CHAPTER

4 Quadratic Equations

Animation 4.1: Completing the square
Source & Credit: 1ucasvb

4.1 Introduction

A *quadratic equation* in x is an equation that can be written in the form $ax^2 + bx + c = 0$; where a, b and c are real numbers and
$a \neq 0$.

Another name for a quadratic equation in x is **2nd Degree Polynomial** in x.

The following equations are the quadratic equations:

i)	$x^2 - 7x + 10 = 0$;	$a = 1$,	$b = -7$,	$c = 10$
ii)	$6x^2 + x - 15 = 0$;	$a = 6$,	$b = 1$,	$c = -15$
iii)	$4x^2 + 5x + 3 = 0$;	$a = 4$,	$b = 5$,	$c = 3$
iv)	$3x^2 - x = 0$;	$a = 3$,	$b = -1$,	$c = 0$
v)	$x^2 = 4$;	$a = 1$,	$b = 0$,	$c = -4$

4.1.1 Solution of Quadratic Equations

There are three basic techniques for solving a quadratic equation:

i) by factorization.
ii) by completing squares, extracting square roots.
iii) by applying the quadratic formula.

By Factorization: It involves factoring the polynomial $ax^2 + bx + c$.

It makes use of the fact that if $ab = 0$, then $a = 0$ or $b = 0$.

For example, if $(x - 2)(x - 4) = 0$, then either $x - 2 = 0$ or $x - 4 = 0$.

Example 1: Solve the equation $x^2 - 7x + 10 = 0$ by factorization.

Solution: $x^2 - 7x + 10 = 0$

$\Rightarrow$ $(x - 2)(x - 5) = 0$

$\therefore$ either $x - 2 = 0$ $\Rightarrow x = 2$

or $x - 5 = 0$ $\Rightarrow x = 5$

$\therefore$ the given equation has two solutions: 2 and 5

$\therefore$ solution set = $\{2, 5\}$

Note: The solutions of an equation are also called its roots.

$\therefore$ 2 and 5 are roots of $x^2 - 7x + 10 = 0$

By Completing Squares, then Extracting Square Roots:

Sometimes, the quadratic polynomials are not easily factorable.

For example, consider $x^2 + 4x - 437 = 0$.

It is difficult to make factors of $x^2 + 4x - 437$. In such a case the factorization and hence the solution of quadratic equation can be found by the method of completing the square and extracting square roots.

Example 2: Solve the equation $x^2 + 4x - 437 = 0$ by completing the squares.

Solution : $x^2 + 4x - 437 = 0$

$$\Rightarrow \quad x^2 + 2\left(\frac{4}{2}\right)x = 437$$

Add $\left(\frac{4}{2}\right)^2 = (2)^2$ to both sides

$x^2 + 4x + (2)^2 = 437 + (2)^2$

$\Rightarrow$ $(x + 2)^2 = 441$

$\Rightarrow$ $x + 2 = \pm\sqrt{441} = \pm 21$

$\Rightarrow$ $x = \pm 21 - 2$

$\therefore$ $x = 19$ or $x = -23$

Hence solution set = $\{-23, 19\}$.

By Applying the Quadratic Formula:

Again there are some quadratic polynomials which are not factorable at all using integral coefficients. In such a case we can always find the solution of a quadratic equation $ax^2+bx+c = 0$ by applying a formula known as quadratic formula. This formula is applicable for every quadratic equation.

Derivation of the Quadratic Formula

Standard form of quadratic equation is

$ax^2 + bx + c = 0$, $a \neq 0$

Step 1. Divide the equation by a

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Step 2. Take constant term to the R.H.S.

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

Step 3. To complete the square on the L.H.S. add $\left(\frac{b}{2a}\right)^2$ to both sides.

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$

$$\Rightarrow \quad \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$\Rightarrow \quad x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \quad x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Hence the solution of the quadratic equation $ax^2+bx+c=0$ is given by

$$\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

which is called **Quadratic Formula.**

Example 3: Solve the equation $6x^2+x-15=0$ by using the quadratic formula.

Solution: Comparing the given equation with $ax^2+bx+c=0$, we get,

$a=6$, $b=1$, $c=-15$

$\therefore$ The solution is given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-1\pm\sqrt{1^2-4(6)(-15)}}{2(6)}$$

$$=\frac{-1\pm\sqrt{361}}{12}=\frac{-1\pm19}{12}$$

i.e., $x=\frac{-1+19}{12}$ or $x=\frac{-1-19}{12}$

$x=\frac{3}{2}$ or Hence soulation set $=\left\{\frac{3}{2},\frac{-5}{3}\right\}$

Example 4: Solve the $8x^2-14x-15=0$ by using the quadratic formula.

Solution: Comparing the given equation with $ax^2+bx+c=0$, we get,

$a=8$, $b=-14$, $c=-15$

By the quadratic formula, we have

$$\therefore \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\therefore \quad x=\frac{-(-14)\pm\sqrt{(-14)^2-4(8)(-15)}}{2(8)}$$

$$=\frac{14\pm\sqrt{676}}{16}=\frac{14\pm26}{16}$$

$\therefore$ either $x=\frac{14+26}{16}\Rightarrow x=\frac{5}{2}$

or $\Rightarrow$

Hence solution set $=\left\{\frac{5}{2},-\frac{3}{4}\right\}$

Exercise 4.1

Solve the following equations by factorization:

1. $3x^2+4x+1=0$ **2.** $x^2+7x+12=0$

3. $9x^2-12x-5=0$ **4.** $x^2-x=2$

5. $x(x+7) = (2x-1)(x+4)$

6. $\frac{x}{x+1} + \frac{x+1}{x} = \frac{5}{2};\ x \neq -1, 0$

7. $\frac{1}{x+1} + \frac{2}{x+2} = \frac{7}{x+5};\ x \neq -1, -2, -5$

8. $\frac{a}{ax-1} + \frac{b}{bx-1} = a+b;\ x \neq \frac{1}{a}, \frac{1}{b}$

Solve the following equations by completing the square:

9. $x^2 - 2x - 899 = 0$ **10.** $x^2 + 4x - 1085 = 0$
11. $x^2 + 6x - 567 = 0$ **12.** $x^2 - 3x - 648 = 0$
13. $x^2 - x - 1806 = 0$ **14.** $2x^2 + 12x - 110 = 0$

Find roots of the following equations by using quadratic formula:

15. $5x^2 - 13x + 6 = 0$ **16.** $4x^2 + 7x - 1 = 0$
17. $15x^2 + 2ax - a^2 = 0$ **18.** $16x^2 + 8x + 1 = 0$
19. $(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$
20. $(a+b)x^2 + (a+2b+c)x + b + c = 0$

4.2 Solution of Equations Reducible to the Quadratic Equation

There are certain types of equations, which do not look to be of degree 2, but they can be reduced to the quadratic form. We shall discuss the solutions of such five types of the equations one by one.

