x which make the formula $p(x)$ true form a set, say P. Then P is the truth set of p. Similarly, the **truth set**, Q, of q may be defined. We can extend this notion and apply it in other cases.

i) **Truth set of ~*p*:** Truth set of $\sim p$ will evidently consist of those values of the variable for which p is false i.e., they will be members of P', the complement of P.

ii) **$p \vee q$:** Truth set of $p \vee q = p(x) \vee q(x)$ consists of those values of the variable for which $p(x)$ is true or $q(x)$ is true or both $p(x)$ and $q(x)$ are true.

Therefore, truth set of $p \vee q$ will be:

$$P \cup Q \quad = \{x \mid p(x) \text{ is true or } q(x) \text{ is true}\}$$

iii) **$p \wedge q$:** Truth set of $p(x) \wedge q(x)$ will consist of those values of the variable for which both $p(x)$ and $q(x)$ are true. Evidently truth set of

$$p \wedge q = P \cap Q$$
$$= \{x \mid p(x) \text{ is true } \wedge q(x) \text{ is true}\}$$

iv) **$p \rightarrow q$:** We know that $p \rightarrow q$ is equivalent to $\sim p \vee q$ therefore truth set of $p \rightarrow q$ will be $P' \cup Q$

v) **$p \leftrightarrow q$:** We know that $p \leftrightarrow q$ means that p and q are simultaneously true or false. Therefore, in this case truth sets of p and q will be the same i.e.,

$$P = Q$$

Note: **(1)** Evidently truth set of a tautology is the relevant universal set and that of an absurdity is the empty set Φ.

(2) With the help of the above results we can express any logical formula in set-theoretic form and vice versa.

We will illustrate this fact with the help of a solved example.

Example 1: Give logical proofs of the following theorems: -
(A, B and C are any sets)

i) $(A \cup B)' = A' \cap B'$ ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution: i) The corresponding formula of logic is

$$\sim(p \vee q) = \sim p \wedge \sim q \qquad (1)$$

We construct truth table of the two sides.

p	p	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

The last two columns of the table establish the equality of the two sides of eq.(1)

(ii) Logical form of the theorem is

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

We construct the table for the two sides of this equation

1	2	3	4	5	6	7	8
p	p	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Comparison of the entries of columns 5 and 8 is sufficient to establish the desired result.

Exercise 2.5

Convert the following theorems to logical form and prove them by constructing truth tables: -

1. $(A \cap B)' = A' \cup B'$ **2.** $(A \cup B) \cup C = A \cup (B \cup C)$

3. $(A \cap B) \cap C = A \cap (B \cap C)$ **4.** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

2.9 Relations

In every-day use *relation* means an abstract type of connection between two persons or objects , for instance, (Teacher, Pupil), (Mother, Son), (Husband, Wife), (Brother, Sister), (Friend, Friend), (House, Owner). In mathematics also some operations determine relationship between two numbers, for example: -

$>$: (5 , 4); square: (25, 5); Square root: (2,4); Equal: (2×2, 4).

Technically a *relation* is a set of ordered pairs whose elements are ordered pairs of related numbers or objects. The relationship between the components of an ordered pair may or may not be mentioned.

i) Let A and B be two non-empty sets, then any subset of the Cartesian product $A \times B$ is called a **binary relation**, or simply a **relation**, from A to B. Ordinarily a relation will be denoted by the letter r.

ii) The set of the first elements of the ordered pairs forming a relation is called its **domain**.

iii) The set of the second elements of the ordered pairs forming a relation is called its **range**.

iv) If A is a non-empty set, any subset of $A \times A$ is called a **relation in *A***. Some authors call it a **relation on *A***.

Example 1: Let c_1, c_2, c_3 be three children and m_1, m_2 be two men such that father of both c_1, c_2 is m_1 and father of c_3 is m_2. Find the relation {(child, father)}

Solution: C = Set of children = $\{c_1, c_2, c_3\}$ and F = set of fathers = $\{m_1, m_2\}$

$C \times F = \{(c_1, m_1), (c_1, m_2), (c_2, m_1), (c_2, m_2), (c_3, m_1), (c_3, m_2)\}$

r = set of ordered pairs (*child, father*).

$= \{(c_1, m_1), (c_2, m_1), (c_3, m_2)\}$

Dom $r = (c_1, c_2, c_3\}$, Ran $r = \{m_1, m_2\}$

The relation is shown diagrammatically in fig. (2.29).

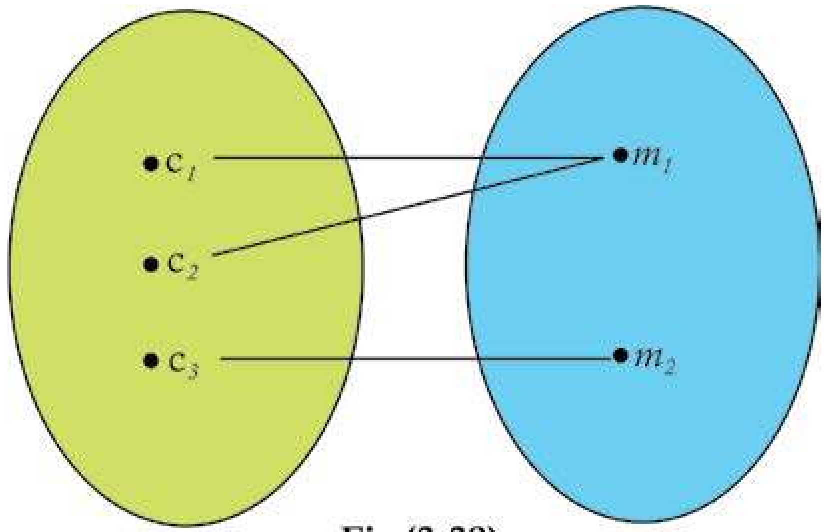


Fig (2.29)

Example 2: Let $A = \{1, 2, 3\}$. Determine the relation r such that xry iff $x < y$.

