CHAPTER

3

Matrices and Determinants

Animation 3.1: Addition of matrix
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3.1 Introduction

While solving linear systems of equations, a new notation was introduced to reduce the amount of writing. For this new notation the word matrix was first used by the English mathematician James Sylvester (1814 - 1897). Arthur Cayley (1821 - 1895) developed the theory of matrices and used them in the linear transformations. Now-a-days, matrices are used in high speed computers and also in
other various disciplines.

The concept of determinants was used by Chinese and Japanese but the Japanese mathematician Seki Kowa (1642 - 1708) and the German Mathematician Gottfried Wilhelm Leibniz (1646 - 1716) are credited for the invention of determinants. G. Cramer (1704 - 1752) applied the determinants successfully for solving the systems of linear equations.

A rectangular array o f numbers enclosed by a pair o f brackets such as:

$$\begin{bmatrix} 2 & -1 & 3 \\ -5 & 4 & 7 \end{bmatrix} \quad \text{(i)} \quad \text{or} \quad \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 4 \\ 3 & 2 & 6 \\ 4 & 1 & -1 \end{bmatrix} \quad \text{(ii)}$$

is called a **matrix**. The horizontal lines of numbers are called **rows** and the vertical lines of numbers are called **columns**. The numbers used in rows or columns are said to be the **entries** or **elements** of the matrix.

The matrix in (i) has two rows and three columns while the matrix in (ii) has 4 rows and three columns. Note that the number of elements of the matrix in (ii) is $4 \times 3 = 12$. Now we give a general definition of a matrix.

Generally, a bracketed rectangular array of $m \times n$ elements
a_{ij} ($i = 1, 2, 3, ..., m$; $j = 1, 2, 3, ..., n$), arranged in m rows and n columns such as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \quad \text{(iii)}$$

is called an m by n matrix (written as $m \times n$ matrix).

$m \times n$ is called the order of the matrix in (iii). We usually use capital letters such as A, B, C, X, Y, etc., to represent the matrices and small letters such as $a, b, c, ... l, m, n, ..., a_{11}, a_{12}, a_{13}$,...., etc., to indicate the entries of the matrices.

Let the matrix in (iii) be denoted by A. The ith row and the jth column of A are indicated in the following tabular representation of A.

$$A = \begin{array}{r c} & \begin{array}{cccccc} & & & & j\text{th column} & \\ & & & & \downarrow & \end{array} \\ \begin{array}{r} \\ \\ \\ \\ i\text{th row} \rightarrow \\ \\ \\ \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{array} \quad \text{(iv)}$$

The elements of the ith row o f A are a_{i1} a_{i2} a_{i3} ...... a_{ij} ...... a_{in}
while the elements of the jth column of A are a_{1j} a_{2j} a_{3j} .... a_{ij} ...... a_{mj} .
We note that a_{ij} is the element of the ith row and jth column of A. The double subscripts are useful to name the elements of the matrices. For example, the element 7 is at a_{23} position in the matrix $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 4 & 7 \end{bmatrix}$

$A = [a_{ij}]_{m \times n}$ or $A = [a_{ij}]$, for $i = 1, 2, 3, ..., m$; $j = 1, 2, 3, ..., n$, where a_{ij} is the element of the ith row and jth column of A.

Note: a_{ij} is also known as the (i, j)th element or entry of A.

The elements (entries) of matrices need not always be numbers but in the study of matrices, we shall take the elements o f the matrices from $\Re$ or from C.

Note: The matrix A is called real if all of its elements are real.

Row Matrix or Row vector: A matrix, which has only one row, i.e., a $1 \times n$ matrix of the form $[a_{i1}\ \ a_{i2}\ \ a_{i3}\ \ \ldots\ldots\ \ a_{in}]$ is said to be a row matrix or a row vector.

Column Matrix or Column Vector: A matrix which has only one column i.e., an $m \times 1$ matrix of the form $\begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{bmatrix}$ is said to be a column matrix or a column vector.

For example $[1\ \ -1\ \ 3\ \ 4]$ is a row matrix having 4 columns and $\begin{bmatrix} \\ \\ \end{bmatrix}$ is a column matrix having 3 rows.

Rectangular Matrix : If m ≠ n, then the matrix is called a rectangular matrix of order $m \times n$, that is, the matrix in which the number of rows is not equal to the number of columns, is said to be a rectangular matrix.

For example; $\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & -3 & 0 \\ 1 & 2 & 4 \\ 3 & -1 & 5 \\ 0 & 1 & 2 \end{bmatrix}$ are rectangular matrices of orders 2×3 and 4×3 respectively.

Square Matrix : If $m = n$, then the matrix of order $m \times n$ is said to be a square matrix of order n or m. i.e., the matrix which has the same number of rows and columns is called a square matrix. For example;

$[\,0\,]$, $\begin{bmatrix} 2 & 5 \\ -1 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 8 \\ 3 & 5 & 4 \end{bmatrix}$ are square matrices of orders 1, 2 and 3 respectively.

Let $A = [a_{ij}]$ be a square matrix o f order n, then the entries a_{11}, a_{22}, a_{33}, ..., a_{nn} form the principal diagonal for the matrix A and the entries a_{1n}, $a_{2\,n-1}$, $a_{3\,n-2}$, ..., $a_{n-1\,2}$, a_{n1} form the secondary diagonal for the

matrix A. For example, $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$, in the matrix the entries of the principal diagonal are a_{11}, a_{22}, a_{33}, a_{44} and the entries of the secondary diagonal are a_{14}, a_{23}, a_{32}, a_{41}.

The principal diagonal of a square matrix is also called the leading diagonal or main diagonal of the matrix.

Diagonal Matrix: Let $A = [a_{ij}]$ be a square matrix of order n.

If $a_{ij} = 0$ for all $i \neq j$ and at least one $a_{ij} \neq 0$ for $i = j$, that is, some elements of the principal diagonal of A may be zero but not all, then the matrix A is called a diagonal matrix. The matrices

$$[7],\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \text{ are diagonal matrices.}$$

Scalar Matrix: Let $A = [a_{ij}]$ be a square matrix of order n.

If $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = k$ (some non-zero scalar) for all $i = j$, then the matrix A is called a scalar matrix of order n. For example;

$$\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix},\ \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \text{ are scalar matrices of order 2, 3 and 4 respectively.}$$

Unit Matrix or Identity Matrix : Let $A = [a_{ij}]$ be a square matrix of order n. If $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = 1$ for all $i = j$, then the matrix A is called a *unit matrix* or *identity matrix* of order n. We denote such matrix by I_n and it is of the form:

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

The identity matrix of order 3 is denoted by I_3, that is, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Null Matrix or Zero Matrix : A square or rectangular matrix whose each element is zero, is called a null or zero matrix. An m $\times$ n matrix with all its elements equal to zero, is denoted by $O_{m\times n}$. Null matrices may be of any order. Here are some examples:

$$[0], [0 \quad 0 \quad 0], \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

O may be used to denote null matrix of any order if there is no confusion.

Equal Matrices: Two matrices of the same order are said to be equal if their corresponding entries are equal. For example, $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are equal, i.e., $A = B$ iff $a_{ij} = b_{ij}$ for $i = 1, 2, 3, \ldots, m$, $j = 1, 2, 3, \ldots, n$. In other words, *A* and *B* represent the same matrix.

3.1.1 Addition of Matrices

Two matrices are conformable for addition if they are of the same order. The sum $A + B$ of two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is the $m \times n$ matrix $C = [c_{ij}]$ formed by adding the corresponding entries of *A* and *B* together. In symbols, we write as $C = A + B$, that is: $[c_{ij}] = [a_{ij} + b_{ij}]$

where $c_{ij} = a_{ij} + b_{ij}$ for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$.

Note that $a_{ij} + b_{ij}$ is the (i, j)th element of $A + B$.

Transpose of a Matrix:

If *A* is a matrix of order $m \times n$ then an $n \times m$ matrix obtained by interchanging the rows and columns of *A*, is called the transpose of *A*. It is denoted by A^t. If $[a_{ij}]_{m\times n}$ then the transpose of *A* is defined as:

$A^t = [a'_{ij}]_{n\times m}$ where $a'_{ij} = a_{ji}$ .. for $i = 1, 2, 3, \ldots, n$ and $j = 1, 2, 3, \ldots, m$

For example, if $B = [b_{ij}]_{3\times 4} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}$, then

$B^t = [b'_{ij}]_{4\times 3}$ where $b'_{ij} = b_{ji}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ i.e.,

$$B^t = \begin{bmatrix} b'_{11} & b'_{12} & b'_{13} \\ b'_{21} & b'_{22} & b'_{23} \\ b'_{31} & b'_{32} & b'_{33} \\ b'_{41} & b'_{42} & b'_{43} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \\ b_{14} & b_{24} & b_{34} \end{bmatrix}$$

Note that the 2nd row of *B* has the same entries respectively as the 2nd column of B^t and the 3rd row of B^t has the same entries respectively as the 3rd column of *B* etc.

Example 1:

If $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \\ 0 & -2 & 1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 3 & -1 & 4 \\ 3 & 1 & 2 & -1 \end{bmatrix}$, then show that

$(A + B)^t = A^t + B^t$

Solution :

$$A + B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \\ 0 & -2 & 1 & 6 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 3 & -1 & 4 \\ 3 & 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1+2 & 0+(-1) & -1+3 & 2+1 \\ 3+1 & 1+3 & 2+(-1) & 5+4 \\ 0+3 & -2+1 & 1+2 & 6+(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 2 & 3 \\ 4 & 4 & 1 & 9 \\ 3 & -1 & 3 & 5 \end{bmatrix}$$

and $\quad (A+B)^t = \begin{bmatrix} 3 & 4 & 3 \\ -1 & 4 & -1 \\ 2 & 1 & 3 \\ 3 & 9 & 5 \end{bmatrix} \qquad (1)$

Taking transpose of A and B, we have

$$A^t = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ -1 & 2 & 1 \\ 2 & 5 & 6 \end{bmatrix} \text{ and } B^t \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \\ 3 & -1 & 2 \\ 1 & 4 & -1 \end{bmatrix}, \text{ so}$$

$$A^t + B^t = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \\ 2 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \\ 3 & -1 & 2 \\ 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 3 \\ -1 & 4 & -1 \\ 2 & 1 & 3 \\ 3 & 9 & 5 \end{bmatrix} \qquad (2)$$

From (1) and (2), we have $(A+B)^t = A^t + B^t$

3.1.2 Scalar Multiplication

If $A = [a_{ij}]$ is $m \times n$ matrix and k is a scalar, then the product of k and A, denoted by kA, is the matrix formed by multiplying each entry of A by k, that is,

$kA = [ka_{ij}]$

Obviously, order of kA is $m \times n$.

Note. If n is a positive integer, then $A + A + A + \ldots$ to n times $= nA$.

If $A = [a_{ij}] \in M_{m\times n}$ (the set of all $m \times n$ matrices over the real field $\Re$ then $ka_{ij} \in \Re$, for all i and j, which shows that $kA \in M_{m\times n}$. It follows that the set $M_{m\times n}$ possesses the closure property with respect to scalar multiplication. If $A, B \in M$ and r,s are scalars, then we can prove that $r(sA) = (rs)A$, $(r+s)A = rA + sA$, $r(A+B) = rA + rB$.

3.1.3 Subtraction of Matrices

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of order $m \times n$, then we define subtraction of B from A as:

$A - B = A + (-B)$

$= [a_{ij}] + [-b_{ij}] = [a_{ij} - b_{ij}]$ for $i = 1, 2, 3, \ldots, m$; $j = 1, 2, 3, \ldots, n$

Thus the matrix $A - B$ is formed by subtracting each entry of B from the corresponding entry of A.

3.1.4 Multiplication of two Matrices

Two matrices A and B are said to be conformable for the product AB if the number of columns of A is equal to the number of rows of B.

Let $A = [a_{ij}]$ be a 2×3 matrix and $B = [b_{ij}]$ be a 3×2 matrix. Then the product AB is defined to be the 2×2 matrix C whose element c_{ij} is the sum of products of the corresponding elements of the ith row of A with elements of jth column of B. The element c_{21} of $\boldsymbol{C}$ is shown in the figure (A), that is

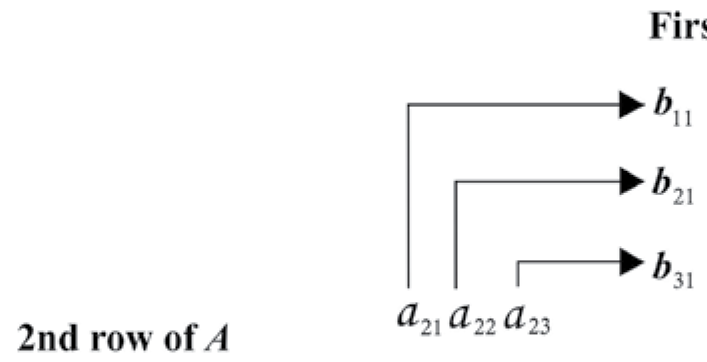


Fig.(A)

$c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$. Thus

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$$

Similarly $\quad BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

$$= \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} & b_{11}a_{13} + b_{12}a_{23} \\ +b_{21}a_{11} \quad b_{22}a_{21} & +b_{21}a_{12} \quad b_{22}a_{22} & +b_{21}a_{13} \quad b_{22}a_{23} \\ b_{31}a_{11} + b_{32}a_{21} & b_{31}a_{12} + b_{32}a_{22} & b_{31}a_{13} + b_{32}a_{23} \end{bmatrix}$$

AB and BA are defined and their orders are 2×2 and 3×3 respectively.