Type I: The equations of the form: $ax^{2n} + bx^n + c = 0;\ a \neq 0$

Put $x^n = y$ and get the given equation reduced to quadratic equation in y.

Example 1: Solve the equation: $x^{\frac{1}{2}} - x^{\frac{1}{4}} - 6 = 0.$

Solution This given equation can be written as $(x^{\frac{1}{4}})^2 - x^{\frac{1}{4}} - 6 = 0$

Let $x^{\frac{1}{4}} = y$

$\therefore$ The given equation becomes

$y^2 - y - 6 = 0$

$\Rightarrow (y-3)(y+2) = 0$

$\Rightarrow y = 3,$ or $y = -2$

$\therefore x^{\frac{1}{4}} = 3$ $\quad x^{\frac{1}{4}} = -2$

$\Rightarrow x = (3)^4$ $\quad \Rightarrow x = (-2)^4$

$\Rightarrow x = 81$ $\quad \Rightarrow x = 16$

Hence solution set is {16, 81}.

Type II: The equation of the form: $(x+a)(x+b)(x+c)(x+d) = k$
where $a + b = c + d$

Example 2: Solve $(x-7)(x-3)(x+1)(x+5) - 1680 = 0$

Solution: $(x-7)(x-3)(x+1)(x+5) - 1680 = 0$

$\Rightarrow [(x-7)(x+5)][(x-3)(x+1)] - 1680 = 0$ (by grouping)

$\Rightarrow (x^2 - 2x - 35)(x^2 - 2x - 3) - 1680 = 0$

Putting $x^2 - 2x = y$, the above equation becomes

$(y-35)(y-3) - 1680 = 0$

$\Rightarrow y^2 - 38y + 105 - 1680 = 0$

$\Rightarrow y^2 - 38y - 1575 = 0$

$\therefore y = \frac{38 \pm \sqrt{1444 + 6300}}{2} = \frac{38 \pm \sqrt{7744}}{2}$ (by quadratic formula)

$= \frac{38 \pm 88}{2}$

$\Rightarrow \quad y = 63 \qquad$ or $\qquad y = -25.$

$\Rightarrow \quad x^2 - 2x = 63 \qquad\qquad \Rightarrow x^2 - 2x = -25$

$\Rightarrow \quad x^2 - 2x - 63 = 0 \qquad\qquad \Rightarrow x^2 - 2x + 25 = 0$

$\Rightarrow \quad (x + 7)(x - 9) = 0 \qquad\qquad \Rightarrow x = \dfrac{2 \pm \sqrt{4 - 100}}{2}$

$\Rightarrow \quad x = -7$ or $x = 9 \qquad\qquad = \dfrac{2 \pm \sqrt{-96}}{2}$

$$= \frac{2 \pm 4\sqrt{6}\,i}{2} = 1 \pm 2\sqrt{6}\,i$$

$\Rightarrow$ or

Hence Solution set = $\{-7, 9, 1 + 2\sqrt{6}\,i, 1 - 2\sqrt{6}\,i\}$

Type III: Exponential Equations: Equations, in which the variable occurs in exponent, are called **exponential equations**. The method of solving such equations is explained by the following examples.

Example 3: Solve the equation: $2^{2x} - 3.2^{x+2} + 32 = 0$

Solution: $2^{2x} - 3.2^{x+2} + 32 = 0$

$\Rightarrow \quad 2^{2x} - 3.2^2 . 2^x + 32 = 0$

$\Rightarrow \quad 2^{2x} - 12.2^x + 32 = 0$

$\Rightarrow \quad y^2 - 12y + 32 = 0 \qquad$ (Putting $2^x = y$)

$\Rightarrow \quad (y - 8)(y - 4) = 0$

$\Rightarrow \quad y = 8 \qquad$ or $\qquad y = 4$

$\Rightarrow \quad 2^x = 8 \qquad\qquad \Rightarrow 2^x = 4$

$\Rightarrow \quad 2^x = 2^3 \qquad\qquad \Rightarrow 2^x = 2^2$

$\Rightarrow \quad x = 3 \qquad\qquad \Rightarrow x = 2$

Hence solution set = {2, 3}.

Example 4: Solve the equation: $4^{1+x} + 4^{1-x} = 10$

Solution: Given that

$4^{1+x} + 4^{1-x} = 10$

$\Rightarrow \quad 4.4^x + 4.4^{-x} = 10$

Let $\quad 4^x = y \Rightarrow 4^{-x} = (4^x)^{-1} = y^{-1} = \dfrac{1}{y}$

$\therefore$ The given equation becomes

$$4y + \frac{4}{y} - 10 = 0$$

$\Rightarrow \quad 4y^2 - 10y + 4 = 0$

$\Rightarrow \quad 2y^2 - 5y + 2 = 0$

$\therefore \quad y = \dfrac{5 \pm \sqrt{25 - 4(2)(2)}}{2(2)} = \dfrac{5 \pm \sqrt{9}}{4} = \dfrac{5 \pm 3}{4}$

$\Rightarrow \quad y = 2 \qquad$ or $\qquad y = \dfrac{1}{2}$

$\therefore \quad 4^x = 2 \qquad\qquad \therefore \quad 4^x = \dfrac{1}{2}$

$\Rightarrow \quad 2^{2x} = 2^1$

$\Rightarrow \quad 2x = 1 \qquad\qquad \Rightarrow 2^{2x} = 2^{-1}$

$\qquad\qquad\qquad\qquad \Rightarrow 2x = -1$

$\Rightarrow \quad x = \dfrac{1}{2} \qquad\qquad \Rightarrow$

Hence Solution set = $\left\{\dfrac{1}{2}, -\dfrac{1}{2}\right\}$.