Solution: $A \times A = \{(1, 1),(1, 2),(1, 3),(2, 1), (2, 2),(2, 3),(3, 1),(3, 2),(3, 3)\}$

Clearly, required relation is:

$r = \{(1, 2), (1, 3), (2, 3)\}$, *Dom* $r = \{1, 2\}$, Ran $r = \{2, 3\}$

Example 3: Let $A = \mathbb{R}$, the set of all real numbers.

Determine the relation r such that $x\, r\, y$ iff $y = x + 1$

Solution: $A \times A = \mathbb{R} \times \mathbb{R}$

$r = \{ (x,y) | \ y = x+1\}$

When $x = 0$, $y = 1$

$x = -1$, $y = 0$,

r is represented by the line passing through the points (0,1), (– 1,0).

Some more points belonging to r are:
$\{(1, 2), (2, 3), (3, 4), (-2, -1), (-3, -2), (-4, -3)\}$
Clearly, Dom $r = \mathbb{R}$, and Ran $r = \mathbb{R}$

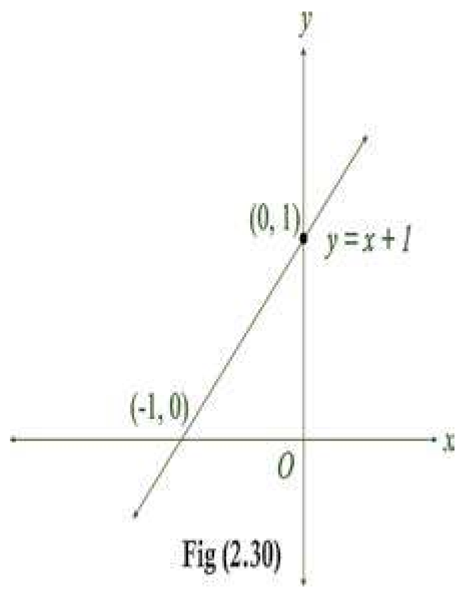


Fig (2.30)

2.10 Functions

A very important special type of relation is a function defined as below: -
Let A and B be two non-empty sets such that:

i) f is a relation from A t o B that is , f is a subset of $A \times B$
ii) Dom $f = A$
iii) First element of no two pairs of f are equal, then f is said to be a function from A to B.

The function f is also written as:

$$f : A \to B$$

which is read: f is a function from A t o B .
If (x, y) in an element of f when regarded as a set of ordered pairs, we write $y = f(x)$. y is called the value of f for x or image of x under f.
In example 1 discussed above

i) r is a subset of $C \times F$
ii) Dom $r = \{c_1, c_2, c_3\} = C$;
iii) First elements of no two related pairs of r are the same.

Therefore, r is a function from C to F.
In Example 2 discussed above

i) r is a subset of $A \times A$;
ii) Dom $r \neq A$

Therefore, the relation in this case is not a function.
In example 3 discussed above

i) r is a subset of $\mathbb{R}$

ii) Dom $r = \mathbb{R}$
iii) Clearly first elements of no two ordered pairs of r can be equal. Therefore, in this case r is a function.

i) Into Function: If a function $f : A \to B$ is such that Ran $f \subset B$ i.e., Ran $f \neq$ B, then f is said to be a function from A **into** B. In fig.(1) f is clearly a function. But Ran $f \neq$ B. Therefore, f is a function from A into B.

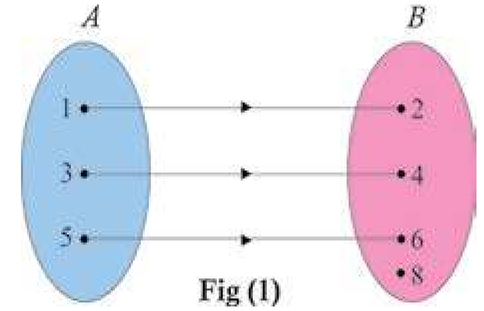


Fig (1)
$f = \{(1,2), (3,4), (5,6)\}$

ii) Onto (Surjective) function: If a function $f : A \to B$ is such that Ran $f = B$ i.e., every element of B is the image of some elements of A, then f is called an **onto** function or a surjective function.

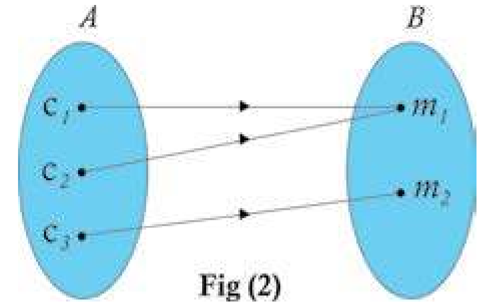


Fig (2)
$f = \{(c_1, m_1), (c_2, m_1), (c_3, m_2)\}$

iii) (1-1) and into (Injective) function: If a function f from A into B is such that second elements of no two of its ordered pairs are equal, then it is called an injective (1 - 1, and into) function. The function shown in fig (3) is such a function.