Note 1. Both products AB and BA are defined but $AB \neq BA$

2. If the product AB is defined, then the order of the product can be illustrated as given below:

Order of A $\quad m\times n$

Order of B $\quad n\times p$

Order of AB $\quad m\times p$

Example 2: If $A=\begin{bmatrix}2 & -1 & 0\\ 1 & 2 & -3\\ 1 & 2 & -2\end{bmatrix}$ and $B=\begin{bmatrix}2 & -2 & 3\\ -1 & -4 & 6\\ 0 & -5 & 5\end{bmatrix}$, then compute A^2B.

Solution :

$$A^2 = AA \begin{bmatrix}2 & -1 & 0\\ 1 & 2 & 3\\ 1 & 2 & -2\end{bmatrix}\begin{bmatrix}2 & -1 & 0\\ 1 & 2 & 3\\ 1 & 2 & -2\end{bmatrix}$$

$$=\begin{bmatrix}4-1+0 & -2-2+0 & 0+3+0\\ 2+2-3 & -1+4-6 & 0-6+6\\ 2+2-2 & -1+4-4 & 0-6+4\end{bmatrix}=\begin{bmatrix}3 & -4 & 3\\ 1 & -3 & 0\\ 2 & -1 & -2\end{bmatrix}$$

$$\therefore A^2B=\begin{bmatrix}3 & -4 & 3\\ 1 & -3 & 0\\ 2 & -1 & -2\end{bmatrix}\begin{bmatrix}2 & -2 & 3\\ -1 & -4 & 6\\ 0 & -5 & 5\end{bmatrix}$$

$$=\begin{bmatrix}6+4+0 & -6+16-15 & 9-24+15\\ 2+3+0 & -2+12+0 & 3-18+0\\ 4+1+0 & -4+4+10 & 6-6-10\end{bmatrix}=\begin{bmatrix}10 & -5 & 0\\ 5 & 10 & -15\\ 5 & 10 & -10\end{bmatrix}$$

Note: Powers of square matrices are defined as:

$A^2 = A\times A,\ A^3 = A\times A\times A,$

$A^n = A\times A\times A\times \ldots$ to n factors.

3.2 Determinant of a 2×2 matrix

We can associate with every square matrix A over $\Re$ or C, a number $|A|$, known as the determinant of the matrix A.

The determinant of a matrix is denoted by enclosing its square array between vertical bars instead of brackets. The number of elements in any row or column is called the order of determinant. For example,

if $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, then the determinant of A is $\begin{vmatrix}a & b\\ c & d\end{vmatrix}$. Its value is defined to be the real number $ad - bc$, that is,

$$|A|=\begin{vmatrix}a & b\\ c & d\end{vmatrix}=ad-bc$$

For example, if $A=\begin{bmatrix}2 & -1\\ 4 & 3\end{bmatrix}$ and $B\begin{bmatrix}1 & 4\\ 2 & 8\end{bmatrix}$, then

$$|A|=\begin{vmatrix}2 & -1\\ 4 & 3\end{vmatrix} = (2)(3)-(-1)(4) = 6+4 = 10$$

and $$|B|=\begin{vmatrix}1 & 4\\ 2 & 8\end{vmatrix} = (1)(8)-(4)(2) = 8-8 = 0$$

Hence the determinant of a matrix is the difference of the products of the entries in the two diagonals.

$$\begin{vmatrix}a & b\\ c & d\end{vmatrix} = ad - bc$$

$-bc \quad ad$

Note: The determinant of a 1×1 matrix $[a_{11}]$ is defined as $|a_{11}| = a_{11}$

3.2.1 Singular and Non-Singular Matrices

A square matrix A is **singular** if $|A| = 0$, otherwise it is a **non singular** matrix. In the above example, $|B| = 0 \Rightarrow \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$ is a singular matrix and $|A| = 10 \neq 0 \Rightarrow A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ is a non-singular matrix.

3.2.2 Adjoint of a 2 × 2 Matrix

The adjoint of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by *adj A* and is defined as: *adj A* = $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

3.2.3 Inverse of a 2 × 2 Matrix

Let A be a non-singualr square matrix of **order 2**. If there exists a matrix B such that

$AB = BA = I_2$ where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then B is called the

multiplicative inverse of A and is usually denoted by A^{-1}, that is, $B = A^{-1}$

$$\boxed{\text{Thus } AA^{-1} = A^{-1}A = I_2}$$

Example 3: For a non-singular matrix A, prove that $A = \frac{1}{|A|} adjA$

Solution : If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, Then:

$AA^{-1} = I_2$, that is,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or $\begin{bmatrix} ap+br & ap+bs \\ cp+dr & cq+ds \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{cases} ap+br = 1....(i) = & ap \quad bs \quad 0....(ii) \\ cp+dr = 0....(iii) = & cq \quad ds \quad 1....(iv) \end{cases}$$

From (iii), $r = \frac{-c}{d}p$

Putting the value of r in (i), we have

$$ap + b\left(\frac{-c}{d}p\right) = 1 \Rightarrow \left(\frac{ad-bc}{d}\right)p = 1 \Rightarrow p = \frac{d}{ad-bc}$$

and $r = \frac{-c}{d}p = \frac{-c}{d}.\frac{d}{ad-bc} = -\frac{c}{ad-bc}$

Similarly, solving (ii) and (iv), we get

$$q = \frac{-b}{ad-bc}, \quad s = \frac{a}{ad-bc}$$

Substituting these values in $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$, we have

$$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thus $A^{-1} = \frac{1}{|A|} AdjA$

Example 4: Find A^{-1} if $A=\begin{bmatrix}5 & 3\\ 1 & 1\end{bmatrix}$ and verify that $AA^{-1} = A^{-1}A$

Solution : $|A|=\begin{vmatrix}5 & 3\\ 1 & 1\end{vmatrix} = 5 - 3 = 2$

Since $|A| \neq 0$, we can find A^{-1}.

$$A^{-1}=\frac{1}{|A|}AdjA$$

$$\Rightarrow \quad A^{-1}=\frac{1}{2}\begin{bmatrix}1 & -3\\ -1 & 5\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & \frac{-3}{2}\\ \frac{-1}{2} & \frac{5}{2}\end{bmatrix}$$

Now $$A.A^{-1}=\begin{bmatrix}5 & 3\\ 1 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{-3}{2}\\ \frac{-1}{2} & \frac{5}{2}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{5}{2}-\frac{3}{2} & \frac{-15}{2}+\frac{15}{2}\\ \frac{1}{2}-\frac{1}{2} & \frac{-3}{2}+\frac{5}{2}\end{bmatrix} \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad \text{(i)}$$

and $$A^{-1}.A=\begin{bmatrix}\frac{1}{2} & \frac{-3}{2}\\ \frac{-1}{2} & \frac{5}{2}\end{bmatrix}\begin{bmatrix}5 & 3\\ 1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}\frac{5}{2}-\frac{3}{2} & \frac{3}{2}-\frac{3}{2}\\ \frac{-5}{2}+\frac{5}{2} & \frac{-3}{2}+\frac{5}{2}\end{bmatrix} \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad \text{(ii)}$$

From (i) and (ii), we have

$$AA^{-1} = A^{-1}A$$

3.3 Solution of simultaneous linear equations by using matrices

Let the system of linear equations be

$$\left.\begin{aligned}a_{11}x_1+a_{12}x_2&=b_1\\ a_{21}x_1+a_{22}x_2&=b_2\end{aligned}\right\} \qquad \text{(i)}$$

where a_{11}, a_{12}, a_{21}, a_{22}, b_1 and b_1 are real numbers.
The system (i) can be written in the matrix form as:

$$\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\end{bmatrix} \quad \text{or} \quad AX=B \qquad \text{(ii)}$$

$$\text{where} \quad A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}, X=\begin{bmatrix}x_1\\ x_2\end{bmatrix} B=\begin{bmatrix}b_1\\ b_2\end{bmatrix}$$

If $|A| \neq 0$, then A^{-1} exists so (ii) gives

$A^{-1}(AX) = A^{-1}B$ (By pre-multiplying (ii) by A^{-1})

or $(A^{-1}A)X = A^{-1}B$ (Matrix multiplication is associative)

$\Rightarrow$ $X = A^{-1}B$ ($\because$ $A^{-1}A = I_2$)

$\Rightarrow$ $X = A^{-1}B$

$$\text{or} \quad \begin{bmatrix}x_1\\ x_2\end{bmatrix}=\frac{1}{|A|}\begin{bmatrix}a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix}$$

$$=\frac{1}{|A|}\begin{bmatrix}a_{22}b_1 - a_{12}b_2\\ -a_{21}b_1 - a_{11}b_2\end{bmatrix} \quad \begin{bmatrix}\frac{b_1a_{22}-a_{12}b_2}{|A|}\\ \frac{a_{11}b_2-b_1a_{21}}{|A|}\end{bmatrix}$$

$$\text{Thus} \quad =\frac{\begin{vmatrix}b_1 & a_{12}\\ b_2 & a_{22}\end{vmatrix}}{|A|} \text{ and } x_2 \frac{\begin{vmatrix}a_{11} & b_1\\ b_{21} & b_2\end{vmatrix}}{|A|}$$

It follows from the above discussion that the system of linear equations such as (i) has a unique solution if $|A| \neq 0$.

Example 5: Solve the following systems of linear equations.

i) $\left.\begin{array}{l} 3x_1 - x_2 = 1 \\ x_1 + x_2 = 3 \end{array}\right\} \begin{array}{l} + \quad = \\ + \quad = \end{array}$ ii) $\left.\begin{array}{lll} x_1 & 2x_2 & 4 \\ 2x_1 & 4x_2 & 12 \end{array}\right\}$

Solution : (i) The matrix form of the system $\left.\begin{array}{l} 3x_1 - x_2 = 1 \\ x_1 + x_2 = 3 \end{array}\right\}$ is

$$\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

or $AX = B$ . . . . (i) where $A == \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$, X $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and B $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix} = 3 + 1 = 4$$

and adj $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$, therefore,

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

(I) becomes $X = A^{-1}B$, that is,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & +\frac{3}{4} \\ -\frac{1}{4} & +\frac{9}{4} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\Rightarrow$ $x_1 = 1$ and $x_2 = 2$

(ii) The matrix form of the system $\left.\begin{array}{l} x_1 + 2x_2 = 4 \\ 2x_1 + 4x_2 = 12 \end{array}\right\}$ is

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$$

and $|A| = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$ = 4 – 4 = 0, so A^{-1} does not exist.

Multiplying the first equation o f the above system by 2, we have

$2x_1 + 4x_2 = 8$ but $2x_1 + 4x_2 = 12$

which is impossible. Thus the system has no solution.

Exercise 3.1

1. If $A == \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ and B $\begin{bmatrix} 1 & 7 \\ 6 & 4 \end{bmatrix}$, then show that

i) $4A - 3A = A$ ii) $3B - 3A = 3(B - A)$

2. If $A = \begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix}$, show that $A^4 = I_2$.

3. Find x and y if

$i)$ $\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix} == \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix}$ $\begin{bmatrix} y & 1 \\ -3 & 2x \end{bmatrix}$

4. If $A == \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ and B $\begin{bmatrix} 0 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix}$, find the following matrices;

i) $4A - 3B = A$ ii) $A + 3(B - A)$

5. Find x and y If $\begin{bmatrix} 2 & 0 & x \\ 1 & y & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & x & y \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$

6. If $A = [a_{ij}]_{3\times 3}$, find the following matrices;

i) $\lambda(\mu A)=(\lambda\mu)A$ ii) $(\lambda+\mu)A=\lambda A+\mu A$ ii) $\lambda A-A=(\lambda-1)A$

7. If $A = [a_{ij}]_{2\times 3}$ and $B = [b_{ij}]_{2\times 3}$, show that $\lambda(A+B)=\lambda A+\lambda B$.

8. If $A=\begin{bmatrix}1 & 2\\ a & b\end{bmatrix}$ and $A^2 \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$, find the values of a and b.

9. If $A=\begin{bmatrix}1 & -1\\ a & b\end{bmatrix}$ and $A^2 \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, find the values of a and b.

10. If $A=\begin{bmatrix}1 & -1 & 2\\ 0 & 3 & 1\end{bmatrix}$ and $B \begin{bmatrix}2 & 3 & 0\\ 1 & 2 & -1\end{bmatrix}$, then show that $(A + B)^t = A^t + B^t$.

11. Find A^3 if $A=\begin{bmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{bmatrix}$

12. Find the matrix X if;.

i) $X\begin{bmatrix}5 & 2\\ -2 & 1\end{bmatrix}\begin{bmatrix}-1 & 5\\ 12 & 3\end{bmatrix}$ ii) $\begin{bmatrix}5 & 2\\ -2 & 1\end{bmatrix}X=\begin{bmatrix}2 & 1\\ 5 & 10\end{bmatrix}$

13. Find the matrix A if,

i) $\begin{bmatrix}5 & -1\\ 0 & 0\\ 3 & 1\end{bmatrix}A=\begin{bmatrix}3 & -7\\ 0 & 0\\ 7 & 2\end{bmatrix}$ ii) $\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}A=\begin{bmatrix}0 & -3 & 8\\ 3 & 3 & -7\end{bmatrix}$

14. Show that $\begin{bmatrix}r\cos\phi & 0 & -\sin\phi\\ 0 & r & 0\\ r\sin\phi & 0 & \cos\phi\end{bmatrix}\begin{bmatrix}\cos\phi & 0 & \sin\phi\\ 0 & 1 & 0\\ -r\sin\phi & 0 & r\cos\phi\end{bmatrix}=rI_3$.