Type IV: Reciprocal Equations: An equation, which remains unchanged when x is replaced by is called a reciprocal equation. In such an equation the coefficients of the terms equidistant from the beginning and end are equal in magnitude. The method of solving such equations is explained through the following example:

Example 5: Solve the equation

$x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$;

Solution: Given that:

$x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$

$\Rightarrow \quad x^2 - 3x + 4 - \frac{3}{x} + \frac{1}{x^2} = 0$ (Dividing by x^2)

$\Rightarrow \quad \left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) + 4 = 0$ (1)

Let $\Rightarrow$

So, the equation (1) reduces to

$y^2 - 2 - 3y + 4 = 0$

$\Rightarrow \quad y^2 - 3y + 2 = 0$

$\Rightarrow \quad (y - 2)(y - 1) = 0$

$\Rightarrow \quad y = 2$ or $y = 1$

$\Rightarrow \quad x + \frac{1}{x} = 2 \qquad \Rightarrow \quad x + \frac{1}{x} = 1$

$\Rightarrow \quad x^2 - 2x + 1 = 0 \qquad \Rightarrow \quad x^2 - x + 1 = 0$

$\Rightarrow \quad (x - 1)^2 = 0 \qquad \Rightarrow$

$\Rightarrow \quad (x - 1)(x - 1) = 0$

$\Rightarrow \quad x = 1, 1 \qquad \Rightarrow$

Hence Solution set

Exercise 4.2

Solve the following equations:

1. $x^4 - 6x^2 + 8 = 0$ **2.** $x^{-2} - 10 = 3x^{-1}$
3. $x^{-6} - 9x^3 + 8 = 0$ **4.** $8x^6 - 19x^3 - 27 = 0$
5. $x^{\frac{2}{5}} + 8 = 6x^{\frac{1}{5}}$ **6.** $(x + 1)(x + 2)(x + 3)(x + 4) = 24$
7. $(x - 1)(x + 5)(x + 8)(x + 2) - 880 = 0$
8. $(x - 5)(x - 7)(x + 6)(x + 4) - 504 = 0$
9. $(x - 1)(x - 2)(x - 8)(x + 5) + 360 = 0$
10. $(x + 1)(2x + 3)(2x + 5)(x + 3) = 945$
Hint: $(x + 1)(2x + 5)(2x + 3)(x + 3) = 945$
11. $(2x - 7)(x^2 - 9)(2x + 5) - 91 = 0$
12. $(x^2 + 6x + 8)(x^2 + 14x + 48) = 105$
13. $(x^2 + 6x - 27)(x^2 - 2x - 35) = 385$
14. $4 \cdot 2^{2x+1} - 9.2^x + 1 = 0$
15. $2^x + 2^{-x+6} - 20 = 0$
16. $4^x - 3.2^{x+3} + 128 = 0$
17. $3^{2x-1} - 12.3^x + 81 = 0$ **18.**
19. $x^2 + x - 4 + \frac{1}{x} + \frac{1}{x^2} = 0$ **20.** $\left(x - \frac{1}{x}\right)^2 + 3\left(x + \frac{1}{x}\right) = 0$
21. $2x^4 - 3x^3 - x^2 - 3x + 2 = 0$ **22.** $2x^4 + 3x^3 - 4x^2 - 3x + 2 = 0$
23. $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$ **24.** $x^4 - 6x^2 + 10 - \frac{6}{x^2} + \frac{1}{x^4} = 0$

Type V: Radical Equations: Equations involving **radical expressions** of the variable are called radical equations. To solve a radical equation, we first obtain an equation free from radicals. Every solution of radical equation is also a solution of the radical-free equation but the new equation have solutions that are not solutions of the original radical equation.

Such extra solutions (roots) are called **extraneous roots**. The method of the solution of different types of radical equations is illustrated by means of the followings examples:

i) The Equations of the form: $l(ax^2+bx)+m\sqrt{ax^2+bx+c}=0$

Example 1: Solve the equation

$$3x^2+15x-2\sqrt{x^2+5x+1}=2$$

Solution : Let $\sqrt{x^2+5x+1}=y$

$\Rightarrow$ $x^2+5x+1=y^2$

$\Rightarrow$ $x^2+5x=y^2-1$

$\Rightarrow$ $3x^2+15x=3y^2-3$

$\therefore$ The given equation becomes $3y^2-3-2y=2$

$\Rightarrow$ $3y^2-2y-5=0$

$\Rightarrow$ $(3y-5)(y+1)=0$

$\Rightarrow$ $y=\frac{5}{3}$ or $y=-1$

$\Rightarrow$ $\sqrt{x^2+5x+1}=\frac{5}{3}$ $\Rightarrow$ $\sqrt{x^2+5x+1}=-1$

$\Rightarrow$ $x^2+5x+1=\frac{25}{9}$ $\Rightarrow$ $x^2+5x+1=1$

$\Rightarrow$ $9x^2+45x+9=25$ $\Rightarrow$ $x^2+5x=0$

$\Rightarrow$ $9x^2+45x-16=0$ $\Rightarrow$ $x(x+5)=0$

$\Rightarrow$ $(3x+16)(3x-1)=0$ $\therefore$ $x=0$ or $x=-5$

$\therefore$ $x=\frac{1}{3}$ or $x=-\frac{16}{3}$

On checking, it is found that 0 and – 5 do not satisfy the given equation. Therefore 0 and –5 being extraneous roots cannot be included in solution set.

Hence solution set

ii) The Equation of the form : $\sqrt{x+a}+\sqrt{x+b}=\sqrt{x+c}$

Example 2: Solve the equation:

Solution: $\sqrt{x+8}+\sqrt{x+3}=\sqrt{12x+13}$

Squaring both sides, we get

$$x+8+x+3+2\sqrt{x+8}\sqrt{x+3}=12x+13$$

$\Rightarrow$ $2\sqrt{x+8}\sqrt{x+3}=10x+2$

$\Rightarrow$ $\sqrt{(x+8)(x+3)}=5x+1$

Squaring again, we have

$x^2+11x+24=25x^2+10x+1$

$\Rightarrow$ $24x^2-x-23=0$

$\Rightarrow$ $(24x+23)(x-1)=0$

$\Rightarrow$ $x=-\frac{23}{24}$ or $x=1$

On checking we find that $-\frac{23}{24}$ is an extraneous root. Hence solution set = {1}.

iii) The Equations of the form:

$$\sqrt{ax^2+bx+c}+\sqrt{px^2+qx+r}=\sqrt{lx^2+mx+n}$$

where ax^2+bx+c, px^2+qx+r and lx^2+mx+n have a common factor.