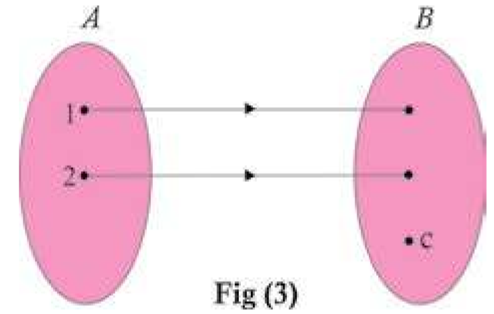


Fig (3)
$f = \{(1, a)(2, b)\}$

iv) (1 –1) and Onto function (bijective function). If f is a function from A onto B such that second elements of no two of its ordered pairs are the same, then f is said to be (1 – 1) function from A onto B.
Such a function is also called a (1 – 1) correspondence between A and B. It is also called a bijective function.
Fig(4) shows a (1-1) correspondence between the sets A and B.

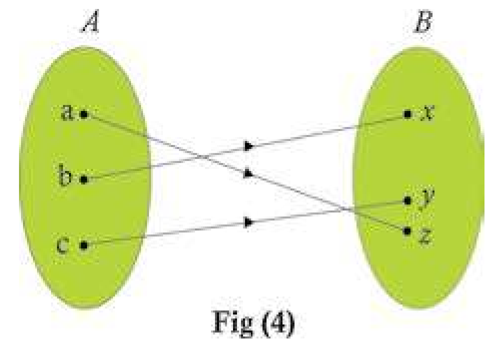


Fig (4)
$f = \{(a, z), (b, x), (c, y)\}$

(a, z), (b, x) and (c, y) are the pairs of corresponding elements i.e., in this case $f = \{(a, z), (b, x), (c, y)\}$ which is a bijective function or (1 - 1) correspondence between the sets A and B.

Set - Builder Notation for a function: We know that set-builder notation is more suitable for infinite sets. So is the case in respect of a function comprising infinite number of ordered pairs. Consider for instance, the function

$f = \{ (1,1), (2,4), (3, 9), (4, 16),...\}$

Dom $f = (1, 2, 3,4, ...\}$.and Ran $f = \{1,4,9, 16, ...\}$

This function may be written as: $f = \{(x, y) \mid y = x^2, x \in N\}$

For the sake of brevity this function may be written as:

f = function defined by the equation $y = x^2$, $x \in N$

Or, to be still more brief: The function x^2 , $x \in N$

In algebra and Calculus the domain of most functions is $\mathbb{R}$ and if evident from the context it is, generally, omitted.

2.10.1 Linear and Quadratic Functions

The function $\{(x, y) \mid y = mx + c\}$ is called a **linear function**, because its graph (geometric representation) is a straight line. Detailed study of a straight line will be undertaken in the next class. For the present it is sufficient to know that an equationof the form

$y = mx + c$ or $ax + by + c = 0$ represents a straight line . This can be easily verified by drawing graphs of a few linear equations with numerical coefficients. The function $\{(x, y) \mid y = ax^2 + bx + c\}$ is called a **quadratic function** because it is defined by a quadratic (second degree) equation in x, y.

Example 4: Give rough sketch of the functions

i) $\{(x, y) \mid 3x + y = 2\}$ ii) $\{(x, y) \mid y = \frac{1}{2}x^2\}$

Solution:

i) The equation defining the function is $3x + y = 2$

$\Rightarrow y = -3x + 2$

We know that this equation, being linear, represents a straight line. Therefore, for drawing its sketch or graph only two of its points are sufficient.

When $x = 0, y = 2$,

When $y = 0, x = \frac{2}{3} = 0.6$ nearly. So two points on the line are A (0, 2) and B = (0.6, 0).

Joining A and B and producing $\overline{AB}$ in both directions, we obtain the line AB i.e., graph of the given function.

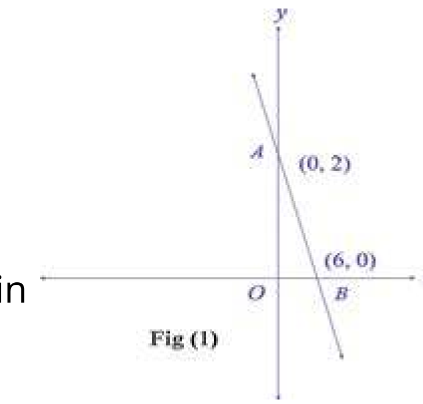


Fig (1)

ii) The equation defining the function is $y = \frac{1}{2}x^2$.

Corresponding to the values 0, ± 1, ±2, ±3 ... of x, values of y are 0, .5, 2, 4.5, ...

We plot the points (0, 0), (±1, .5), (±2, 2), (±3, 4.5), ...

Joining them by means of a smooth curve and extending it upwards we get the required graph. We notice that:

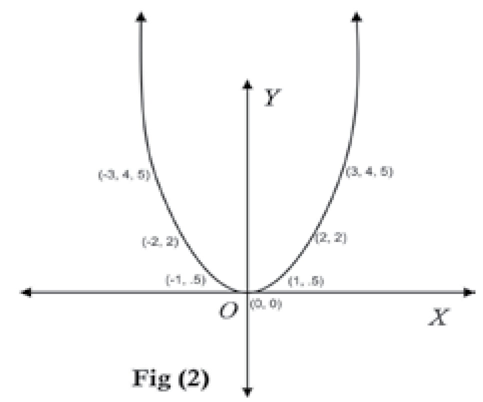


Fig (2)

i) The entire graph lies above the x-axis.

ii) Two equal and opposite values of x correspond to every value of y (but not vice versa).

iii) As x increases (numerically) y increases and there is no end to their ncrease. Thus the graph goes infinitely upwards. Such a curve is called a parabola. The students will learn more about it in the next class.