3.4 Field

A set F is called a **field** if the operations of addition '+ ' and multiplication '. ' on F satisfy the following properties written in tabular form as:

Addition		Multiplication
	Closure	
i) For any $a,b \in F$, $a + b \in F$		For any $a,b \in F$, $a.b \in F$
	Commutativity	
ii) For any $a,b \in F$, $a + b = b + a$		For any $a,b \in F$, $a.b = b.a$
	Associativity	
iii) For any $a,b,c \in F$, $(a + b) + c = a + (b + c)$		For any $a,b,c \in F$, $(a.b).c = a.(b.c)$
	Existence of Identity	
iv) For any $a \in F, \exists\ 0 \in F$ such that $a + 0 = 0 + a = a$		For any $a \in F, \exists\ 1 \in F$ such that $a.1 = 1.a = a$
	Existence of Inverses	
v) For any $a \in F, \exists -a \in F$ such that $a + (-a) = (-a) + a = 0$		v) For any $a \in F$, $a \neq 0$ $\exists \frac{1}{a} \in F$ such that $a.(\frac{1}{a})=(\frac{1}{a}).a=1$
	Distributivity	
vi) For any $a,b,c \in F$,	D_1 : $a(b + c) = ab + ac$ D_2 : $(b + c)a = ba + ca$	

All the above mentioned properties hold for Q, $\Re$, and C.

3.5 Properties of Matrix Addition , Scalar Multiplication and Matrix Multiplication.

If A, B and C are $n\times n$ matrices and c and d are scalers, the following properties are true:

1. **Commutative property w.r.t. addition:** $A + B = B + A$

Note: w.r.t. is used for "with respect to".

2. **Associative property w.r.t. addition:** $(A + B) + C - A + (B + C)$
3. **Associative property of scalar multiplication:** $(cd)A = c(dA)$
4. **Existance of additive identity:** $A + O = O + A - A$ (O is null matrix)
5. **Existance of multiplicative identity:** $IA = AI = A$ (I is unit/identity matrix)
6. **Distributive property w.r.t scalar multiplication:**
   (a) $c(A + B) = cA + cB$ **(b)** $(c + d)A = cA + dA$
7. **Associative property w.r.t. multiplication:** $A(BC) = (AB)C$
8. **Left distributive property:** $A(B + C) = AB + AC$
9. **Right distributive property:** $(A + B)C = AC + BC$
10. $c(AB) = (cA)B = A(cB)$

Example 1: Find AB and BA if $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 4 & 2 \\ 3 & 0 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \\ 1 & -2 & 3 \end{bmatrix}$

Solution : $AB = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 4 & 2 \\ 3 & 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \\ 1 & -2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 2\times1+0\times2+1\times1 & 2\times(-1)+0\times3+1\times(-2) & 2\times0+0\times(-1)+1\times3 \\ 1\times1+4\times2+2\times1 & 1\times(-1)+4\times3+2\times(-2) & 1\times0+4\times(-1)+2\times3 \\ 3\times1+0\times2+6\times1 & 3\times(-1)+0\times3+6\times(-2) & 3\times0+0\times(-1)+6\times3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 & 3 \\ 11 & 7 & 2 \\ 9 & -15 & 18 \end{bmatrix} \qquad (1)$$

$$BA = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 1 & 4 & 2 \\ 3 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times2+(-1)\times1+0\times3 & 1\times0+(-1)\times4+0\times0 & 1\times1+(-1)\times2+0\times6 \\ 2\times2+3\times1+(-1)\times3 & 2\times0+3\times4+(-1)\times0 & 2\times1+3\times2+(-1)\times6 \\ 1\times2+(-2)\times1+3\times3 & 1\times0+(-2)\times4+3\times0 & 1\times1+(-2)\times2+3\times6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 & -1 \\ 4 & 12 & 2 \\ 9 & -8 & 15 \end{bmatrix} \qquad (2)$$

Thus from (1) and (2), $AB \neq BA$

Note: Matrix multiplication is not commutative in general

Example 2: If $= \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & 2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$,then find AA^t and (A^t).

Solution : Taking transpose of A, we have

$$A^t = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}, \text{ so}$$

$$AA^t = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & 2 \\ -3 & 5 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+9+0 & 2+0+12+0 & -6-5+6+0 \\ 2+0+12+0 & 1+0+16+4 & -3+0+8+2 \\ -6-5+6+0 & -3+0+8+2 & 9+25+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 14 & -5 \\ 14 & 21 & 7 \\ -5 & 7 & 39 \end{bmatrix}$$

As $A^t = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}$, so $\left(A^t\right)^t \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & 2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$ which is A,

That is, $(A^t)^t = A$. (Note that this rule holds for any matrix A.)

Exercise 3.2

1. If $A = [a_{ij}]_{3\times 4}$, then show that

i) $I_3A = A$ ii) $AI_4 = A$

2. Find the inverses of the following matrices.

i) $\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ ii) $\begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix}$ iii) $\begin{bmatrix} 2i & i \\ i & -i \end{bmatrix}$ iv) $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$

3. Solve the following system of linear equations.

i) $\left.\begin{array}{l} 2x_1 - 3x_2 = 5 \\ 5x_1 + x_2 = 4 \end{array}\right\}$ ii) $\left.\begin{array}{l} 4x_1 + 3x_2 = 5 \\ 3x_1 - x_2 = 7 \end{array}\right\}$ iii) $\left.\begin{array}{l} 3x_1 - 5y = 1 \\ -2x + y = -3 \end{array}\right\}$

4. If $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -2 & 5 \\ -1 & 0 & 4 \end{bmatrix}$, $B \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 & -2 \\ 1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}$, then find

i) $A - B$ ii) $B - A$ iii) $(A - B) - C$ iv) $A - (B - C)$

5. If $A = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}$, $B = \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}$ and $C = \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$, then show that

i) $(AB)C = A(BC)$ ii) $(A + B)C = AC + BC$

6. If A and B are square matrices of the same order, then explain why in general;

i) $(A + B)^2 \neq A^2 + 2AB + B^2$ ii) $(A - B)^2 \neq A^2 - 2AB + B^2$

iii) $(A + B)(A - B) \neq A^2 - B^2$

7. If $A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & 2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$ then find AA^t and A^tA.

8. Solve the following matrix equations for X:

i) $3X - 2A = B$ if $A = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 1 & 5 \end{bmatrix}$ and $B \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix}$

ii) $2X - 3A = B$ if $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 5 \end{bmatrix}$ and $B \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$

9. Solve the following matrix equations for A:

(i) $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A - \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 3 & 6 \end{bmatrix}$ (ii) $A \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix}$

3.6 Determinants

The determinants of square matrices of order $n \geq 3$, can be written by following the same pattern as already discussed. For example, if $n = 4$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \text{ then the determinat of } A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

Now our aim is to compute the determinants of various orders. But before describing a method for computation o f determinants of order $n \geq 3$, we introduce the following definitions.

3.6.1 Minor and Cofactor of an Element of a Matrix or its Determinant

Minor of an Element: Let us consider a square matrix A of order 3 .Then the minor of an element a_{ij}, denoted by .M_{ij} is the determinant of the $(3-1)\times(3-1)$ matrix formed by deleting the ith row and the jth column of A(or|A|).

For example, if

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix},$$ then the matrix obtained by deleting the first row and the second column of A is $\begin{bmatrix} a_{21} & a_{23}\\ a_{31} & a_{33}\end{bmatrix}$ (see adjoining figure) $\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$ and its determinant is the minor of an, that is,

$$M_{12}=\begin{vmatrix} a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix}$$

Cofactor of an Element: The cofactor of an element a_{ij} denoted by A_{ij} is defined by $A_{ij}=(-1)^{i+j}\times M_{ij}$

where M_{ij} is the minor of the element a_{ij} of A or $|A|$.

For example, $A_{12}=(-1)^{1+2}M_{12}=(-1)^{3}\begin{vmatrix} a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix}=-\begin{vmatrix} a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix}$

3.6.2 Determinant of a Square Matrix of Order $n\geq 3$:

The determinant of a square matrix of order n is the sum of the products of each elem ent of row (or column) and its cofactor.

If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n}\\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n}\\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n}\\ \vdots & \vdots & \vdots & & \vdots & & \vdots\\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in}\\ \vdots & \vdots & \vdots & & \vdots & & \vdots\\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nj} & \cdots & a_{nn}\end{bmatrix}$, then

$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} + \dots + a_{in}A_{in}$ for $i = 1, 2, 3,\dots, n$

or $|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j} + \dots + a_{nj}A_{nj}$ for $j = 1, 2, 3,\dots, n$

Putting $i = 1$, we have

$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + \dots + a_{1n}A_{1n}$ which is called the expansion of |A|by the first row.

If A is a matrix of order 3, that is, $A=\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$, then:

$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} + \dots + a_{in}A_{in}$ for $i = 1, 2, 3$ (1)

or $|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j} + \dots + a_{nj}A_{nj}$ for $j = 1, 2, 3$ (2)

For example, for $i = 1$, $j = 1$ and $j = 2$, we have

$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$ (i)

or $|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$ (ii)

or $|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$ (iii)

(iii) can be written as: $|A| = a_{12}(-1)^{1+2}M_{12} + a_{22}(-1)^{2+2}M_{22} + a_{32}(-1)^{3+2}M_{32}$

i.e., $|A| = -a_{12}M_{12} + a_{22}M_{22} - a_{32}M_{32}$ (iv)

Similarly (i) can be written as $|A| = a_{11}M_{11} - a_{12}M_{12} - a_{13}M_{13}$ (v)

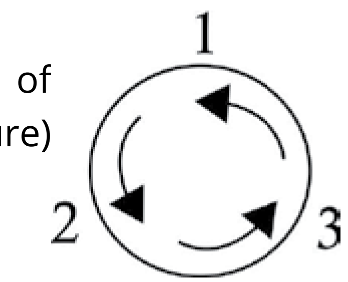
Putting the values of $M_{11}.M_{12}$: and M_{13} in (iv). we obtain

$$|A|=a_{11}\begin{vmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}$$

or $|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$ (vi)

or $|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$ (vi)'

The second scripts of positive terms are in circular order of anti-clockwise direction i.e., these are as 123, 231, 312 (adjoining figure) while the second scripts of negative terms are such as 132, 213, 321.



An alternative way to remember the expansion of the determinant $|A|$ given in (vi)' is shown in the figure below.

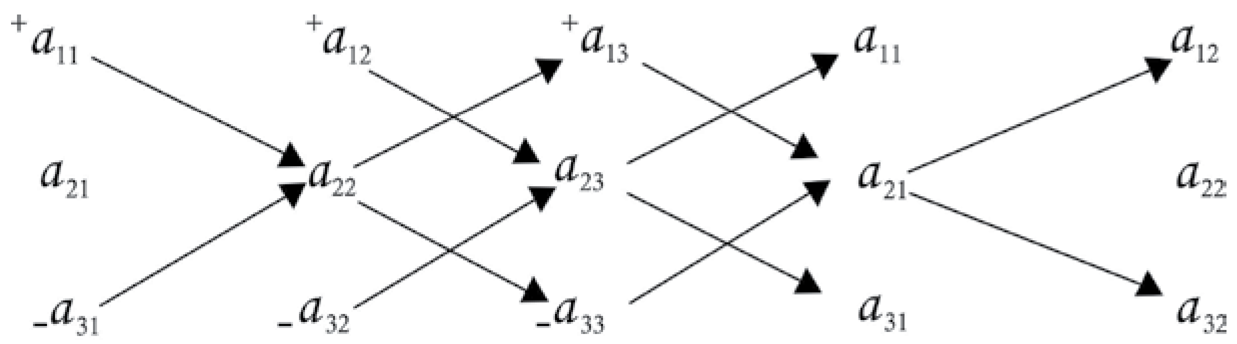


Fig.

Example 1: Evaluate the determinant of $A=\begin{bmatrix}1 & -2 & 3\\ -2 & 3 & 1\\ 4 & -3 & 2\end{bmatrix}$

Solution : $|A|=\begin{vmatrix}1 & -2 & 3\\ -2 & 3 & 1\\ 4 & -3 & 2\end{vmatrix}$

Using the result (v) of the Art.3.6.2, that is,

$|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$, we get,

$$|A|=1\begin{vmatrix}3 & 1\\ -3 & 2\end{vmatrix}-(-2)\begin{vmatrix}-2 & 1\\ 4 & 2\end{vmatrix}+3\begin{vmatrix}-2 & 3\\ 4 & -3\end{vmatrix}$$

$= 1[6 - 1(-3)] + 2[(-2).2 - 1.4] + 3[(-2)(-3) - 12]$

$= (6 + 3) + 2(-4 - 4) + 3(6 - 12)$

$= 9 - 16 - 18 = -25$

Example 2: Find the cofactors A_{12}, A_{22} and A_{32} if $A=\begin{bmatrix}1 & -2 & 3\\ -2 & 3 & 1\\ 4 & -3 & 2\end{bmatrix}$ and find $|A|$.