Example3: Solve the equation: $\sqrt{x^2+4x-21}+\sqrt{x^2-x-6}=\sqrt{6x^2-5x-39}$

Solution: Consider that:

$x^2+4x-21=(x+7)(x-3)$

$x^2-x-6=(x+2)(x-3)$

$6x^2-5x-39=(6x+13)(x-3)$

$\therefore$ The given equation can be written as

$$\sqrt{(x+7)(x-3)}+\sqrt{(x+2)(x-3)}=\sqrt{(6x+13)(x-3)}$$

$\Rightarrow$ $\sqrt{x-3}\left[\sqrt{x+7}+\sqrt{x+2}-\sqrt{6x+13}\right]=0$

$\therefore$ Either $\sqrt{x-3}=0$ or $\sqrt{x+7}+\sqrt{x+2}-\sqrt{6x+13}=0$

$\sqrt{x-3}=0 \Rightarrow x-3 = 0 \Rightarrow x = 3$

Now solve the equation $\sqrt{x+7}+\sqrt{x+2}-\sqrt{6x+13}=0$

$\Rightarrow \quad \sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$

$\Rightarrow \quad x+7+x+2+2\sqrt{(x+7)(x+2)}=6x+13$ (Squaring both sides)

$\Rightarrow \quad 2\sqrt{(x+7)(x+2)}=4x+4$

$\Rightarrow \quad \sqrt{x^2+9x+14}=2x+2$

$\Rightarrow \quad x^2+9x+14 = 4x^2+8x+4$ (Squaring both sides again)

$\Rightarrow \quad 3x^2-x-10=0$

$\Rightarrow \quad (3x+5)(x-2)=0$

$\Rightarrow \quad x=-\frac{5}{3},2$

Thus possible roots are 3, 2, $-\frac{5}{3}$.

On verification, it is found that $-\frac{5}{3}$ is an extraneous root. Hence solution set = {2, 3}

iv) The Equations of the form: $\sqrt{ax^2+bx+c}+\sqrt{px^2+qx+r}=mx+n$
where, $(mx+n)$ is a factor of $(ax^2+bx+c)-(px^2+qx+r)$

Example 4: Solve the equation: $\sqrt{3x^2-7x-30}-\sqrt{2x^2-7x-5}=x-5$

Solution: Let $\sqrt{3x^2-7x-30}=a$ and $\sqrt{2x^2-7x-5}=b$

Now $a^2-b^2=(3x^2-7x-30)-(2x^2-7x-5)$

$a^2-b^2=x^2-25$ (i)

The given equation can be written as:

$a-b=x-5$ (ii)

$\frac{(a+b)(a-b)}{a-b}=\frac{(x+5)(x-5)}{x-5}$ [From (i) and (ii)]

$\Rightarrow \quad a+b=x+5$ (iii)

$2a=2x$ [From (ii) and (iii)]

$\Rightarrow \quad a=x$

$\therefore \quad \sqrt{3x^2-7x-30}=x$

$\Rightarrow \quad 3x^2-7x-30=x^2$

$\Rightarrow \quad 2x^2-7x-30=0$

$\Rightarrow \quad (2x+5)(x-6)=0$

$\Rightarrow \quad x=-\frac{5}{2},6$

On checking, we find that $-\frac{5}{2}$ is an extraneous root.
Hence solution set = { 6 }

Exercise 4.3

Solve the following equations:

1. $3x^2+2x-\sqrt{3x^2+2x-1}=3$
2. $x^2---7=x-3\sqrt{2x^2-3x+2}$
3. $\sqrt{2x+8}+\sqrt{x+5}=7$
4. $\sqrt{3x+4}=2+\sqrt{2x-4}$
5. $\sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$
6. $\sqrt{x^2+x+1}-\sqrt{x^2+x-1}=1$
7. $\sqrt{x^2+2x-3}+\sqrt{x^2+7x-8}=\sqrt{5(x^2+3x-4)}$
8. $\sqrt{2x^2-5x-3}+3\sqrt{2x+1}=\sqrt{2x^2+25x+12}$
9. $\sqrt{3x^2-5x+2}+\sqrt{6x^2-11x+5}=\sqrt{5x^2-9x+4}$
10. $(x+4)(x+1)=\sqrt{x^2+2x-15}+3x+31$
11. $\sqrt{3x^2-2x+9}+\sqrt{3x^2-2x-4}=13$
12. $\sqrt{5x^2+7x+2}-\sqrt{4x^2+7x+18}=x-4$

4.3 Three Cube Roots of Unity

Let x be a cube root of unity

$\therefore \quad x=\sqrt[3]{1}=(1)^{\frac{1}{3}}$

$\Rightarrow \quad x^3=1$

$\Rightarrow \quad x^3-1=0$

$\Rightarrow \quad (x-1)(x^2+x+1)=0$

Either $x-1=0 \Rightarrow x=1$

or $x^2+x+1=0$

$$\therefore \quad x=\frac{-1\pm\sqrt{1-4}}{2} \quad =\frac{-1\pm\sqrt{-3}}{2}$$

$$\Rightarrow \quad x=\frac{-1\pm\sqrt{3}i}{2} (\because \sqrt{-1}=i)$$

Thus the three cube roots of unity are:

$$1, \frac{-1+\sqrt{3}i}{2} \text{ and } \frac{-1-\sqrt{3}i}{2}$$

Note: We know that the numbers containing i are called **complex** numbers. So $\frac{-1+\sqrt{3}i}{2}$ and $\frac{-1-\sqrt{3}i}{2}$ are called complex or imaginary cube roots of unity.

*By complex root we mean, a root containing non-zero imaginary part.