2.11 Inverse of a function

If a relation or a function is given in the tabular form i.e., as a set of ordered pairs, its inverse is obtained by interchanging the components of each ordered pair. The inverse of r and f are denoted r^{-1} and f^{-1} respectively.

If r or f are given in set-builder notation the inverse of each is obtained by interchanging x and y in the defining equation. The inverse of a function may or may not be a function.

The inverse of the linear function

$\{(x, y) \mid y = mx + c\}$ is $\{(x, y) \mid x = my + c\}$ which is also a linear function. Briefly, we may say that the **inverse of a line is a line.**

The line $y = x$ is clearly self-inverse. The function defined by this equation i.e., the function $\{(x, y) \mid y = x\}$ is called the identity function.

Example 6: Find the inverse of

i) $\{(1, 1), (2,4), (3,9), (4, 16),\ldots x \in Z^+\}$,

ii) $\{(x, y) \mid y = 2x + 3, x \in \mathbb{R}\}$ iii) $\{(x, y) \mid x^2 + y^2 = a^2\}$.

Tell which of these are functions.

Solution:

i) The inverse is:

$\{(2,1), (4, 2), (9, 3), (16, 4)\ldots\}$.

This is also a function.

Note: Remember that the equation

$y = \sqrt{x}, x \geq 0$

defines a function but the equation $y^2 = x, x \geq 0$ does not define a function.

The function defined by the equation

$y = \sqrt{x}, x \geq 0$

is called the **square root function.**

The equation $y^2 = x \Rightarrow y = \pm\sqrt{x}$

Therefore, the equation $y^2 = x$ $(x \geq 0)$ may be regarded as defining the union of the functions defined by

$y = \sqrt{x}, x \geq 0$ and $y = -\sqrt{x}, x \geq 0$.

ii) The given function is a **linear function**. Its inverse is:

$\{(x, y) \mid x = 2y + 3\}$

which is also a linear function.

Points (0, 3), (–1.5, 0) lie on the given line and points (3, 0), (0, –1.5) lie on its inverse. (Draw the graphs yourselves).

The lines *l*, *i'* are symmetric with respect to the line $y = x$. This quality of symmetry is true not only about a linear *n* function and its inverse but is also true about any function of a higher degree and its inverse (why?).

Exercise 2.6

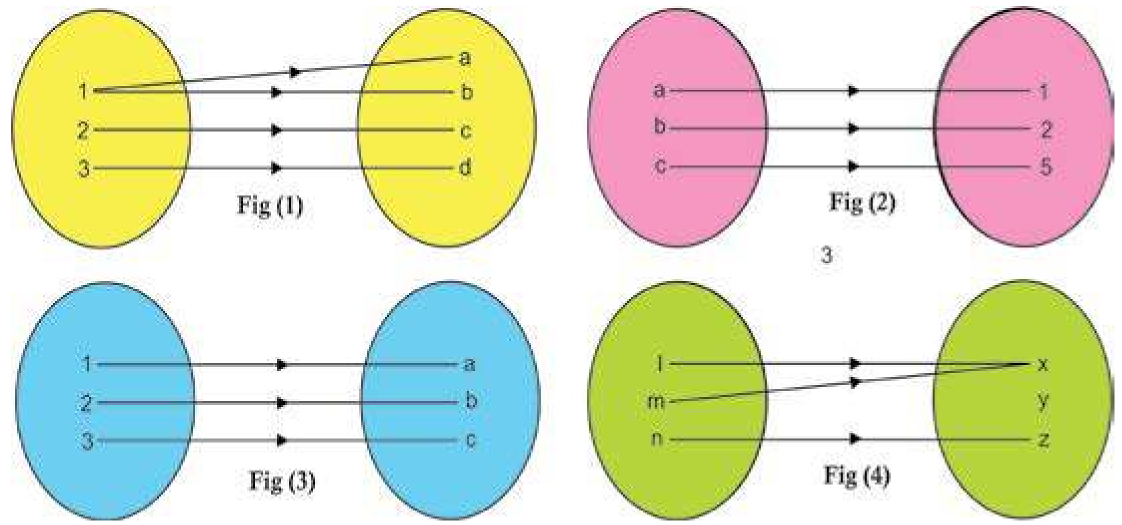
1. For $A = \{1, 2, 3, 4\}$, find the following relations in A. State the domain and range of each relation. Also draw the graph of each.

i) $\{(x, y) \mid y = x\}$ ii) $\{(x, y) \mid y + x = 5\}$

ii) $\{(x, y) \mid x + y < 5\}$ iv) $\{(x, y) \mid x + y > 5\}$

2. Repeat Q -1 when $A = \mathbb{R}$, the set of real numbers. Which of the real lines are functions.

3. Which of the following diagrams represent functions and of which type?



4. Find the inverse of each of the following relations. Tell whether each relation and its inverse is a function or not: -

i) $\{(2,1), (3,2), (4,3), (5,4), (6,5)\}$ ii) $\{(1,3), (2,5), (3,7), (4,9), (5,11)\}$

iii) $\{(x, y) \mid y = 2x + 3, x \in \mathbb{R}\}$ iv) $\{(x, y) \mid y^2 = 4ax, x \geq 0\}$

v) $\{(x, y) \mid x^2 + y^2 = 9, |x| \leq 3, |y| \leq 3\}$

2.12 Binary Operations

In lower classes we have been studying different number systems investigating the properties of the operations performed on each system. Now we adopt the opposite course. We now study certain operations which may be useful in various particular cases.

An operation which when performed on a single number yields another number of the same or a different system is called a **unary operation.**

Examples of *Unary operations* are negation of a given number, extraction of square roots or cube roots of a number, squaring a number or raising it to a higher power.