Solution : We first find M_{12}, M_{22} and M_{32},

$$M_{12}=\begin{vmatrix}-2 & 1\\ 4 & 2\end{vmatrix}=-4-4=-8;\quad M_{22}=\begin{vmatrix}1 & 3\\ 4 & 2\end{vmatrix}=2-12=-10 \text{ and } M_{32}=\begin{vmatrix}1 & 3\\ -2 & 1\end{vmatrix}=1-(-6)=7$$

Thus $A_{12} = (-1)^{1+2}M_{12} = (-1)(-8) = 8$; $A_{22} = (-1)^{2+2}M_{22} = 1(-10) = -10$

$A_{32} = (-1)^{3+2}M_{32} = (-1)(7) = -7$;

and $|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} = (-2)8 + 3(-10) + (-3)(-7)$

$= -16 - 30 + 21 = -25$

Note that $a_{11}A_{12} + a_{21}A_{22} + a_{31}A_{32} = 1(8) + (-2)(-10) + 4(-7)$

$= 8 + 20 - 28 = 0$

and $a_{13}A_{12} + a_{23}A_{22} + a_{33}A_{32} = 3(8) + 1(-10) + 2(-7)$

$= 24 - 10 - 14 = 0$

Similarly we can show that $a_{11}A_{13} + a_{21}A_{23} + a_{31}A_{33} = 0$;

$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$; and $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$;

3.7 Properties of Determinants which Help in their Evaluation

1. For a square matrix A, $|A| = |A^t|$
2. If in a square matrix A, two rows or two columns are interchanged, the determinant of the resulting matrix is $-|A|$.
3. If a square matrix A has two identical rows or two identical columns, then $|A| = 0$.
4. If all the entries of a row (or a column) of a square matrix A are zero, then $|A| = 0$.
5. If the entries of a row (or a column) in a square matrix A are multiplied by a number $k \in \Re$, then the determinant of the resulting matrix is $k\,|A|$.
6. If each entry of a row (or a column) of a square matrix consists of two terms, then its determinant can be written as the sum of two determinants, i.e., if

$$B=\begin{bmatrix}a_{11}+b_{11} & a_{12} & a_{13}\\ a_{21}+b_{21} & a_{22} & a_{23}\\ a_{31}+b_{31} & a_{32} & a_{33}\end{bmatrix},$$ then

$$|B| = \begin{vmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix}$$

7. If to each entry of a row (or a column) of a square matrix A is added a non-zero multiple of the corresponding entry of another row (or column), then the determinant of the resulting matrix is $|A|$.
8. If a matrix is in triangular form, then the value of its determinant is the product of the entries on its main diagonal.

Now we prove the above mentioned properties of determinants.

Proporty 1: If the rows and columns of a determinant are interchanged, then the value of the determinant does not change. For example.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = a_{11}a_{22} - a_{21}a_{12} = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} \text{ (rows and columns are interchanged)}$$

Property 2: The value of a determinant changes sign if any two rows (columns) are interchanged. For example,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

and $\begin{vmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{vmatrix} = a_{12}a_{21} - a_{11}a_{22} = -(a_{11}a_{22} - a_{12}a_{21})$ (columns are interchanged)

Property 3: If all the entries in any row (column) are zero, the value of the determinant is zero. For example,

$$\begin{vmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} = 0\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - 0\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + 0\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = 0 \quad \text{(expanding by C)}$$

Property 4: If any two rows (columns) of a determinant are identical, the value of the determinant is zero. For example,

$$\begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} = 0, \quad \text{(it can be proved by expanding the determinant)}$$

Property 5: If any row (column) of a determinant is multiplied by a non-zero number k, the value of the new determinant becomes equal to k times the value of original determinant. For example,

$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, multiplying first row by a non-zero number k, we get

$$\begin{vmatrix} ka_{11} & ka_{12} \\ a_{21} & a_{22} \end{vmatrix} = ka_{11}a_{22} - ka_{12}a_{21} = k(a_{11}a_{22} - a_{12}a_{21}) = k\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Property 6: If any row (column) of a determinant consists of two terms, it can be written as the sum of two determinants as given below:

$$\begin{vmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{(proof is left for the reader)}$$

Property 7: If any row (column) o f a determinant is multiplied by a non-zero number k and the result is added to the corresponding entries of another row (column), the value of the determinant does not change. For example,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12}+ka_{11} \\ a_{21} & a_{22}+ka_{21} \end{vmatrix} \quad (k \text{ multiple of } C_1 \text{ is added to } C_2)$$

It can be proved by expanding both the sides. Proof is left for the reader.

Example 3: If $A = \begin{bmatrix} 2 & -2 & 3 & 4 \\ 3 & 1 & 5 & -1 \\ -5 & -3 & 1 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix}$, evaluate $|A|$

Solution :

$$|A| = \begin{vmatrix} 2 & -2 & 3 & 4 \\ 3 & 1 & 5 & -1 \\ -5 & -3 & 1 & 0 \\ 1 & -1 & 0 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 3 & 0 \\ 0 & 4 & 5 & -7 \\ 0 & -8 & 1 & 10 \\ 1 & -1 & 0 & 2 \end{vmatrix} \quad \text{By } R_1 + (-2)R_4,\ R_2 + (-3)R_4 \text{ and } R_3 + 5R_4$$

Expanding by first column, we have

$|A| = 0.A_{11} + 0.A_{21} + 0.A_{31} + 1.A_{41}$

$$= (-1)^{4+1} \times \begin{vmatrix} 0 & 3 & 0 \\ 4 & 5 & -7 \\ -8 & 1 & 10 \end{vmatrix} = (-1)\begin{vmatrix} 0 & 3 & 0 \\ 4 & 5 & -7 \\ -8 & 1 & 10 \end{vmatrix}$$

$= (-1)(-3)[4 \times 10 - (-7)(-8)] = 3(40 - 56) = -48$

Example 4: Without expansion, show that $\begin{vmatrix} x & a+x & b+c \\ x & b+x & c+a \\ x & c+x & a+b \end{vmatrix} = 0$

Solution : Multiplying each entry of C_1 by -1 and adding to the corresponding entry of C_2 i.e., by $C_2 + (-1)C_1$, we get

$$\begin{vmatrix} x & a+x & b+c \\ x & b+x & c+a \\ x & c+x & a+b \end{vmatrix} = \begin{vmatrix} x & a+x+(-1)x & b+c \\ x & b+x+(-1)x & c+a \\ x & c+x+(-1)x & a+b \end{vmatrix}$$

$$= \begin{vmatrix} x & a & b+c \\ x & b & c+a \\ x & c & a+b \end{vmatrix} = x\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} \left(\begin{array}{c}\text{by property 5 or} \\ \text{taking x common} \\ \text{from } C_1\end{array}\right)$$

$$= x\begin{vmatrix} 1 & a+(b+c) & b+c \\ 1 & b+(c+a) & c+a \\ 1 & c+(a+b) & a+b \end{vmatrix}, \left(\begin{array}{c}\text{adding the entries of } C_3 \text{ to the} \\ \text{corresponding entries of } C_2\end{array}\right)$$

$$= x(a+b+c)\begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix}, \text{ (by property 5)}$$

$= x(a + b + c).0$ ($\because$ C_1 and C_2 are identical or by property 3)

$= 0$

Example 5: Solve the equation $\begin{vmatrix} x & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & -2 & 3 & 4 \\ -2 & x & 1 & -1 \end{vmatrix} = 0$

Solution: By $C_3 + C_2$ and $C_4 + C_2$, we have

$$\begin{vmatrix} x & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 2 \\ -2 & x & x+1 & x-1 \end{vmatrix} = 0$$

Expanding by R_2, we get $\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & 2 \\ -2 & x+1 & x-1 \end{vmatrix} = 0$ ($\because$ $(-1)^{2+2} = 1$)

By $R_3 + 2R_2$, we get $\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & 2 \\ 0 & x+3 & x+3 \end{vmatrix} = 0$

or $\quad (x+3)\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = 0$ (by taking $x + 3$ common from R_3)

$\Rightarrow \quad x + 3 = 0 \quad$ or $\quad \begin{vmatrix} x & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = 0$

$\Rightarrow \quad x = -3 \quad$ or $\quad x = 0 \quad$ ($\because$ R_1 and R_3 are identical if $x = 0$)

Thus the solution set is $\{-3, 0\}$.

3.8 Adjoint and Inverse of a Square Matrix of Order $n \geq 3$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then the matrix of co-factors of $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$,

and $\operatorname{adj} A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$,

Inverse of a Square Matrix of Order n≥3: Let A be a non singular square matrix of order n. If there exists a matrix B such that $AB = BA = I_n$, then B is called the multiplicative inverse of A and is denoted by A^{-1}. It is obvious that the order of A^{-1} is $n \times n$.

Thus $AA^{-1} = I_n$ and $A^{-1}A = I_n$.

If A is a non singular matrix, then

$$A^{-1} = \frac{1}{|A|}\operatorname{adj}A$$

Example 6: Find A^{-1} if $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

Solution: We first find the cofactors of the elements of A.

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = 1.(2+1) = 3, \qquad A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = (-1)(-1) = 1$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} = 1.(0-2) = -2, \qquad A_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} = (-1)(0+2) = -2$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1.(1-2) = -1, \qquad A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = (-1)(-1-0) = 1$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} = 1.(0-4) = -4, \qquad A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = (-1)(1-0) = -1$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 1.(2-0) = 2$$

Thus $[A_{ij}]_{3\times 3} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ -2 & -1 & 1 \\ -4 & -1 & 2 \end{bmatrix}$

and $\operatorname{adj} A = [A'_{ij}]_{3\times 3} = \begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix}$ $(\because A'_{ij} = A_{ji}$ for i, j=1, 2, 3)

Since $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$= 1(3) + 0(1) + 2(-2)$

$= 3 + 0 - 4 = -1,$

So $A^{-1} = \frac{1}{|A|}\operatorname{adj}A = \frac{1}{-1}\begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 4 \\ -1 & 1 & 1 \\ 2 & -1 & -2 \end{bmatrix}$

Example 7: If $A = \begin{bmatrix} -1 & 2 \\ 1 & 4 \\ 2 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ then verify that

$(AB)^t = B^tA^t$

Solution: so $AB = \begin{bmatrix} -1 & 2 \\ 1 & 4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1-4 & -3+2 \\ 1-8 & 3+4 \\ 2+2 & 6-1 \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ -7 & 7 \\ 4 & 5 \end{bmatrix}$,

$$(AB)^t = \begin{bmatrix} -5 & -7 & 4 \\ -1 & 7 & 5 \end{bmatrix}$$

and $B^tA^t = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 2 \\ 2 & 4 & -1 \end{bmatrix} \left(\because A^t = \begin{bmatrix} -1 & 1 & 2 \\ 2 & 4 & -1 \end{bmatrix} and\ B^t = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\right)$

$$= \begin{bmatrix} -1-4 & 1-8 & 2+2 \\ -3+2 & 3+4 & 6-1 \end{bmatrix} \quad \begin{bmatrix} -5 & -7 & 4 \\ -1 & 7 & 5 \end{bmatrix}$$

Thus $(AB)^t = B^tA^t$

Exercise 3.3

Evaluate the following determinants.

1\. i) $\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$ ii) $\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$ iii) $\begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix}$

iv) $\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}$ v) $\begin{vmatrix} 1 & 2 & -2 \\ -1 & 1 & -3 \\ 2 & 4 & -1 \end{vmatrix}$ vi) $\begin{vmatrix} 2a & a & a \\ b & 2b & b \\ c & c & 2c \end{vmatrix}$

2\. Without expansion show that

i) $\begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = 0$ ii) $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix} \quad 0$ iii) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \quad 0$

3\. Show that

i) $\begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$

ii) $\begin{vmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1 \end{vmatrix} = 9\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix}$ iii) $\begin{vmatrix} a+l & a & a \\ a & a \quad l+ & a \\ a & a & a+l \end{vmatrix} \quad l^2(3a \quad l)$

iv) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$ v) $\begin{vmatrix} b+c & a & a \\ b & c \quad a & b \\ c & c & a+b \end{vmatrix} \quad 4abc$

vi) $\begin{vmatrix} b & -1 & a \\ a & b & 0 \\ 1 & a & b \end{vmatrix} = a^3 \quad b^3$ vii) $\begin{vmatrix} r\cos\phi & 1 & -\sin\phi \\ 0 & 1 & 0 \\ r\sin\phi & 0 & \cos\phi \end{vmatrix} \quad r$

viii) $\begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix} = a^3+b^3+c^3-3abc$

ix) $\begin{vmatrix} a+\lambda & b & c \\ a & b+\lambda & c \\ a & b & c+\lambda \end{vmatrix} = \lambda^2(a+b+c+\lambda)$

x) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

xi) $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$

4\. If If $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 0 \\ -2 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 & 5 \\ 3 & 1 & 4 \\ -2 & 1 & -2 \end{bmatrix}$, then find;

i) A_{12}, A_{22}, A_{32} and $|A|$ ii) B_{21}, B_{22}, B_{23} and $|B|$

5\. Without expansion verify that

i) $\begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \gamma+\alpha & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix} = 0$ ii) $= \begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix} \quad 0$ iii) $\begin{vmatrix} 1 & a^2 & \frac{a}{bc} \\ 1 & b^2 & \frac{b}{ca} \\ 1 & c^2 & \frac{c}{ab} \end{vmatrix} \quad 0$

iv) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

v) $\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix} = 0$

vi) $\begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix} = \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$

vii) $\begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix}$

viii) $\begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix} + \begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$ ix) $\begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix}$