4.3.1 Properties of Cube Roots of Unity

i) Each complex cube root of unity is square of the other

Proof: (a) $\left(\frac{-1+\sqrt{3}\,i}{2}\right)^2 = \frac{(-1)^2+(\sqrt{3}\,i)^2+2(-1)(\sqrt{3}\,i)}{4}$

$$== \frac{1-3-2\sqrt{3}\,i}{4} \quad \frac{-2-2\sqrt{3}\,i}{4}$$

$$= 2\left(\frac{-1-\sqrt{3}\,i}{4}\right)$$

$$= \frac{-1-\sqrt{3}\,i}{2}$$

(b) $\left(\frac{-1-\sqrt{3}i}{2}\right)^2 = \left[-\frac{(1+\sqrt{3}i)}{2}\right]^2$

$$= \frac{(1)^2+(\sqrt{3}\,i)^2+(2)(1)(\sqrt{3}\,i)}{4}$$

$$== \frac{1-3+2\sqrt{3}i}{4} \quad \frac{-2+2\sqrt{3}i}{4}$$

$$= 2\left(\frac{-1+\sqrt{3}\,i}{4}\right)$$

$$= \frac{-1+\sqrt{3}\,i}{2}$$

Hence each complex cube root of unity is square of the other.

Note: if $\frac{-1+\sqrt{3}\,i}{2} = \omega$, then $\frac{-1-\sqrt{3}\,i}{2} = \omega^2$,

and if $\frac{-1-\sqrt{3}\,i}{2} = \omega$, then $\frac{-1+\sqrt{3}\,i}{2} = \omega^2$ [ω is read as omega]

ii) The Sum of all the three cube roots of unity is zeroi.e. 1 + ω + ω^2 = 0

Proof: We know that cube roots of unity are

$$1, \frac{-1+\sqrt{3}\,i}{2} \text{ and } \frac{-1-\sqrt{3}\,i}{2}$$

Sum of all the three cube roots = $1+\frac{-1+\sqrt{3}\,i}{2} + \frac{-1-\sqrt{3}\,i}{2}$

$$= \frac{2-1+\sqrt{3}\,i-1-\sqrt{3}\,i}{2} = \frac{0}{2} = 0$$

if $\omega == \frac{-1+\sqrt{3}i}{2}$, then ω^2 $\frac{-1-\sqrt{3}i}{2}$

Hence sum of cube roots of unity $= 1+\omega+\omega^2 = 0$

iii) The product of all the three cube roots of unity is unity i.e., $\omega^3 = 1$

Proof: Let $\frac{-1+\sqrt{3}i}{2} = \omega$ and $\frac{-1-\sqrt{3}i}{2} \quad \omega^2$

$$\therefore \quad 1.\omega.\omega^2 = \left(\frac{-1+\sqrt{3}i}{2}\right)\left(\frac{-1-\sqrt{3}i}{2}\right)$$

$$= \frac{(-1)^2 - (\sqrt{3}i)^2}{4}$$

$$= \frac{1-(-3)}{4} = \frac{1+3}{4}$$

$$\Rightarrow \quad \omega^3 = 1$$

$\therefore$ Product of the complex cube roots of unity $= \omega^3 = 1$.

iv) For any $n \in z, \omega^n$ is equivalent to one of the cube roots of unity.

With the help of the fact that $\omega^3 = 1$, we can easily reduce the higher exponent of ω to its lower equivalent exponent.

$$\begin{aligned}
\text{e.g. } \omega^4 &= \omega^3.\omega = 1.\omega = \omega \\
\omega^5 &= \omega^3.\omega^2 = 1.\omega^2 = \omega^2 \\
\omega^6 &= (\omega^3)^2 = (1)^2 = 1 \\
\omega^{15} &= (\omega^3)^5 = (1)^5 = 1 \\
\omega^{27} &= (\omega^3)^9 = (1)^9 = 1 \\
\omega^{11} &= \omega^9.\omega^2 = (\omega^3)^3.\ \omega^2 = (1)^3.\omega^2 = \omega^2 \\
\omega^{-1} &= \omega^{-3}.\omega^2 = (\omega^3)^{-1}.\omega^2 = \omega^2 \\
\omega^{-5} &= \omega^{-6}.\omega = (\omega^3)^{-2}.\omega = \omega \\
\omega^{-12} &= (\omega^3)^{-4} = (1)^{-4} = 1
\end{aligned}$$

Example 1: Prove that: $(x^3 + y^3) = (x+y)(x+\omega y)(x+\omega^2 y)$

Solution :

$$\begin{aligned}
\text{R.H.S} &= (x+y)(x+\omega y)(x+\omega^2 y) \\
&= (x+y)[x^2 + (\omega+\omega^2)yx + \omega^3 y^2] \\
&= (x+y)(x^2 - xy + y^2) = x^3 + y^3 \qquad \{\because \omega^3 = 1, \omega+\omega^2 = -1\} \\
&= \text{L.H.S.}
\end{aligned}$$

Example 2: Prove that: $= (-1+\sqrt{-3})^4 + (-1-\sqrt{-3})^4 = -16$

Solution:

$$\begin{aligned}
\text{L.H.S} &= (-1+\sqrt{-3})^4 + (-1-\sqrt{-3})^4 \\
&= \left[2\left(\frac{-1+\sqrt{-3}}{2}\right)\right]^4 \quad \left[2\left(\frac{-1-\sqrt{-3}}{2}\right)\right]^4 \\
&= (2\omega)^4 + (2\omega^2)^4 \\
&= 16\omega^4 + 16\omega^8 \\
&= 16(\omega^4 + \omega^8) \\
&= 16\left[\omega^3.\ \omega + \omega^6.\ \omega^2\right] \\
&= 16(\omega+\omega^2) \\
&= 16(-1) \\
&= -16 = \text{R.H.S}
\end{aligned}$$

$\left\{ \text{Let } \frac{-1+\sqrt{-3}}{2} = \omega \quad \therefore \frac{-1-\sqrt{-3}}{2} = \omega^2 \right.$

$\because \omega^3 = \omega^6 = 1$

$\because \omega + \omega^2 = \quad 1$

4.4 Four Fourth Roots of Unity

Let x be the fourth root of unity

$$\therefore \quad x = = \sqrt[4]{1} \quad (1)^{\frac{1}{4}}$$

$$\Rightarrow \quad x^4 = 1$$

$$\Rightarrow \quad x^4 - 1 = 0$$

$$\Rightarrow \quad (x^2-1)(x^2+1) = 0$$

$$\Rightarrow \quad x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

and $x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm i$.

Hence four fourth roots of unity are:

+ 1 ,– 1 , + i, – i.