We now consider binary operation, of much greater importance, operation which requires two numbers. We start by giving a formal definition of such an operation.

A *binary operation* denoted as ※ (read as star) on a non-empty set *G* is a function which associates with each ordered pair (*a, b*), of elements of *G*, a unique element, denoted as a ※ *b* of *G*.

In other words, a binary operation on a set *G* is a function from the set $G \times G$ to the set *G*. For convenience we often omit the word *binary* before *operation*.

Also in place of saying ※ is an operation on *G*, we shall say *G* is closed with respect to ※ .

Example 1: Ordinary addition, multiplication are operations on *N*. i.e., *N* is closed with respect to ordinary addition and multiplication because

$$\forall\, a, b \in N,\ a + b \in N \wedge a.b \in N$$

($\forall$ stands for" for all" and $\wedge$ stands for" and")

Example 2: Ordinary addition and multiplication are operations on *E*, the set of all even natural numbers. It is worth noting that addition is not an operation on *O*, the set of old natural numbers.

Example 3: With obvious modification of the meanings of the symbols, let *E* be any even natural number and *O* be any odd natural number, then

$E \oplus E = E$ (Sum of two even numbers is an even number).

$E \oplus O = O$

and $O \oplus O = E$

$\oplus$	E	O
E	E	O
O	O	E

These results can be beautifully shown in the form of a table given above:

This shows that the set {*E, O*} is closed under (ordinary) addition.

The table may be read (horizontally).

$E \oplus E = E,\quad E \oplus O = O;$

$O \oplus O = E,\quad O \oplus E = O$

Example 4: The set (1 ,–1, *i*, –*i* } where $i = \sqrt{-1}$ is closed w.r.t multiplication (but not *w. r. t* addition). This can be verified from the adjoining table.

$\otimes$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Note: The elements of the set of this example are the **fourth roots** of unity.

Example 5: It can be easily verified that ordinary multiplication (but not addition) is an operation on the set $\{1, \omega, \omega^2\}$ where $\omega^3 = 1$. The adjoining table may be used for the verification of this fact.

(ω is pronounced omega)

$\otimes$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

Operations on Residue Classes Modulo *n*.

Three consecutive natural numbers may be written in the form:

$3n$, $3n + 1$, $3n + 2$ When divided by 3 they give remainders 0, 1, 2 respectively.

Any other number, when divided by 3, will leave one of the above numbers as the reminder. On account of their special importance (in theory of numbers) the remainders like the above are called **residue classes Modulo 3.** Similarly, we can define **Residue classes Modulo 5** etc. An interesting fact about residue classes is that ordinary addition and multiplication are operations on such a class.

Example 6: Give the table for addition of elements of the set of residue classes modulo 5.

Solution: Clearly {0,1,2,3,4} is the set of residues that we have to consider. We add pairs of elements as in ordinary addition except that when the sum equals or exceeds 5, we divide it out by 5 and insert the remainder only in the table. Thus 4 + 3 = 7 but in place of 7 we insert 2(= 7—5) in the table and in place of 2 + 3 = 5 , we insert 0(= 5—5).

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Example 7: Give the table for addition of elements of the set of residue classes modulo 4.

Solution: Clearly {0,1,2,3}is the set of residues that we have to consider. We add pairs of elements as in ordinary addition except that when the sum equals or exceeds 4, we divide it out by 4 and insert the remainder only in the table. Thus 3 + 2 = 5 but in place

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

of 5 we insert 1(= 5−4) in the table and in place of 1 + 3 = 4 , we insert 0(= 4−4).

Example 8: Give the table for multiplication of elemnts of the set of residue classes modulo 4.

Solution: Clearly {0,1,2,3} is the set of residues that we have to consider. We multiply pairs of elements as in ordinary multiplcation except that when the product equals or exceeds 4, we divide it out by 4 and insert the remainder only in the table. Thus 3×2=6 but in place of 6 we insert 2 (= 6−4)in the table and in place of 2×2=4, we insert 0(= 4−4).

⊗	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Example 9: Give the table for multiplication of elements of the set of residue classes modulo 8.

Solution: Table is given below:

⊗	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

Note: For performing multiplication of residue classes 0 is generally omitted.

2.12.1 Properties of Binary Operations

Let S be a non-empty set and $\divideontimes$ a binary operation on it. Then $\divideontimes$ may possess one or more of the following properties: -

i) Commutativity: $\divideontimes$ is said to be commutative if

$$a \divideontimes b = b \divideontimes a \ \forall \ a,b \in S.$$

ii) Associativity: $\divideontimes$ is said to be associative on S if

$$a \divideontimes (b \divideontimes c) = (a \divideontimes b) \divideontimes c \ \forall \ a,b,c \in S.$$

iii) Existence of an identity element: An element $e \in S$ is called an identity element w.r.t $\divideontimes$ if

$$a \divideontimes e = e \divideontimes a = a, \ \forall \ a \in S.$$

iv) Existence of inverse of each element: For any element $a \in S, \exists$ an element $a' \in S$ such that

$$a \divideontimes a' = a' \divideontimes a = e \qquad \text{(the identity element)}$$

Note: (1) The Symbol $\exists$ stands for 'there exists'.

(2) Some authors include closure property in the properties of an operation. Since this property S is already included in the definition of *operation* we have considered it unnecessary to mention it in the above list.