6. Find values of x if

i) $\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = 30$ ii) $\begin{vmatrix} 1 & x-1 & 3 \\ 1 & x & 2 \\ 2 & -2 & x \end{vmatrix} = 0$ iii) $\begin{vmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$

7. Evaluate the following determinants:

i) $\begin{vmatrix} 3 & 4 & 2 & 7 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix}$ ii) $\begin{vmatrix} 2 & 3 & 1 & -1 \\ 4 & 0 & 2 & 1 \\ 5 & 2 & -1 & 6 \\ 3 & -7 & 2 & -2 \end{vmatrix}$ iii) $\begin{vmatrix} -3 & 9 & 1 & 1 \\ 0 & 3 & -1 & 2 \\ 9 & 7 & -1 & 1 \\ -2 & 0 & 1 & -1 \end{vmatrix}$

8. Show that $\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = (x+3)(x-1)^3$

9. Find $|AA^t|$ and $|A^tA|$ if

i) $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$ ii) $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$

10. If A is a square matrix of order 3, then show that $|KA| = k^3|A|$.

11. Find the value of λ if A and B singular.

$A = \begin{bmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{bmatrix}$

12. Which of the following matrices are singular and which of them are non singular?

i) $\begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$ ii) $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 2 \\ 3 & -1 & 3 & 4 \end{bmatrix}$

13. Find the inverse of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$ and show that $A^{-1}A = I_3$

14. Verify that $(AB)^{-1} = B^{-1}A^{-1}$ if

i) $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix}$ ii) $A = \begin{bmatrix} 5 & 1 \\ 2 & 2 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

15. Verify that $(AB)^t = B^t A^t$ and if

$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$

16. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ verify that $(A^{-1})^t = (A^t)^{-1}$

17. If A and B are non-singular matrices, then show that

i) $(AB)^{-1} = B^{-1}A^{-1}$ ii) $(A^{-1})^{-1} = A$

3.9 Elementary Row and Column Operations on a Matrix

Usually a given system of linear equations is reduced to a simple equivalent system by applying in turn a finite number of elementary operations which are stated as below:

1. Interchanging two equations.
2. Multiplying an equation by a non-zero number.
3. Adding a multiple of one equation to another equation.

Note: The systems of linear equations involving the same variables, are equivalent if they have the same solution.

Corresponding to these three elementary operations, the following elementary row operations are applied to matrices to obtain equivalent matrices.

i) Interchanging two rows
ii) Multiplying a row by a non-zero number
iii) Adding a multiple of one row to another row.

Note: Matrices A and B are equivalent if B can be obtained by applying in turn a finite number of row operations on A.

Notations that are used to represent row operations for I to III are given below:
Interchanging R_i and R_j is expressed as $R_i \leftrightarrow R_j$.
k times R_i, is denoted by $kR_i \to R'_i$
Adding k times R_j to R_i is expressed as $R_i + kR_j \to R'_i$
(R'_i is the new row obtained after applying the row operation).

For equivalent matrices A and B, we write $A \underset{\sim}{R} .B$.

If $A \underset{\sim}{R} B$ then $B \underset{\sim}{R} A$. Also if $A \underset{\sim}{R} B$ and $B \underset{\sim}{R} C$, then $A \underset{\sim}{R} C$. Now we state the elementary column operations and notations that are used for them.

i) Interchanging two columns $C_i \leftrightarrow C_j$
ii) Multiplying a column by a non-zero number $kC_i \to C'_i$
iii) Adding a multiple of one column to another column $C_i + kC_j \to C'_i$

Consider the system of linear equations;

$$\left.\begin{aligned} x+y+2z&=1\\ 2x-y=8z&=12\\ 3x+5y+4z=&\ \ 3\end{aligned}\right\}$$ which can be written in matrix forms as

$$\begin{bmatrix}1&1&2\\2&-1&8\\3&5&4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\12\\-3\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}x&y&z\end{bmatrix}\begin{bmatrix}1&2&3\\1&-1&5\\2&8&4\end{bmatrix}=\begin{bmatrix}1&12&-3\end{bmatrix}$$

that is, $AX = B$ (i) $X^t A^t = B^t$ (ii)

where $A=\begin{bmatrix}1&1&2\\-2&1&8\\3&5&4\end{bmatrix}, X \begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B \begin{bmatrix}1\\12\\-3\end{bmatrix}$

A is called the matrix of coefficients.

Appending a column of constants on the left of A, we get the augmented matrix of the given system, that is,

$$\left[\begin{array}{ccc:c}1&1&2&1\\2&-1&8&12\\3&5&4&-3\end{array}\right]$$ (Appended column is separated by a dotted line segment)

Now we explain the application of elementary operations on the system-of linear equations and the application of elementary row operations on the augmented matrix of the system writing them side by side.

$$\left.\begin{aligned} x+y+2z&=1\\ 2x+-y+8z&=12\\ 3x+5y+4z=&\ \ 3\end{aligned}\right\} \qquad \left[\begin{array}{ccc:c}1&1&2&1\\2&1&8&12\\3&5&4&-3\end{array}\right]$$

Adding –2 times the first equation to the second and –3 times the first equation to the third, we get (By $R_2 + (-2)R_1 \to R'_2$ and $R_3 + (-3)R_1 \to R'_3$, we get)

$$\left.\begin{aligned} x+y+2z&=1\\ -3y+4z&=10\\ 2y-2z=&\ \ 6\end{aligned}\right\} \qquad \underset{\sim}{R}\left[\begin{array}{ccc:c}1&1&2&1\\0&3&4&10\\0&2&-2&-6\end{array}\right]$$

Interchanging the second and third equations, we have (By $R_2 \leftrightarrow R_3$, we get)

$$\left.\begin{aligned} x+y+2z&=1\\ 2y-2z&=6\\ -3y+4z&=10 \end{aligned}\right\} \qquad \underset{\sim}{R}\begin{bmatrix} 1 & 1 & 2 & \vdots & 1\\ 0 & 2 & -2 & \vdots & -6\\ 0 & 3 & 4 & \vdots & 10 \end{bmatrix}$$

Multiplying the second equation by $\frac{1}{2}$, we get By $\frac{1}{2}R_2 \to R'_2$, we get.

$$\left.\begin{aligned} x+y+2z&=1\\ y-z&=-3\\ -3y+4z&=10 \end{aligned}\right\} \qquad \underset{\sim}{R}\begin{bmatrix} 1 & 1 & 2 & \vdots & 1\\ 0 & 1 & -1 & \vdots & -3\\ 0 & -3 & 4 & \vdots & 10 \end{bmatrix}$$

Adding 3 times the second equation to By $R_3 + 3R_2 \to R'_3$, we obtain, the third, we obtain,

$$\left.\begin{aligned} x+y+2z&=1\\ y-z&=-3\\ z&=1 \end{aligned}\right\} \qquad \underset{\sim}{R}\begin{bmatrix} 1 & 1 & 2 & \vdots & 1\\ 0 & 1 & -1 & \vdots & -3\\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix}$$

The given system is reduced to the triangular form which is so called because on the left the coefficients (of the terms) within the dotted triangle are zero.

Putting $z = 1$ in $y - z = -3$, we have $y - 1 = -3 \Rightarrow y = -2$

Substiliting $z = 1$, $y = -2$ in the first equation, we get

$$x + (-2) + 2(1) = 1 \Rightarrow x = 1$$

Thus the solution set o f the given system is $\{(1, -2, 1)\}$.

Appending a row of constants below the matrix A^t, we obtain the

augmented matrix for the matrix equation (ii), that is $\begin{bmatrix} 1 & 2 & 3\\ 1 & -1 & 5\\ 2 & 8 & 4\\ \cdots & \cdots & \cdots\\ 1 & 12 & -3 \end{bmatrix}$

Now we apply elementary column operations to this augmented matrix.

$$\begin{bmatrix} 1 & 2 & 3\\ 1 & -1 & 5\\ 2 & 8 & 4\\ \cdots & \cdots & \cdots\\ 1 & 12 & -3 \end{bmatrix} \underset{\sim}{C} \begin{bmatrix} 1 & 0 & 0\\ 1 & -3 & 2\\ 2 & 4 & -2\\ \cdots & \cdots & \cdots\\ 1 & 10 & -6 \end{bmatrix} \text{ By } C_2 + (-2)C_1 \to C'_2 \text{ and } C_3 + (-3)C_1 \to C'_3$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0\\ 1 & 2 & -3\\ 2 & -2 & 4\\ \cdots & \cdots & \cdots\\ 1 & -6 & 10 \end{bmatrix} \text{ By } C_2 \leftrightarrow C_3 \qquad \underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0\\ 1 & 1 & -3\\ 2 & -1 & 4\\ \cdots & \cdots & \cdots\\ 1 & -3 & 10 \end{bmatrix} \text{ By } \frac{1}{2}C_2 \to C'_2$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0\\ 1 & 1 & 0\\ 2 & -1 & 1\\ \cdots & \cdots & \cdots\\ 1 & -3 & 1 \end{bmatrix} \text{ By } C_3 + 3C_2 \to C'_3$$

Thus $\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} 1 & 0 & 0\\ 1 & 1 & 0\\ 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \end{bmatrix}$

or $\begin{bmatrix} x+y+2z & y-z & z \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \end{bmatrix}$

$$\Rightarrow \quad \left.\begin{aligned} x+y+2z&=1\\ y-z&=-3\\ z&=1 \end{aligned}\right\}$$

Upper Triangular Matrix: A square matrix $A = [a_{ij}]$ is called upper triangular if all elements below the principal diagonal are zero, that is,

$$a_{ij} = 0 \text{ for all } i > j$$

Lower Triangular Matrix: A square matrix $A = [a_{ij}]$ is said to be lower triangular if all elements above the principal diagonal are zero, that is,

$$a_{ij} = 0 \text{ for all } i < j$$

Triangular Matrix: A square matrix *A* is named as triangular whether it is upper triangular or lower triangular. For example, the matrices

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 4 & 1 & 5 & 0 \\ -1 & 2 & 3 & 1 \end{bmatrix}$ are triangular matrices of order 3 and 4

respectively. The first matrix is upper triangular while the second is lower triangular.

Note: Diagonal matrices are both upper triangular and lower triangular.

Symmetric Matrix: A square matrices $A = [a_{ij}]_{n \times n}$ is called symmetric if $A^t = A$.
From $A^t = A$, it follows that $[a'_{ij}]_{n \times n} = [a_{ij}]_{n \times n}$
which implies that $a'_{ij} = a_{ij}$ for $i, j = 1, 2, 3, \ldots, n$.
but by the definition of transpose, $a'_{ij} = a_{ji}$ for $i, j = 1, 2, 3, \ldots, n$.
Thus $a_{ij} = a_{ji}$ for $i, j = 1, 2, 3, \ldots, n$.
and we conclude that a square matrix $A = [a_{ij}]_{n \times n}$ is symmetric if $a_{ij} = a_{ji}$.
For example, the matrices

$\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$, $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 2 & -1 \\ 3 & 0 & 5 & 6 \\ 2 & 5 & 1 & -2 \\ -1 & 6 & -2 & 3 \end{bmatrix}$ are symmetric.

Skew Symmetric Matrix : A square matrix A = $[a_{ij}]_{n \times n}$ is called skew symmetric or anti-symmetric if $A^t = -A$.

From $A^t = -A$, it follows that $[a'_{ij}]$ = for $i, j = 1, 2, 3, \ldots, n$
which implies that $a'_{ij} = -a_{ij}$ for $i, j = 1, 2, 3, \ldots, n$
but by the definition of transpose $a'_{ij} = a_{ji}$ for $i, j = 1, 2, 3, \ldots, n$
Thus $-a_{ij} = a_{ji}$ or $a_{ij} = -a_{ji}$

Alternatively we can say that a square matrix $A = [a_{ij}]_{nxn}$ is anti-symmetric if $a_{ij} = -a_{ji}$.

For diagonal elements $j = i$, so
$a_{ii} = -a_{ii}$ *or* $2a_{ii} = 0 \Rightarrow$ $a_{ii} = 0$ for $i = 1, 2, 3, \ldots, n$

For example if B = $\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$,then

$$B^t = \begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix} = (-1)\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix} = -B$$

Thus the matrix *B* is skew-symmetric.

Let $A = [a_{ij}]$ be an *n* x *m* matrix with complex entries, Then the *n* x *m* matrix $[\bar{a}_{ij}]$ where $\bar{a}_{ij}$ is the complex conjugate of a_{ij} for all *i*, *j*, is called *conjugate* of *A* and is denoted by $\bar{A}$. For example, if

$$A = \begin{bmatrix} 3-i & -i \\ 2i & 1+i \end{bmatrix}, \text{then } \bar{A} \begin{bmatrix} \overline{3-i} & \overline{-i} \\ \overline{2i} & \overline{1+i} \end{bmatrix} \begin{bmatrix} 3+i & i \\ -2i & 1-i \end{bmatrix}$$

Hermitian Matrix: A square matrix $A = [a_{ij}]_{nxn}$ with complex entries, is called *hermitian* if $(\bar{A})^t = A$.

From, $(\bar{A})^t = A$ it follows that $[\bar{a}'_{ij}]_{n\times n} = [a_{ij}]_{nxn}$ which implies that $\bar{a}'_{ij} = a_{ij}$ for $i, j = 1, 2, 3, \ldots, n$ but by the definition of transpose, $\bar{a}'_{ij} = \bar{a}_{ij}$ for $i, j = 1, 2, 3, \ldots, n$.