4.4.1 Properties of four Fourth Roots of Unity

We have found that the four fourth roots of unity are:

$+1, -1, +i, -i$

i) Sum of all the four fourth roots of unity is zero

$\because \quad +1+(-1)+i+(-i)=0$

ii) The real fourth roots of unity are additive inverses of each other

+1 and –1 are the real fourth roots of unity

and +1 + (–1) = 0 = (–1) + 1

iii) Both the complex/imaginary fourth roots of unity are conjugate of each other

i and $-i$ are complex / imaginary fourth roots of unity, which are obviously conjugates of each other.

iv) Product of all the fourth roots of unity is –1

$\therefore \quad 1\times(-1)\times i\times(-i)=-1$

Exercise 4.4

1. Find the three cube roots of: 8, – 8, 27, –27, 64.

2. Evaluate:

i) $(1+\omega-\omega^2)^8$ ii) $\omega^{28}+\omega^{29}+1$ iii) $(1+\omega-\omega^2)(1-\omega+\omega^2)$

iv) $\left(\frac{-1+\sqrt{-3}}{2}\right)^9+\left(\frac{-1-\sqrt{-3}}{2}\right)^7$ v) $(-1+\sqrt{-3})^5+(-1-\sqrt{3})^5$

3. Show that:

i) $x^3-y^3=(x-y)(x-\omega y)(x-\omega^2 y)$

ii) $x^3+y^3+z^3-3xyz=(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$

iii) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)....2n \text{ factors}=1$

Hint: $1+\omega^4=1+\omega^3.\omega=1+\omega=-\omega^2,\ 1+\omega^8=1+\omega^6.\omega^2=1+\omega^2=-\omega$

4. If ω is a root of $x^2+x+1=0$, show that its other root is ω^2 and prove that $\omega^3=1$.

5. Prove that complex cube roots of –1 are $\frac{1+\sqrt{3}i}{2}$ and $\frac{1-\sqrt{3}i}{2}$ and hence prove that

$$\left(\frac{1+\sqrt{-3}}{2}\right)^9+\left(\frac{1-\sqrt{-3}}{2}\right)^9 = 2..$$

6. If ω is a cube root of unity, form an equation whose roots are 2ω and $2\omega^2$.

7. Find four fourth roots of 16, 81, 625.

8. Solve the following equations:

i) $2x^4-32=0$ ii) $3y^5-243y=0$

iii) $x^3+x^2+x+1=0$ iv) $5x^5-5x=0$

4.5 Polynomial Function:

A polynomial in x is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1}+....+a_{1n}x+a_0, \qquad a_n\neq 0 \qquad \text{(i)}$$

where n is a non-negative integer and the coefficients $a_n, a_{n-1},....,a_1$ and a_0 are real numbers. It can be considered as a **Polynomial function** of x. The highest power of x in polynomial in x are called the **degree** of the polynomial. So the expression (i), is a polynomial of degree n. The polynomials $x^2-2x+3,\ 3x^3+2x^2-5x+4$ are of degree 2 and 3 respectively.

Consider a polynomial; $3x^3-10x^2+13x-6$.

If we divide it by a linear factor $x-2$ as shown below, we get a quotient x^2-4x+5 and a remainder 4 .

$$
\begin{array}{rl}
 & 3x^2 - 4x + 5 \\
\text{divisor} \rightarrow x-2 & \overline{)\,3x^3 - 10x^2 + 13x - 6} \quad \leftarrow \text{dividend} \\
 & 3x^3 - 6x^2 \\
 & -\quad + \\
\hline
 & -4x^2 + 13x \\
 & -4x^2 + 8x \\
 & +\quad - \\
\hline
 & 5x - 6 \\
 & 5x - 10 \\
 & -\quad + \\
\hline
 & 4 \quad \leftarrow \text{remainder}
\end{array}
$$

Hence we can write: $3x^3 - 10x^2 + 13x - 6 = (x-2)(3x^2 - 4x + 5) + 4$

i.e., dividend = (divisor) (quotient) + remainder.

4.6 Theorems:

Remainder Theorem: If a polynomial $f(x)$ of degree $n \geq 1$, n is non-negative integer is divided by $x - a$ till no x-term exists in the remainder, then $f(a)$ is the remainder.

Proof: Suppose we divide a polynomial $f(x)$ by $x - a$. Then there exists a unique quotient $q(x)$ and a unique remainder R such that $f(x) = (x - a)(qx) + R$ (i)

Substituting $x = a$ in equation (i), we get

$f(a) = (a - a)q(a) + R$

$\Rightarrow$ $f(a) = R$

Hence remainder = $f(a)$

Note: Remainder obtained when $f(x)$ is divided by $x - a$ is same as the value of the polynomial $f(x)$ at $x = a$.

Example 1: Find the remainder when the polynomial $x^3 + 4x^2 - 2x + 5$ is divided by $x - 1$.

Solution: Let $f(x) = x^3 + 4x^2 - 2x + 5$ and $x - a = x - 1 \Rightarrow a = 1$

Remainder = $f(1)$ (By remainder theorem)

$= (1)^3 + 4(1)^2 - 2(1) + 5$

$= 1 + 4 - 2 + 5$

$= 8$

Example 2: Find the numerical value of k if the polynomial $x^3 + kx^2 - 7x + 6$ has a remainder of – 4, when divided by $x + 2$.

Solution: Let $f(x) = x^3 + kx^2 - 7x + 6$ and $x - a = x + 2$, we have, $a = -2$

Remainder = $f(-2)$ (By remainder theorem)

$= (-2)^3 + k(-2)^2 - 7(-2) + 6$

$= -8 + 4k + 14 + 6$

$= 4k + 12$

Given that remainder = – 4

$\therefore$ $4k + 12 = -4$

$\Rightarrow$ $4k = -16$

$\Rightarrow$ $k = -4$

Factor Theorem: The polynomial $x - a$ is a factor of the polynomial $f(x)$ if and only if $f(a) = 0$ i.e.; $(x - a)$ is a factor of $f(x)$ if and only if $x = a$ is a root of the polynomial equation $f(x) = 0$.

Proof: Suppose $g(x)$ is the quotient and R is the remainder when a polynomial $f(x)$ is divided by $x - a$, then by **Remainder Theorem**

$f(x) = (x - a)\, g(x) + R$

Since $f(a) = 0$ $\Rightarrow R = 0$

$\therefore$ $f(x) = (x - a)\, g(x)$

$\therefore$ $(x - a)$ is a factor of $f(x)$.