(3) Some authors define *left identity* and *right identity* and also *left inverse* and *right inverse* of each element of a set and prove uniqueness of each of them. The following theorem gives their point of view: -

Theorem:

i) In a set S having a binary operation $\divideontimes$ a left identity and a right identity are the same.

ii) In a set having an associative binary Operation left inverse of an element is equal to its right inverse.

Proof:

ii) Let e' be the left identity and e'' be the right identity. Then

$$e' \divideontimes e'' = e' \qquad (\because e'' \text{ is a right identity})$$
$$= e'' \qquad (\because e' \text{ is a left identity})$$

Hence $e' = e'' = e$

Therefore, e is the unique identity of S under $\divideontimes$

ii) For any $a \in S$, let a', a'' be its left and right inverses respectively then

$$a' \divideontimes (a \divideontimes a'') = a' \divideontimes e \qquad (\because a'' \text{ is right inverse of } a)$$
$$= a' \qquad (\because e \text{ is the identity})$$

Also $(a' \divideontimes a) \divideontimes a'' = e \divideontimes a'' \qquad (\because a' \text{ is left inverse of } a)$

$$= a''$$

But $a' \divideontimes (a \divideontimes a'') = (a' \divideontimes a) \divideontimes a''$ $\divideontimes$ is associative as supposed)

$$\because \ a' = a''$$

Inverse of a is generally written as a^{-1}.

Example 10: Let A =(1,2,3,...., 20}, the set of first 20 natural numbers.
Ordinary addition is not a binary operation on A because the set is not closed w.r.t. addition. For instance, $10 + 25 = 25 \notin A$

Example 11: Addition and multiplication are commutative and associative operations on the sets

$N, Z, Q, \mathbb{R}$, (usual notation),

e.g. $4\times 5 = 5\times 4$, $2 + (3 + 5) = (2 + 3) + 5$ etc.

Example 12: Verify by a few examples that subtraction is not a binary operation on N but it is an operation on Z, the set of integers.

Exercise 2.7

1. Complete the table, indicating by a tick mark those properties which are satisfied by the specified set of numbers.

Set of numbers → / Property ↓		Natural	Whole	Integers	Rational	Reals
Closure	$\oplus$					
	$\otimes$					
Associative	$\oplus$					
	$\otimes$					
Identity	$\oplus$					
	$\otimes$					
Inverse	$\oplus$					
	$\otimes$					
Commutative	$\oplus$					
	$\otimes$					

2. What are the field axioms? In what respect does the field of real numbers differ from that of complex numbers?

3. Show that the adjoining table is that of multiplication of the elements of the set of residue classes modulo 5.

※	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

4. Prepare a table of addition of the elements of the set of residue classes modulo 4.
5. Which of the following binary operations shown in tables (a) and (b) is commutative?

※	a	b	c	d
a	a	c	b	d
b	b	c	b	a
c	c	d	b	c
d	a	a	b	b

(a)

※	a	b	c	d
a	a	c	b	d
b	c	d	b	a
c	b	b	a	c
d	d	a	c	d

(b)

6. Supply the missing elements of the third row of the given table so that the operation ※ may be associative.

※	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	-	-	-	-
d	d	c	c	d

7. What operation is represented by the adjoining table? Name the identity element of the relevant set, if it exists. Is the operation associative? Find the inverses of 0,1,2,3, if they exist.

※	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

2.13 Groups

We have considered, at some length, binary operations and their properties. We now use our knowledge to classify sets according to the properties of operations defined on them.

First we state a few preliminary definitions which will culminate in the definition of a group.

Groupoid: A **groupoid** is a non-empty set on which a binary operation ※ is defined.

Some authors call the system (S, ※) a groupoid. But, for the sake of brevity and convenience we shall call S a **groupoid**, it being understood that an operation ※ is defined on it.

In other words, a closed set with respect to an operation ※ is called a groupoid.

Example 1: The set $\{E, O\}$ where E is any even number and O is any odd number, (as already seen) are closed w.r.t. addition.

It is, therefore, a groupoid.

Example 2: The set of Natural numbers is not closed under operation of subtraction e.g.,

For $4, 5 \in N$, $4 - 5 = -1 \notin N$

Thus (N, –) is not a groupoid under subtraction.

Example 3: As seen earlier with the help of a table the set $\{1, -1, i, -i\}$, is closed w.r.t. . multiplication (but not w.r.t. addition). So it is also a groupoid w.r.t $\times$.

Semi-group: A non-empty set S is semi-group if;

i) It is **closed** with respect to an operation ※ and

ii) The operation ※ is **associative.**

As is obvious from its very name, a semi-group satisfies half of the conditions required for a group.

Example 4: The set of natural numbers, N, together with the operation of addition is a semi-group. N is clearly closed w.r.t. addition (+).

Also $\forall\ a,b,c \in N$, $\quad a + (b + c) = (a + b) + c$

Therefore, both the conditions for a semi-group are satisfied.

Non-commutative or non-abelian set: A set A is non-commutative if commutative law does not hold for it.

For example a set A is non-commutative or non-abilian set under ※ when is defined as:

$\forall\ x, y \in x$ ※ $y = x$.

Clearly x ※ $y = x$ and y ※ $x = y$ indicates that A is non-commutative or non-abilian set.

Example 5: Consider Z, the set of integers together with the operation of multiplication. Product of any two integers is an integer.

Also product of integers is associative because $\forall a,b,c \in Z \quad a.(b.c) = (a.b).c$

Therefore, (Z,.) is a semi-group.

Example 6: Let $P(S)$ be the power-set of S and let $A,B,C,$ ... be the members of P. Since union of any two subsets of S is a subset of S, therefore P is closed with respect to $\cup$. Also the operation is associative.