Thus $a_{ij} = \bar{a}_{ij}$ for $i, j = 1, 2, 3, \ldots, n$ and we can say that a square matrix $A = [a_{ij}]_{nxn}$ is *hermitian* if $a_{ij} = \bar{a}_{ji}$ for $i, j = 1, 2, 3, \ldots, n$.

For diagnal elements, $j = i$ so $a_{ii} = \bar{a}_{ii}$ which implies that a_{ii} is real for $i = 1, 2, 3, \ldots, n$

For example, if A = $\begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$, then

$$\bar{A} = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix} \Rightarrow (\bar{A})^t = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix} = A$$

Thus *A* is hermitian.

Skew Hermitian Matrix: A square matrix $A = [a_{ij}]_{nxn}$ with complex entries, is called skew-hermitian or anti-hermitian if $\left(\bar{A}\right)^t = -A$.

From $\left(\bar{A}\right)^t = -A$, it follows that $\left[\bar{a}_{ij}^{t}\right]_{n\times n} = [-a_{ij}]_{nxn}$

which implies that $\bar{a}_{ij}^{t} = -a_{ij}$ for $i, j = 1, 2, 3, \ldots, n$.

but by the definition of transpose, $\bar{a}_{ij}^{t} = \bar{a}_{ji}$ for $i, j = 1, 2, 3, \ldots, n$.

Thus $-a_{ij} = \bar{a}_{ji}$ or $a_{ij} = -\bar{a}_{ji}$, for $i, j = 1, 2, 3, \ldots, n$.

and we can conclude that a square matrix $A = [a_{ij}]_{nxn}$ is anti-hermitian if $a_{ij} = -\bar{a}_{ji}$

For diagonal elements $j = i$, so $a_{ii} = -\bar{a}_{ii} \Rightarrow a_{ii} + \bar{a}_{ii} = 0$

which holds if $a_{ii} = 0$ or $a_{ii} = i\lambda$ where λ is real

because $0 + 0 = 0$ or $i\lambda + i\bar{\lambda} = i\lambda - i\lambda = 0$

For example, if $A = \begin{bmatrix} 0 & 2+3i \\ -2+3i & 0 \end{bmatrix}$, then

$$\bar{A} = \begin{bmatrix} 0 & 2+3i \\ -2+3i & 0 \end{bmatrix}$$

$$\Rightarrow \left(\bar{A}\right)^t = \begin{bmatrix} 0 & -2+3i \\ 2+3i & 0 \end{bmatrix} = (-1)\begin{bmatrix} 0 & 2-3i \\ -2-3i & 0 \end{bmatrix} = -A$$

Thus A is skew-hermitian.

3.10 Echelon and Reduced Echelon Forms of Matrices

In any non-zero row of a matrix, the first non-zero entry is called the *leading entry* of that row. The zeros before the leading entry of a row are named as the *leading zero entries* of the row.

Echelon Form of a Matrix: An $m \times n$ matrix A is called in **(row) echelon form** if

i) In each successive non-zero row, the number of zeros before the leading entry is greater than the number of such zeros in the preceding row,

ii) Every non-zero row in A precedes every zero row (if any),

iii) The first non-zero entry (or leading entry) in each row is 1.

Note: Some authors do not require the condition (iii)

The matrices $\begin{bmatrix} 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are in echelon form

but the matrices $\begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 4 \end{bmatrix}$ are not in echelon form.

Reduced Echelon Form of a Matrix: An $m \times n$ matrix A is said to be in **reduced (row) echelon form** if it is in (row) echelon form and if the first non-zero entry (or leading entry) in R_i lies in C_j, then all other entries of C_j are zero.

The matrices $\begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are in (row) reduced echelon form.

Example 1: Reduce the following matrix to (row) echelon and reduced (row) echelon form,

$$\begin{bmatrix} 2 & 3 & -1 & 9 \\ 1 & -1 & 2 & -3 \\ 3 & 1 & 3 & 2 \end{bmatrix}$$

Solution: $\begin{bmatrix} 2 & 3 & -1 & 9 \\ 1 & -1 & 2 & -3 \\ 3 & 1 & 3 & 2 \end{bmatrix} \quad \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 3 & -1 & 9 \\ 3 & 1 & 3 & 2 \end{bmatrix}$ By $R_1 \leftrightarrow R_2$

$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -5 & 15 \\ 0 & 4 & -3 & 11 \end{bmatrix}$ By $R_2 + (-2)R_1 \to R'_2$ and $R_3 + (-3)R_1 \to R'_3$ $\quad \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 3 \\ 0 & 4 & -3 & 11 \end{bmatrix}$ By $\frac{1}{5}R_2 \to R'_2$

$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & -1 \end{bmatrix}$ By $R_3 + (-4)R_2 \to R'_3$ $\quad \underset{\sim}{R} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ By $R_1 + 1.R_2 \to R'_1$

$$\underset{\sim}{R}\begin{bmatrix}1&0&0&1\\0&1&0&2\\0&0&1&-1\end{bmatrix}\quad \begin{matrix}\text{By } R_1+(-1)R_3\to R'_1\\ \text{and } R_2+1.R_3\to R'_2\end{matrix}$$

Thus $\begin{bmatrix}1&-1&2&3\\0&1&-1&3\\0&0&1&-1\end{bmatrix}$ and $\begin{bmatrix}1&0&0&1\\0&1&0&2\\0&0&1&-1\end{bmatrix}$ are (row) echelon and reduced (row) echelon forms of the given matrix respectively.

Let *A* be a non-singular matrix. If the application of elementary row operations on $A \vdots I$ in succession reduces *A* to I, then the resulting matrix is $I \vdots A^{-1}$.

Similarly if the application of elementary column operations on $\begin{matrix}A\\ \cdots\\ I\end{matrix}$ in succession reduces *A* to I, then the resulting matrix is $\begin{matrix}I\\ \hline A^{-1}\end{matrix}$

Thus $A \vdots I \ \underset{\sim}{R}\ I \vdots A^{-1}$ and $\begin{matrix}A\\ \cdots\\ I\end{matrix}\ \underset{\sim}{C}\ \begin{matrix}I\\ \cdots\\ A^{-1}\end{matrix}$

Example 2: Find the inverse of the matrix A = $\begin{bmatrix}2&5&-1\\3&4&2\\1&2&-2\end{bmatrix}$

Solution: $|A| = \begin{vmatrix}2&5&-1\\3&4&2\\1&2&-2\end{vmatrix}$ = 2(–8 – 4) – 5(–6 – 2) –1(6 – 4) = –24 + 40 – 2

= 40 – 26 = 14 As $|A| \neq 0$, so *A* is non-singular.

Appending I_3 on the left of the matrix *A*, we have $\begin{bmatrix}2&5&-1&\vdots&1&0&0\\3&4&2&\vdots&0&1&0\\1&2&-2&\vdots&0&0&1\end{bmatrix}$

Interchanging R_1 and R_3 we get..

$$\begin{bmatrix}1&2&-2&\vdots&0&0&1\\3&4&2&\vdots&0&1&0\\2&5&-1&\vdots&1&0&0\end{bmatrix}\underset{\sim}{R}\begin{bmatrix}1&2&-2&\vdots&0&0&1\\0&-2&8&\vdots&0&1&-3\\0&1&3&\vdots&1&0&-2\end{bmatrix}\begin{matrix}\text{By } R_2+(-3)R_1\to R'_2\\ \text{and } R_3+(-2)R_1\to R'_3\end{matrix}$$

By $-\frac{1}{2}R_2\to R'_2$, we get

$$\begin{bmatrix}1&2&-2&\vdots&0&0&1\\0&1&-4&\vdots&0&-\frac{1}{2}&\frac{3}{2}\\0&1&3&\vdots&1&0&-2\end{bmatrix}\underset{\sim}{R}\begin{bmatrix}1&0&6&\vdots&0&0&-2\\0&1&-4&\vdots&0&-\frac{1}{2}&\frac{3}{2}\\0&0&7&\vdots&1&\frac{1}{2}&-\frac{7}{2}\end{bmatrix}\begin{matrix}\text{By } R_3+(-1)R_2\to R'_3\\ \text{and } R_1+(-2)R_2\to R'_1\end{matrix}$$

By $\frac{1}{7}R_3\to R'_3$, we have

$$\begin{bmatrix}1&0&6&\vdots&0&1&-2\\0&1&-4&\vdots&0&-\frac{1}{2}&\frac{3}{2}\\0&0&1&\vdots&\frac{1}{7}&\frac{1}{14}&-\frac{1}{2}\end{bmatrix}\underset{\sim}{R}\begin{bmatrix}1&0&0&\vdots&-\frac{6}{7}&\frac{4}{7}&1\\0&1&0&\vdots&\frac{4}{7}&-\frac{3}{14}&-\frac{1}{2}\\0&0&1&\vdots&\frac{1}{7}&\frac{1}{14}&-\frac{1}{2}\end{bmatrix}\begin{matrix}\text{By } R_1+(-6)R_3\to R'_1\\ \text{and } R_2+4R_3\to R'_2\end{matrix}$$

Thus the inverse of *A* is $\begin{bmatrix}-\frac{6}{7}&\frac{4}{7}&1\\-\frac{4}{7}&-\frac{3}{14}&-\frac{1}{2}\\\frac{1}{7}&\frac{1}{14}&-\frac{1}{2}\end{bmatrix}$

Appending I_3 below the matrices *A*, we have

$$\begin{bmatrix}2&5&-1\\3&4&2\\1&2&-2\\ \cdots&\cdots&\cdots\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Interchanging C_1 and C_3, we get

$$\left[\begin{array}{ccc} 2 & 5 & -1 \\ 3 & 4 & 2 \\ 1 & 2 & -2 \\ \hdashline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \underset{\sim}{C} \left[\begin{array}{ccc} -1 & 5 & 2 \\ 2 & 4 & 3 \\ -2 & 2 & 1 \\ \hdashline 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right] \underset{\sim}{C} \left[\begin{array}{ccc} 1 & 5 & 2 \\ -2 & 4 & 3 \\ 2 & 2 & 1 \\ \hdashline 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{array}\right] \text{By } (-1)C_1 \to C'_1$$

By $C_2 + (-5)C_1 \to C'_2$ and $C_3 + (-2)C_1 \to C'_3$, we have

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ -2 & 14 & 7 \\ 2 & -8 & -3 \\ \hdashline 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 5 & 2 \end{array}\right] \underset{\sim}{C} \left[\begin{array}{ccc} 1 & 0 & 0 \\ -2 & 1 & 7 \\ 2 & -\frac{4}{7} & -3 \\ \hdashline 0 & 0 & 1 \\ 0 & \frac{1}{14} & 0 \\ -1 & \frac{5}{14} & 2 \end{array}\right] \quad \text{By } \frac{1}{14}C_2 \to C'_2$$

By $C_1 + (2)C_2 \to C'_1$ and $C_3 + (-7)C_2 \to C'_3$ we have

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{6}{7} & -\frac{4}{7} & 1 \\ \hdashline 0 & 0 & 1 \\ \frac{1}{7} & \frac{1}{14} & \frac{1}{2} \\ -\frac{2}{7} & \frac{5}{14} & -\frac{1}{2} \end{array}\right] \underset{\sim}{C} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hdashline -\frac{6}{7} & \frac{4}{7} & 1 \\ \frac{4}{7} & -\frac{3}{14} & -\frac{1}{2} \\ \frac{1}{7} & \frac{1}{14} & -\frac{1}{2} \end{array}\right] \begin{array}{l} \text{By } C_1 + \left(-\frac{6}{7}\right)C_3 \to C'_1 \\ \text{and } C_2 + \left(\frac{4}{7}\right)C_3 \to C'_2 \end{array}$$

Thus the inverse of A is $\begin{bmatrix} -\frac{6}{7} & \frac{4}{7} & 1 \\ \frac{4}{7} & -\frac{3}{14} & -\frac{1}{2} \\ \frac{1}{7} & \frac{1}{14} & -\frac{1}{2} \end{bmatrix}$

Rank of a Matrix: Let A be a non-zero matrix. If r is the number of non-zero rows when it is reduced to the reduced echelon form, then r is called the (row) **rank** of the matrix A.

Example 3: Find the rank of the matrix $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 0 & 7 & -7 \\ 3 & 1 & 12 & -11 \end{bmatrix}$

Solution: $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 0 & 7 & -7 \\ 3 & 1 & 12 & -11 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 2 & 3 & -1 \\ 0 & 4 & 6 & -2 \end{bmatrix} \begin{array}{l} \text{By } R_2 + (-2)R_1 \to R'_2 \\ \text{and } R_3 + (-3)R_1 \to R'_3 \end{array}$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 2 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 4 & 6 & -2 \end{bmatrix} \text{By } \frac{1}{2}R_2 \to R'_2 \quad \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{By } R_3 + (-4)R_2 \to R'_3$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & \frac{7}{2} & -\frac{7}{2} \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{By } R_1 + 1.R_2 \to R'_1$$

As the number of non-zero rows is 2 when the given matrix is reduced to the reduced echelon form, therefore, the rank of the given matrix is 2.

Exercise 3.4

1. If $A = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 3 & -1 \\ 5 & -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix}$,then show that $A + B$ is symmetric.

2. If $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}$, show that

i) $A + A^t$ is symmetric ii) $A - A^t$ is skew-symmetric.