Conversely, if $(x - a)$ is a factor of $f(x)$, then

$R = f(a) = 0$

which proves the theorem.

Note: To determine if a given linear polynomial $x - a$ is a factor of $f(x)$, all we need to check whether $f(a) = 0$.

Example 3: Show that $(x - 2)$ is a factor of $x^4 - 13x^2 + 36$.

Solution: Let $f(x) = x^4 - 13x^2 + 36$ and $x - a = x - 2 \Rightarrow a = 2$

Now $f(2) = (2)^4 - 13(2)^2 + 36$

$= 16 - 52 + 36$

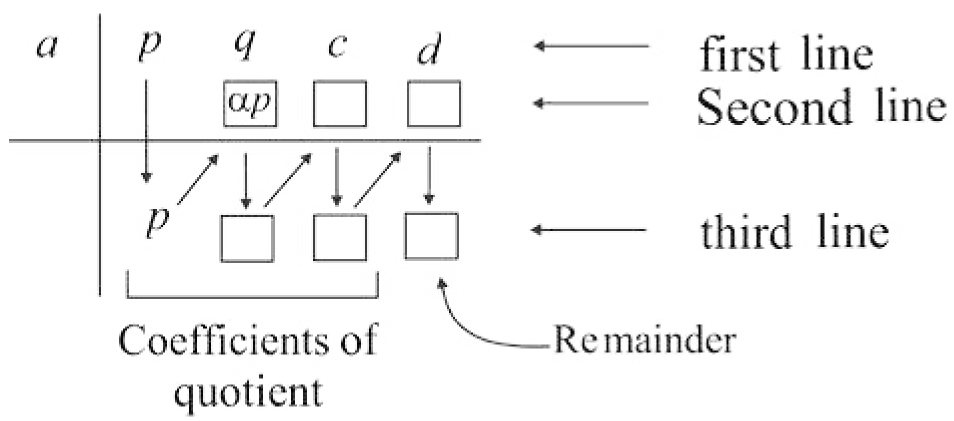
$= 0$ = remainder

$\Rightarrow (x - 2)$ is a factor of $x^4 - 13x^2 + 36$

4.7 Synthetic Division

There is a nice shortcut method for long division of a polynomial $f(x)$ by a polynomial of the form $x - a$. This process of division is called **Synthetic Division**.

To divide the polynomial $px^3 + qx^2 + cx + d$ by $x - a$



Out Line of the Method:

i) Write down the coefficients of the dividend $f(x)$ from left to right in decreasing order of powers of x. Insert 0 for any missing terms.

ii) To the left of the first line, write a of the divisor $(x - a)$.

iii) Use the following patterns to write the second and third lines:

Vertical pattern $(\downarrow)$ Add terms

Diagonal pattern $(\nearrow)$ Multiply by a.

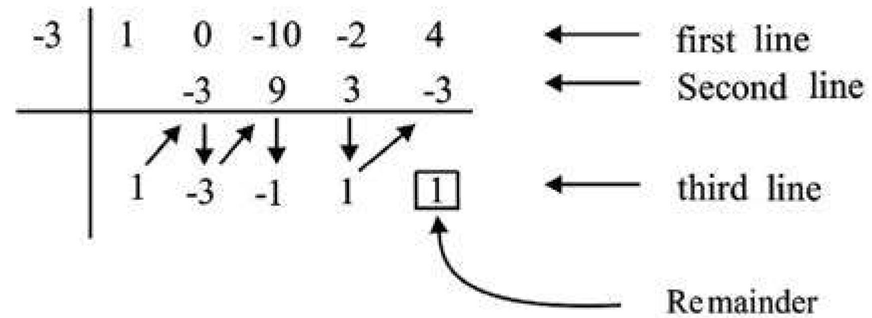
Example 4: Use synthetic division to find the quotient and the remainder when the polynomial $x^4 - 10x^2 - 2x + 4$ is divided by $x + 3$.

Solution: Let $f(x) = x^4 - 10x^2 - 2x + 4$

$= x^4 + 0x^3 - 10x^2 - 2x + 4$

and $x - a = x + 3 = x - (-3) \Rightarrow x = -3$

Dividend $x^4 - 10x^2 - 2x + 4$



$\therefore$ Quotient $= x^3 - 3x^2 - x + 1$

Remainder $= 1$

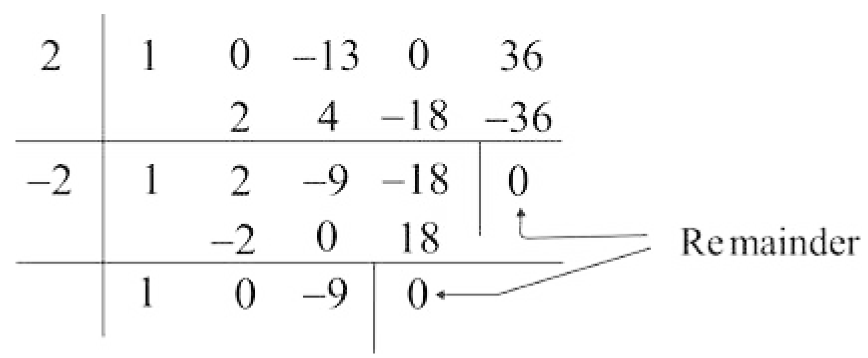
Example 5: If $(x - 2)$ and $(x + 2)$ are factors of $x^4 - 13x^2 + 36$. Using synthetic division, find the other two factors.

Solution: Let $f(x) = x^4 - 13x^2 + 36$

$= x^4 + 0x^3 - 13x^2 - 0x + 36$

Here $x - a = x - 2 \Rightarrow x = 2$ and $x - a = x + 2 = x - (-2) \Rightarrow x = -2$

By synthetic Division:



$\therefore$ Quotient $= x^2 + 0x - 9$

$= x^2 - 9$

$= (x+3)(x-3)$

$\therefore$ Other two factors are $(x+3)$ and $(x-3)$.

Example 6: If $x+1$ and $x-2$ are factors of x^3+px^2+qx+2. By use of synthetic division find the values of p and q.