(e.g. $A \cup (B \cup C) = (A \cup B) \cup C$, which is true in general),

Therefore, $\quad (P(S), \cup)$ is a semi-group.

Similarly $\quad (P(S), \cap)$ is a semi-group.

Example 7: Subtraction is non-commutative and non-associative on N.

Solution: For 4, 5, 6, $\in N$, we see that

$$4-5 = -1 \text{ and } 5-4 = 1$$

Clearly $\quad 4-5 \neq 5-4$

Thus subtraction is non-commutative on N.

Also $5-(4-1) = 5-(3) = 2$ and $(5-4)-1 = 1-1 = 0$

Clearly $5-(4-1) \neq (5-4)-1$

Thus subtraction is non-associative on N.

Example 8: For a set A of distinct elements, the binary operation ※ on A defined by

x ※ $y = x$, $\forall\ x, y \in A$

is non commutative and assocaitve.

Solution : Consider

x ※ $y = x$ and y ※ $x = y$

Clearly x ※ $y \neq y$ ※ x

Thus ※ is non-commutative on A.

Monoid: A semi-group having an identity is called a **monoid** i.e., a monoid is a set S;

i) which is closed w.r.t. some operation ※ .

ii) the operation ※ is associative and

iii) it has an identity.

Example 9: The power-set $P(S)$ of a set S is a **monoid** w.r.t. the operation $\cup$,because, as seen above, it is a semi-group and its identity is the empty-set ϕ because if A is any subset of S,

$\phi \cup A = A \cup \quad = A$

Example 10: The set of all non negative integers i.e., $Z^+ \cup \{0\}$

i) is clearly closed w.r.t. addition,

ii) addition is also associative, and

iii) 0 is the identity of the set.

$(a + 0 = 0 + a = a \qquad \forall\ a \in Z^+ \cup \{0\})$

$\therefore$ the given set is a monoid w.r.t. addition.

Note: It is easy to verify that the given set is a monoid w.r.t. multiplication as well but not w.r.t. subtraction

Example 11: The set of natural numbers, N. w.r.t. $\otimes$

i) the product of any two natural numbers is a natural number;

ii) Product of natural numbers is also associative i.e.,

$\forall\ a, b, c \in N \qquad a.(b.c) = (a.b).c$

iii) $1 \in N$ is the identity of the set.

$\therefore N$ is a monoid w.r.t. multiplication

Note: N is not a monoid w.r.t. addition because it has no identity w.r.t. addition.

Definition of Group: A *monoid* having inverse of each of its elements under ※ is called a group under ※. That is a group under ※ is a set G (say) if

i) G is closed w.r.t. some operation ※

ii) The operation of ※ is associative;

iii) G has an identity element w.r.t. ※ and

iv) Every element of G has an inverse in G w.r.t. ※.

If G satisfies the additional condition:

v) For every $a, b \in G$

a ※ $b = b$ ※ a

then G is said to be an Abelian* or commutative group under ※

Example 12: The set N w.r.t. +

Condition (i) colsure: satisfied i.e., $\forall\ a, b \in N, a + b \in N$

(ii) Associativity: satisfied i.e.,

$\forall\ a,b,c \in N, a + (b + c) = (a + b) + c$

(iii) and (iv) not satisfied i.e., neither identity nor inverse of any element exists.

$\therefore N$ is only a semi-group. Neither *monoid* nor a *group w.r.t.* +.

Example 13: N w.r.t $\otimes$

Condition: (i) Closure: satisfied

$\forall\ a, b \in N, \qquad a, b \in N$

(ii) Associativity: satisfied

$\forall\ a,b,c \in N, \quad a.(b.c) = (a.b).c$

(iii) Identity element, yes, 1 is the identity element

(iv) Inverse of any element of N does not exist in N, so N is a *monoid* but not a group under multiplication.

Example 14: Consider $S = \{0,1,2\}$ upon which operation $\oplus$ has been performed as shown in the following table. Show that S is an abelian group under $\oplus$.

Solution :

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

i) Clearly S as shown under the operation is closed.

ii) The operation is associative e.g

$0 + (1 + 2) = 0 + 0 = 0$

$(0 + 1) + 2 = 1 + 2 = 0$ etc.

iii) Identity element 0 exists.

iv) Inverses of all elements exist, for example

$0 + 0 = 0,\ 1 + 2 = 0,\ 2 + 1 = 0$

$\Rightarrow 0^{-1} = 0 \quad 1^{-1} = 2,\ \ 2^{-1} = 1$

v) Also $\oplus$ is clearly commutative e.g., $1 + 2 = 0 = 2 + 1$

Hence the result,

Example 15: Consider the set $S = \{1,-1,i\ -i\}$. Set up its multiplication table and show that the set is an abelian group under multiplication

Solution :

$\otimes$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

i) S is evidently closed w.r.t.$\otimes$.
ii) Multiplication is also associative
(Recall that multiplication of complex numbers is associative)
iii) Identity element of S is 1.
iv) Inverse of each element exists.
Each of 1 and -1 is self inverse.
i and - i are inverse of each other.
v) $\otimes$ is also commutative as in the case of C, the set of complex numbers. Hence given set is an *Abelian group*.

Example : Let G be the set of all 2×2 non-singular real matrices, then under the usual multiplication of matrices, G is a non-abelian group.
Condition (i) Closure: satisfied; i.e., product of any two 2×2 matrices is again a matrix of order 2×2.
(ii) Associativity: satisfied
For any matrices A , B and C conformable for multiplication.

$$A \times (B \times C) = (A \times B) \times C .$$

So, condition of associativity is satisfied for 2×2 matrices

(iii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an identity matrix.