3. If A is any square matrix of order 3, show that

i) $A + A^t$ is symmetric and ii) $A - A^t$ is skew-symmetric.

4. If the matrices A and B are symmetric and $AB = BA$, show that AB is symmetric.

5. Show that AA^t and A^tA are symmetric for any matrix of order 2 x 3.

6. If $A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$,show that

i) $A + \left(\overline{A}\right)^t$ is hermitian ii) $A - \left(\overline{A}\right)^t$ is skew-hermitian.

7. If A is symmetric or skew-symmetric, show that A^2 is symmetric.

8. If $A = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}$, find $A\left(\overline{A}\right)^t$.

9. Find the inverses of the following matrices. Also find their inverses by using row and column operations.

i) $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

10. Find the rank of the following matrices

i) $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -6 & 5 & 1 \\ 3 & 5 & 4 & -3 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & -4 & -7 \\ 2 & -5 & 1 \\ 1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix}$ iii) $\begin{bmatrix} 3 & -1 & 3 & 0 & -1 \\ 1 & 2 & -1 & -3 & -2 \\ 2 & 3 & 4 & 2 & 5 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix}$

3.11 System of Linear Equations

An equation of the form:

$$ax + by = k \qquad \text{(i)}$$

where $a \neq 0$, $b \neq 0$, $k \neq 0$

is called a non-homogeneous linear equation in two variables x and y.

Two linear equations in the same two variables such as:

$$\left.\begin{aligned} a_1x + b_1y = k_1 \\ a_2x + b_2y = k_2 \end{aligned}\right\} \qquad \text{(I)}$$

is called a system of non-homogeneous linear equations in the two variables x and y if constant terms k_1, k_2 are not both zero.

If in the equation (i), $k = 0$, that is, $ax + by = 0$, then it is called a homogeneous linear equation in x and y.

If in the system (I), $k_1 = k_2 = 0$, then it is said to be a system of homogenous linear equations in x and y.

An equation of the form:

$$ax + by + cz = k \qquad \text{. . . . (ii)}$$

is called a non-homogeneous linear equation in three variables x, y and z if $a \neq 0$, $b \neq 0$, $c \neq 0$ and $k \neq 0$. Three linear equations in three variables such as:

$$\left.\begin{aligned} a_1x + b_1y + c_1z = k_1 \\ a_2x + b_2y + c_2z = k_2 \\ a_3x + b_3y + c_3z = k_3 \end{aligned}\right\} \qquad \text{(II)}$$

is called a system of non-homogeneous linear equations in the three variables x, y and z, if constant terms k_1, k_2 and k_3 are not all zero.

If in the equations (ii) $k = 0$ that is, $ax + by + cz = 0$.
then it is called a homogeneous linear equation it x, y and z.

If in the system (II), $k_1 = k_2 = k_3 = 0$, then it is said to be a system of homogeneous linear equations in x ; y and z.

A system of linear equations is said to be consistent if the system has a unique solution or it has infinitely many solutions.

A system of linear equations is said to be inconsistent if the system has no solution.

The system (II), consists of three equations in three variables so it is called 3 x 3 linear system but a system of the form:

$$\left.\begin{aligned} x - y + 2z &= 6 \\ 2x + y + 3z &= 4 \end{aligned}\right\}$$

is named as 2 x 3 linear system.

Now we solve the following three 3 x 3 linear systems to determine the criterion for a system to be consistent or for a system to be inconsistent.

$$\left.\begin{aligned} 2x + 5y - z &= 5 \\ 3x + 4y + 2z &= 11 \\ x + 2y - 2z &= -3 \end{aligned}\right\} \quad \text{....(1),} \qquad \left.\begin{aligned} x + y + 2z &= 1 \\ 2x - y + 7z &= 11 \\ 3x + 5y + 4z &= -3 \end{aligned}\right\} \quad \text{(2)}$$

and
$$\left.\begin{aligned} x - y + 2z &= 1 \\ 2x - 6y + 5z &= 7 \\ 3x + 5y + 4z &= -3 \end{aligned}\right\} \quad \text{....(3)}$$

The augmented matrix of the system (1) is

$$\begin{bmatrix} 2 & 5 & -1 & \vdots & 5 \\ 3 & 4 & 2 & \vdots & 11 \\ 1 & 2 & -2 & \vdots & -3 \end{bmatrix}$$

We apply the elementary row operations to the above matrix to reduce it to the equivalent reduced (row) echelon form, that is,

$$\begin{bmatrix} 2 & 5 & -1 & \vdots & 5 \\ 3 & 4 & 2 & \vdots & 11 \\ 1 & 2 & -2 & \vdots & -3 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 2 & -2 & \vdots & -3 \\ 3 & 4 & 2 & \vdots & 11 \\ 2 & 5 & -1 & \vdots & 5 \end{bmatrix} \text{ BY } R_1 \leftrightarrow R_3$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & -2 & \vdots & -3 \\ 0 & -2 & 8 & \vdots & 20 \\ 2 & 5 & -1 & \vdots & 5 \end{bmatrix} \text{BY } R_2 + (-3)R_1 \to R_2' \; \underset{\sim}{R} \begin{bmatrix} 1 & 2 & -2 & \vdots & -3 \\ 0 & -2 & 8 & \vdots & 20 \\ 0 & 1 & 3 & \vdots & 11 \end{bmatrix} \text{BY } R_3 + (-2)R_1 \to R_3'$$

$\text{BY } -\frac{1}{2}R_2 \to R_2',$ we get

$$\begin{bmatrix} 1 & 2 & -2 & \vdots & -3 \\ 0 & 1 & -4 & \vdots & -10 \\ 0 & 1 & 3 & \vdots & 11 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 0 & 6 & \vdots & 17 \\ 0 & 1 & -4 & \vdots & -10 \\ 0 & 0 & 7 & \vdots & 21 \end{bmatrix} \begin{array}{l} \text{By } R_1 + (-2)R_2 \to R'_1 \\ \text{and } R_3 + (-1), R_2 \to R'_3 \end{array}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 6 & \vdots & 17 \\ 0 & 1 & -4 & \vdots & -10 \\ 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \text{BY } \frac{1}{7}R3 \to R'3 \; \underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & \vdots & -1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \begin{array}{l} By\ R_1 + (-6)R_3 \to R'_1 \\ \text{and } R_2 + 4R_3, R'_2 \end{array}$$

Thus the solution is $x = -1$, $y = 2$ and $z = 3$.

The augmented matrix for the system (2) is

$$\begin{bmatrix} 1 & 1 & 2 & \vdots & 1 \\ 2 & -1 & 7 & \vdots & 11 \\ 3 & 5 & 4 & \vdots & -3 \end{bmatrix}$$

Adding $(-2)R_1$ to R_2 and $(-3)R_1$ to R_3, we get

$$\begin{bmatrix} 1 & 1 & 2 & \vdots & 1 \\ 2 & -1 & 7 & \vdots & 11 \\ 3 & 5 & 4 & \vdots & -3 \end{bmatrix} \qquad \underset{\sim}{R} \begin{bmatrix} 1 & 1 & 2 & \vdots & 1 \\ 0 & -3 & 3 & \vdots & 9 \\ 0 & 2 & -2 & \vdots & -6 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 2 & \vdots & 1 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & 2 & -2 & \vdots & -6 \end{bmatrix} \text{By } -\frac{1}{3}R_2 \to R_2' \; \underset{\sim}{R} \begin{bmatrix} 1 & 0 & 3 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \begin{array}{l} \text{BY } R_1 + (-1)R_2 \to R_1' \\ \text{and } R_3 + (-2)R_2 \to R_3' \end{array}$$

The system (2) is reduced to equivalent system

$$x + 3z = 4$$
$$y - z = -3$$
$$0z = 0$$

The equation $0z = 0$ is satisfied by any value of z.

From the first and second equations, we get

$$x = -3z + 4 \qquad \text{.......(a)}$$

and $$y = z - 3 \qquad \text{........(b)}$$

As z is arbitrary, so we can find infinitely many values of x and y from equation (a) and (b) or the system (2), is satisfied by $x = 4 - 3t$, $y = t - 3$ and $z = t$ for any real value of t.

Thus the system (2) has infinitely many solutions and it is consistent.

The augmented matrix of the system (3) is $\begin{bmatrix} 1 & -1 & 2 & \vdots & 1 \\ 2 & -6 & 5 & \vdots & 7 \\ 3 & 5 & 4 & \vdots & -3 \end{bmatrix}$

Adding $(-2)R_1$, to R_2 and $(-3)R_1$, to R_3 we have

$$\begin{bmatrix} 1 & -1 & 2 & \vdots & 1 \\ 2 & -6 & 5 & \vdots & 7 \\ 3 & 5 & 4 & \vdots & -3 \end{bmatrix} \qquad \underset{\sim}{R}\begin{bmatrix} 1 & -1 & 2 & \vdots & 1 \\ 0 & -4 & 1 & \vdots & 5 \\ 0 & 8 & -2 & \vdots & -6 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 2 & \vdots & 1 \\ 0 & 1 & -\frac{1}{4} & \vdots & -\frac{5}{4} \\ 0 & 8 & -2 & \vdots & -6 \end{bmatrix} \text{By } -\frac{1}{4}R_2 \to R_2' \quad \underset{\sim}{R}\begin{bmatrix} 1 & 0 & \frac{7}{4} & \vdots & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{4} & \vdots & -\frac{5}{4} \\ 0 & 0 & 0 & \vdots & 4 \end{bmatrix} \begin{array}{l} \text{By } R_1 + 1.R_2 \to R_1' \\ \text{and } R_3 + (-8)R_2 \to R_3' \end{array}$$

Thus the system (3) is reduced to the equivalent system

$$x + \frac{7}{4}z = \frac{1}{4}$$
$$y - \frac{1}{4}z = \frac{5}{4}$$
$$0z = 4$$

The third equation $0z = 4$ has no solution, so the system as a whole has no solution. Thus the system is inconsistent.

We see that in the case of the system (1), the (row) rank of the augmented matrix and the coefficient matrix of the system is the same, that is, 3 which is equal to the number of the variables in the system (1).

Thus a linear system is consistent and has a unique solution if the (row) rank of the coefficient matrix is the same as that of the augmented matrix of the system.

In the case of the system (2), the (row) rank of the coefficient matrix is the same as that of the augmented matrix of the system but it is 2 which is less than the number of variables in the system (2).

Thus a system is consistent and has infinitely many solutions if the (row) ranks of the coefficient matrix and the augmented matrix of the system are equal but the rank is less than the number of variables in the system.

In the case of the system (3), we see that the (row) rank of the coefficient matrix is not equal to the (row) rank of the augmented matrix of the system.

Thus we conclude that a system is inconsistent if the (row) ranks of the coefficient matrix and the augmented matrix of the system are different.

3.11.1 Homogeneous Linear Equations

Each equation of the system of following linear equations:

$$\left.\begin{array}{ll} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0 & \text{.......(i)} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0 & \text{.......(ii)} \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0 & \text{.......(iii)} \end{array}\right\}$$

is always satisfied by $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$, so such a system is always consistent. The solution (0, 0, 0) of the above homogeneous equations (i), (ii), and (iii) is called the trivial solution. Any other solution of equations (i), (ii) and (iii) other than the trivial solution is called a **non-trivial**
solution. The above system can be written as

$$AX = O \text{, where } O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $|A| \neq 0$, then A is non-singular and A^{-1} exists, that is,

$$A^{-1}(AX) = A^{-1}O = O$$

or $\quad (A^{-1}A)X = O \Rightarrow X = O$, i.e., $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

In this case the system of homogeneous equations possesses only the trivial solution.

Now we consider the case when the system has a non-trivial solution.

Multiplying the equations (i), (ii) and (iii) by A_{11}, A_{21} and A_{31} respectively and adding the resulting equations (where A_{11}, A_{21} and A_{31} are cofactors of the corresponding elements of A), we have

$(a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31})x_1 + (a_{12}A_{11} + a_{22}A_{21} + a_{32}A_{31})x_2 + (a_{13}A_{11} + a_{23}A_{21} + a_{33}A_{31})x_3 = 0$, that is, $|A|x_1 = 0$. Similarly, we can get $|A|x_2 = 0$ and $|A|x_3 = 0$

For a non-trivial solution, at least one of x_1, x_2 and x_3 is different from zero. Let $x_1 \neq 0$, then from $|A|x_1 = 0$, we have $|A| = 0$.

For example, the system

$$\left.\begin{array}{lll} x_1 + x_2 + x_3 & = 0 & (\text{I}) \\ x_1 - x_2 + 3x_3 & = 0 & (\text{II}) \\ x_1 + 3x_2 - x_3 & = 0 & (\text{III}) \end{array}\right\}$$

has a non-trivial solution because

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 1 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & 2 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 2 \\ 2 & -2 \end{vmatrix} = 0$$

Solving the first two equations of the system, we have

$2x_1 + 4x_3 = 0 \qquad$ (adding (I) and (II))

$\Rightarrow \quad x_1 = -2x_3$

and $2x_2 - 2x_3 = 0 \qquad$ (subtracting (II) from (I))

$\Rightarrow \quad x_2 = x_3$

Putting $x_1 = -2x_3$ and $x_2 = x_3$ in (III), we see that $(-2x_3) + 3(x_3) - x_3 = 0$, which shows that the equation (I), (II) and (III) are satisfied by

$x_1 = -2t$, $x_2 = t$ and $x_3 = t$ for any real value of t.