(iv) As G contains non-singular matrices only so, it contains inverse of each of its elements.
(v) We know that $AB \neq BA$ in general. Particularly for G, $AB \neq BA$.
Thus G is a non-abilian or non-commutative gorup.

Finite and Infinite Gorup: A gorup G is said to be a **finite group** if it contains finite number of elements. Otherwise G is an **infinite group**.
The given examples of groups are clearly distinguishable whether finite or infinite.
Cancellation laws: If a,b,c are elements of a group G, then
i) $ab = ac \Rightarrow b = c$ (Left cancellation Law)
ii) $ba = ca \Rightarrow b = c$ (Right cancellation Law)

Proof: (i) $ab = ac \Rightarrow a^{-1}(ab) = a^{-1}(ac)$
$\Rightarrow (a^{-1}a)b = (a^{-1}a)c$ (by associative law)
$\Rightarrow eb = ec$ $(\therefore a^{-1}a = e)$
$\Rightarrow b = c$
ii) Prove yourselves.

2.14 Solution of linear equations

a,b being elements of a group G, solve the following equations:
i) $ax = b$, ii) $xa = b$

Solution : (i) Given: $ax = b \Rightarrow a^{-1}(ax) = a^{-1}b$

$\Rightarrow (a^{-1}a)\, x = a^{-1}b$ (by associativity)
$\Rightarrow ex = a^{-1}b$
$\Rightarrow x = a^{-1}b$ which is the desired solution.
ii) Solve yourselves.

Note: Since the inverse (left or right) of any element a of a group is unique, from the above procedure, it follows that the above solution is also unique.

2.15 Reversal law of inverses

If a,b are elements of a group G, then show that

$$(ab)^{-1} = b^{-1}a^{-1}$$

Proof: $(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1}$ (Associative law)
$= a\,e\,a^{-1}$
$= aa^{-1}$
$= e$
$\therefore a\,b$ and $b^{-1}a^{-1}$ are inverse of each other.

Note: The rule can obviously be extended to the product of three or more elements of a group.

Theorem: If (G, $\divideontimes$) is a group with e its identity, then e is unique.

Proof: Suppose the contrary that identity is not unique. And let e' be another identity.

e, e' being identities, we have

$e' \divideontimes e = e \divideontimes e' = e'$ (e is an identity) (i)

$e' \divideontimes e = e \divideontimes e' = e$ (e' is an identity) (ii)

Comparing (i) and (ii)

$e' = e$.

Thus the identity of a group is always unique.

Examples:

i) $(Z, +)$ has no identity other then 0 (zero).

ii) $(\mathbb{R} - \{0\}, \times)$ has no identity other than 1.

iii) $(C,+)$ has no identity other than $0 + 0i$.

iv) $(C,.)$ has no identity other than $1 + 0i$.

v) $(M_2,.)$ has no identity other than $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

where M_2 is a set of 2×2 matrices.

Theoram: If $(G, \divideontimes)$ is a group and $a \in G$, there is a unique inverse of a in G.

Proof: Let $(G, \divideontimes)$ be a group and $a \in G$.

Suppose that a' and a'' are two inverses of a in G. Then

$a' = a' \divideontimes e = a' \divideontimes (a \divideontimes a'')$ (a'' is an inverse of a w.r.t. $\divideontimes$)

$= (a' \divideontimes a) \divideontimes a''$ (Associative law in G).

$= e \divideontimes a''$ (a' is an inverse of a).

$= a''$ (e is an identity of G).

Thus inverse of a is unique in G.

Examples 16:

i) in group $(Z, +)$, inverse of 1 is -1 and inverse of 2 is –2 and so on.

ii) in group $(\mathbb{R} - \{0\}, \times)$ inverse of 3 is $\frac{1}{3}$ etc.

Exercise 2.8

1. Operation $\oplus$ performed on the two-member set $G = \{0,1\}$ is shown in the adjoining table. Answer the questions: -

$\oplus$	0	1
0	0	1
1	1	0

i) Name the identity element if it exists?

ii) What is the inverse of 1 ?

iii) Is the set G, under the given operation a group? Abelian or non-Abelian?

2. The operation $\oplus$ as performed on the set $\{0,1,2,3\}$ is shown in the adjoining table, show that the set is an Abelian group?

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

3. For each of the following sets, determine whether or not the set forms a group with respect to the indicated operation.

	Set	Operation
i)	The set of rational numbers	$\times$
ii)	The set of rational numbers	$+$
iii)	The set of positive rational numbers	$\times$
iv)	The set of integers	$+$
v)	The set of integers	$\times$

4. Show that the adjoining table represents the sums of the elements of the set $\{E, O\}$. What is the identity element of this set? Show that this set is an *abelian group*.

$\oplus$	E	O
E	E	O
O	O	E

5. Show that the set $\{1, \omega, \omega^2\}$, when $\omega^3=1$, is an Abelian group w.r.t. ordinary multiplication.

6. If G is a group under the operation and $a, b \in G$, find the solutions of the equations: $a \divideontimes x = b$, $x \divideontimes a = b$

7. Show that the set consisting of elements of the form $a + \sqrt{3}\, b$ (a, b being rational), is an abelian group w.r.t. addition.

8. Determine whether, $(P(S), \divideontimes)$, where $\divideontimes$ stands for intersection is a semi-group, a monoid or neither. If it is a monoid, specify its identity.

9. Complete the following table to obtain a semi-group under ※

※	a	b	c
a	c	a	b
b	a	b	c
c	-	-	a

10. Prove that all 2×2 non-singular matrices over the real field form a non-abelian group under multiplication.