Thus the system consisting of (I), (II) and (III) has infinitely many solutions. But the system

$$\left.\begin{array}{ll} x_1 + x_2 + x_3 & = 0 \\ x_1 - x_2 + 3x_3 & = 0 \\ x_1 + 3x_2 - 2x_3 & = 0 \end{array}\right\}$$ has only the trivial solution,

because in this case

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 1 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} -2 & 2 \\ 2 & -3 \end{vmatrix} = 6 - 4 = 2 \neq 0$$

Solving the first two equations of the above system, we get $x_1 = -2x_3$ and $x_2 = x_3$. Putting $x_1 = -2x_3$ and $x_2 = x_3$ in the expression.

$x_1 + 3x_2 - 2x_3$, we have $-2x_3 + 3(x_3) - 2x_3 = -x_3$, that is,

the third equation is not satisfied by putting $x_1 = -2x_3$ and $x_2 = x_3$ but it is satisfied only if $x_3 = 0$. Thus the above system has only the trivial solution.

3.11.2 Non-Homogeneous Linear Equations

Now we will solve the systems of non-homogeneous linear equations with help of the following methods.

i) Using matrices, that is, $AX = B \Rightarrow X = A^{-1}B$.

ii) Using echelon and reduced echelon forms

iii) Using Cramer's rule.

Example 1: Use matrices to solve the system $\left.\begin{array}{ll} x_1 - 2x_2 + x_3 & - = 4 \\ 2x_1 - 3x_2 + 2x_3 & - = 6 \\ 2x_1 + 2x_2 + x_3 & = 5 \end{array}\right\}$

Solution: The matrix form of the given system is

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \\ 5 \end{bmatrix}$$

or $\quad AX = B \qquad$ (i)

where $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} -4 \\ 6 \\ 5 \end{bmatrix}$

As $\quad |A| = \begin{vmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{vmatrix}$ By $R_2 + (-2)R_1 \to R_2'$

$$= (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = (1-2) = -1 \text{ , that is,}$$

$|A| \neq 0$, so the inverse of A exists and (i) can be written as

$X = A^{-1}B \qquad$ (ii)

Now we find adj A.

Since $\quad [A_{ij}]_{3\times 3} = \begin{bmatrix} -7 & 2 & 10 \\ 4 & -1 & -6 \\ -1 & 0 & 1 \end{bmatrix}, \left(\begin{array}{l} \because A_{11} = -7, A_{12} = 2, A_{13} = 10, A_{21} = 4 \\ A_{22} = -1, A_{23} = -6, A_{31} = 1, A_{32} = 0, A_{33} = 1 \end{array} \right)$

So $\quad$ adj $A = \begin{bmatrix} -7 & 4 & -1 \\ 2 & -1 & 0 \\ 10 & -6 & 1 \end{bmatrix}$

and $\quad A^{-1} = \dfrac{1}{|A|}\text{adj}A = \dfrac{1}{-1}\begin{bmatrix} -7 & 4 & -1 \\ 2 & -1 & 0 \\ 10 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -4 & 1 \\ 2 & 1 & 0 \\ -10 & 6 & -1 \end{bmatrix}$

Thus $\quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}\begin{bmatrix} -4 \\ -6 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 & -4 & 1 \\ -2 & 1 & 0 \\ -10 & 6 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ -6 \\ 5 \end{bmatrix} = \begin{bmatrix} -28+24+5 \\ 8-6+0 \\ 40-36-5 \end{bmatrix}$, i.e.,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Hence $x_1 = 1$, $x_2 = 2$ and $x_3 = -1$.

Example 2: Solve the system;

$$\left.\begin{array}{r} x_1 + 3x_2 + 2x_3 = 3 \\ 4x_1 + 5x_2 - 3x_3 = -3 \\ 3x_1 - 2x_2 + 17x_3 = 42 \end{array}\right\},$$

by reducing its augmented matrix to the echelon form and the reduced echelon form.

Solution: The augmented matrix of the given system is

$$\left[\begin{array}{ccc:c} 1 & 3 & 2 & 3 \\ 4 & 5 & -3 & -3 \\ 3 & -2 & 17 & 42 \end{array}\right]$$

We reduce the above matrix by applying elementary row operations, that is,

$$\left[\begin{array}{ccc:c} 1 & 3 & 2 & 3 \\ 4 & 5 & -3 & -3 \\ 3 & -2 & 17 & 42 \end{array}\right] \underset{\sim}{R} \left[\begin{array}{ccc:c} 1 & 3 & 2 & 3 \\ 0 & -7 & -11 & -15 \\ 0 & -11 & 11 & 33 \end{array}\right] \begin{array}{l} \text{By } R_2 + (-4)R_1 \to R_2' \\ \text{and } R_3 + (-3)R_1 \to R_3' \end{array}$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} 1 & 3 & 2 & 3 \\ 0 & -11 & 11 & 33 \\ 0 & -7 & -11 & -15 \end{array}\right] \text{By } R_2 \leftrightarrow R_3$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} 1 & 3 & 2 & 3 \\ 0 & 1 & -1 & -3 \\ 0 & -7 & -11 & -15 \end{array}\right] \text{By } \left(-\frac{1}{11}\right)R_2 \to R_2' \;\; \underset{\sim}{R} \left[\begin{array}{ccc:c} 1 & 3 & 2 & 3 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & -18 & -36 \end{array}\right] \text{By } R_3 + 7R_2 \to R_3'$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:c} 1 & 3 & 2 & 3 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \text{By} \left(-\frac{1}{18}\right)R_3 \to R_3'$$

The equivalent system in the (row) echelon form is

$$\left.\begin{aligned} x_1 + 3x_2 + 2x_3 &= 3 \\ x_2 - x_3 -&= 3 \\ x_3 &= 2 \end{aligned}\right\}$$

Substituting $x_3 = 2$ in the second equation gives: $x_2 - 2 = -3 \Rightarrow x_2 = -1$
Putting $x_2 = -1$ and $x_3 = 2$ in the first equation, we have

$$x_1 + 3(-1) + 2(2) = 3 \Rightarrow x_1 = 3 + 3 - 4 = 2.$$

Thus the solution is $x_1 = 2$, $x_2 = -1$ and $x_3 = 2$

Now we reduce the matrix $\begin{bmatrix} 1 & 3 & 2 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & 0 & -1 & \vdots & 2 \end{bmatrix}$ to reduced (row) echelon form, i.e.,

$$\begin{bmatrix} 1 & 3 & 2 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 0 & 5 & \vdots & 12 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix} \quad \text{By } R_1 + (-3)R_2 \to R_1'$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix} \quad \begin{aligned} &\text{By } R_1 + (-5)R_3 \to R_1' \\ &\text{and} R_2 + 1.R_3 \to R_2' \end{aligned}$$

The equivalent system in the reduced (row) echelon form is

$x_1 = 2$
$x_2 = -1$
$x_3 = 2$

which is the solution o f the given system.

3.12 Cramer's Rule

Consider the system of equations,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \qquad (1)$$

These are three linear equations in three variables x_1, x_2, x_3 with coefficients and constant terms in the real field R. We write the above system of equations in matrix form as: $\quad AX = B \qquad (2)$

where $\quad A = [a_{ij}]_{3\times3}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

We know that
the matrix equation (2) can be written as: $X = A^{-1}B$ (if A^{-1} exists)

Note: $A^{-1}(AX) \neq BA^{-1}$

We have already proved that $A^{-1} = \dfrac{1}{|A|}\text{adj } A$ and

$$adj\ A = [A_{ij}']_{3\times3} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \qquad (\because A_{ij}' \quad A_{ji})$$

Thus $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \dfrac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \dfrac{1}{|A|}\begin{bmatrix} A_{11}b_1 + A_{21}b_2 + A_{31}b_3 \\ A_{12}b_1 + A_{22}b_2 + A_{32}b_3 \\ A_{13}b_1 + A_{23}b_2 + A_{33}b_3 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \dfrac{A_{11}b_1 + A_{21}b_2 + A_{31}b_3}{|A|} \\ \dfrac{A_{12}b_1 + A_{22}b_2 + A_{32}b_3}{|A|} \\ \dfrac{A_{13}b_1 + A_{23}b_2 + A_{33}b_3}{|A|} \end{bmatrix}$$

Hence $x_1 = \dfrac{b_1A_{11} + b_2A_{21} + b_3A_{31}}{|A|} = \dfrac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|A|} \qquad$ (i)

$$x_2 = \frac{b_1A_{12}+b_2A_{22}+b_3A_{32}}{|A|} = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{|A|} \qquad \text{(ii)}$$

$$x_3 = \frac{b_1A_{13}+b_2A_{23}+b_3A_{33}}{|A|} = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{|A|} \qquad \text{(iii)}$$

The method of solving the system with the help of results (i), (ii) and (iii) is often referred to as **Cramer's Rule.**

Example 3: Use Crammer's rule to solve the system. $\left.\begin{array}{r} 3x_1+x_2-x_3=-4 \\ x_1+x_2-2x_3=-4 \\ -x_1+2x_2-x_3=1 \end{array}\right\}$

Solution: Here $|A| = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 3(-1+4)-1.(-1-2)-1.(2+1)$

= 9 + 3 – 3 = 9

So $$x_1 = \frac{\begin{vmatrix} -4 & 1 & -1 \\ -4 & 1 & -2 \\ 1 & 2 & -1 \end{vmatrix}}{9} = \frac{-4(-1+4)-1(4+2)-1(-8-1)}{9}$$

$$= \frac{-12-6+9}{9} = \frac{-9}{9} = 1$$

$$x_2 = \frac{\begin{vmatrix} 3 & -4 & -1 \\ 1 & -4 & -2 \\ -1 & 1 & -1 \end{vmatrix}}{9} = \frac{3(4+2)+4(-1-2)-1(1-4)}{9}$$

$$= \frac{18-12+3}{9} = \frac{9}{9} = 1$$

$$x_3 = \frac{\begin{vmatrix} 3 & 1 & -4 \\ 1 & 1 & -4 \\ -1 & 2 & 1 \end{vmatrix}}{9} = \frac{3(1+8)-1(1-4)-(2+1)}{9}$$

$$= \frac{27+3-12}{9} = \frac{18}{9} = 2$$

Hence $x_1 = -1$, $x_2 = 1$, $x_3 = 2$

Exercise 3.5

1. Solve the following systems of linear equations by Cramer's rule.

i) $\left.\begin{array}{r} 2x+2y+z=3 \\ 3x-2y-2z=1 \\ 5x+y-3z=2 \end{array}\right\}$ ii) $\left.\begin{array}{r} 2x_1-x_2+x_3=5 \\ 4x_1+2x_2+3x_3=8 \\ 3x_1-4x_2-x_3=3 \end{array}\right\}$ iii) $\left.\begin{array}{r} 2x_1-x_2+x_3=8 \\ x_1+2x_2+2x_3=6 \\ x_1-2x_2-x_3=1 \end{array}\right\}$

2. Use matrices to solve the following systems:

i) $\left.\begin{array}{r} x-2y+z=-1 \\ 3x+y-2z=4 \\ y-z=1 \end{array}\right\}$ ii) $\left.\begin{array}{r} 2x_1+x_2+3x_3=3 \\ x_1+x_2-2x_3=0 \\ -3x_1-x_2+2x_3=-4 \end{array}\right\}$ iii) $\left.\begin{array}{r} x+y=2 \\ 2x-z=1 \\ 2y-3z=-1 \end{array}\right\}$

3. Solve the following systems by reducing their augmented matrices to the echelon form and the reduced echelon forms.

i) $\left.\begin{array}{r} x_1-2x_2-2x_3=1 \\ 2x_1+3x_2+x_3=1 \\ 5x_1-4x_2-3x_3=1 \end{array}\right\}$ ii) $\left.\begin{array}{r} x+2y+z=2 \\ 2x+y+2z=-1 \\ 2x+3y-z=9 \end{array}\right\}$ iii) $\left.\begin{array}{r} x_1+4x_2+2x_3=2 \\ 2x_1+x_2-2x_3=9 \\ 3x_1+2x_2-2x_3=12 \end{array}\right\}$

4. Solve the following systems of homogeneous linear equations.

i) $\left.\begin{array}{r} x+2y-2z=0 \\ 2x+y+5z=0 \\ 5x+4y+8z=0 \end{array}\right\}$ ii) $\left.\begin{array}{r} x_1+4x_2+2x_3=0 \\ 2x_1+x_2-3x_3=0 \\ 3x_1+2x_2-4x_3=0 \end{array}\right\}$ iii) $\left.\begin{array}{r} x_1-2x_2-x_3=0 \\ x_1+x_2+5x_3=0 \\ 2x_1-x_2+4x_3=0 \end{array}\right\}$

5. Find the value of λ for which the following systems have non-trivial solutions. Also solve the system for the value of λ.

i) $$\left.\begin{aligned} x+y+z&=0\\ 2x+y-\lambda z&=0\\ x+2y-2z&=0 \end{aligned}\right\}$$

ii) $$\left.\begin{aligned} x_1+4x_2+\lambda x_3&=0\\ 2x_1+x_2-3x_3&=0\\ 3x_1+\lambda x_2-4x_3&=0 \end{aligned}\right\}$$

6. Find the value of λ for which the following system does not possessa unique solution. Also solve the system for the value of λ.

$$\left.\begin{aligned} x_1+4x_2+\lambda x_3&=2\\ 2x_1+x_2-2x_3&=11\\ 3x_1+2x_2-2x_3&=16 \end{aligned}\right\}$$