Solution: Here $x-a = x+1 \Rightarrow a=-1$ and $x-a = x-2 \Rightarrow a = 2$

Let $f(x) = x^3+px^2+qx+2$

By Synthetic Division:

-1	1	p	q	2
		-1	$-p+1$	$-q+p-1$
2	1	$p-1$	$q-p+1$	$1-q+p$ ← Remainder
		2	$2p+2$	
	1	$p+1$	$p+q+3$ ← Remainder	

Since $x+1$ and $x-2$ are the factors of $f(x)$

$\therefore$ $p-q+1 = 0$ (i)

and $p+q+3 = 0$ (ii)

Adding (i) & (ii) we get $2p+4=0$ $\Rightarrow p = -2$

from (i) $-2-q+1 = 0$ $\Rightarrow q = -1$

Example 7: By the use of synthetic division, solve the equation $x^4-5x^2+4=0$ if -1 and 2 are its roots.

-1	1	0	-5	0	4
		-1	1	4	-4
2	1	-1	-4	4	0 ← Remainder
		2	2	-4	
	1	1	-2	0 ← Remainder	

Solution: $f(x) = x^4-0x^3-5x^2+0x+4$

Depressed Equation:

$x^2+x-2=0$

$\Rightarrow (x+2)(x-1) = 0$ $\Rightarrow x=-2$ or $x = 1$

Hence Solution set = $\{-2, -1, 1, 2\}$.

Exercise 4.5

Use the remainder theorem to find the remainder when the first polynomial is divided by the second polynomial:

1. x^2+3x+7, $x+1$ **2.** x^3-x^2+5x+4, $x-2$

3. $3x^4+4x^3+x-5$, $x+1$ **4.** x^3-2x^2+3x+3, $x-3$

Use the factor theorem to determine if the first polynomial is *a* factor of the second polynomial.

5. $x-1$, x^2+4x-5 **6.** $x-2$, x^3+x^2-7x+1

7. $\omega+2$, $2\omega^3+\omega^2-4\omega+7$ **8.** $x-a$, x^n-a^n where n is a positive integer

9. $x+a$, x^n+a^n where n is an odd integer.

10. When $x^4+2x^3+kx^2+3$ is divided by $x-2$ the remainder is 1. Find the value of k.

11. When the polynomial x^3+2x^2+kx+4 is divided by $x-2$ the remainderis 14. Find the value of k.

Use Synthetic division to show that x is the solution of the polynomial and use the result to factorize the polynomial completely.

12. $x^3-7x+6=0$, $x=2$ **13.** $x^3-28x-48=0, - x = 4$

14. $2x^4+7x^3-4x^2-27x-18$, $x=2, - x = 3$

15. Use synthetic division to find the values of p and q if $x+1$ and $x-2$ are the factors of the polynomial x^3+px^2+qx+6.

16. Find the values of a and b if –2 and 2 are the roots of the polynomial x^3-4x^2+ax+b.

4.8 Relations Between the Roots and the Coefficients of a Quadratic Equation

Let α,β are the roots of $ax^2+bx+c=0, a\neq 0$ such that

$$\frac{-b+\sqrt{b\ -\ ac}}{} \quad \text{and} \quad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\therefore\ \alpha+\beta = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a} = -\frac{2b}{2a} = \frac{b}{a}$$

$$\text{and} \quad \alpha\beta=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$=\frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2}$$

$$=\frac{b^2-b^2+4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

Sum of the roots = S $-\frac{b}{a}=-\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$

Product of the roots = $P=\frac{c}{a}=\frac{\text{constant term}}{\text{coefficient of } x^2}$

The above results are helpful in expressing symmetric functions of the roots in terms of the coefficients of the quadratic equations.

Example 1: If α,β are the roots of $ax^2+bx+c=0, a\neq 0$, find the values of

i) $\alpha^2+\beta^2$ ii) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$ iii) $(\alpha-\beta)^2$

Solution: Since α,β are the roots of $ax^2+bx+c=0$

$$\therefore\ \alpha+\beta=-\frac{b}{a} \quad \text{and} \quad \alpha\beta=\frac{c}{a}$$

i) $\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta$

$$=\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)=\frac{b^2}{a^2}-\frac{2c}{a}=\frac{b^2-2ca}{a^2}$$

ii) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha} = \frac{\alpha^3+\beta^3}{\alpha\beta}=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}$

$$=\frac{\left(-\frac{b}{a}\right)^3-3\frac{c}{a}\left(-\frac{b}{c}\right)}{\frac{c}{a}} = \frac{\frac{-b^3+3abc}{a^3}}{\frac{c}{a}}$$

$$=\frac{-b^3+3abc}{a^2c}$$

iii) $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta$

$$=\left(-\frac{b}{a}\right)^2-4\left(\frac{c}{a}\right)=\frac{b^2}{a^2}-4\frac{c}{a}=\frac{b^2-4ac}{a^2}$$

Example 2: Find the condition that one root of $ax^2+bx+c=0, a\neq 0$ is square of the other.

Solution: As one root of $ax^2+bx+c=0$ is square of the other, let the roots be α and α^2

Sum of roots $a+a^2=-\frac{b}{a}$ (i)

Product of roots= $a\,.\,a^2=\frac{c}{a}$ $\Rightarrow \alpha^3=\frac{c}{a}$ (ii)

Cubing both sides of (i), we get

$$a^3 + a^6 + 3aa^2(a + a^2) = -\frac{b^3}{a^3}$$

$$\Rightarrow \quad a^3 + (a^3)^2 + 3a^3(a + a^2) = \frac{b^3}{a^3}$$

$$\Rightarrow \quad \frac{c}{a} + \left(\frac{c}{a}\right)^2 + 3\frac{c}{a}\left(-\frac{b}{a}\right) = \frac{b^3}{a^3}$$ (From (i), (ii))

$$\Rightarrow \quad a^2c + ac^2 - 3abc = b^3$$

4.9 Formation of an Equation Whose Roots are Given

$\therefore \quad (x-\alpha)(x-\beta) = 0$ has the roots α and β

$\Rightarrow \quad x^2 - (\alpha+\beta)x + \alpha\beta = 0$ has the roots α and β.

For S = Sum of the roots and P = Product of the roots.

Thus $x^2 - Sx + P = 0$

Example 3: If α, β are the root of $ax^2 + bx + c = 0$ form the equation whose roots are double the roots of this equation.

Solution: $\because$ α and β are the root of $ax^2 + bx + c = 0$

$$\therefore \quad \alpha + \beta - = \frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

The new roots are 2α and 2β.

$\therefore$ Sum of new roots = $2\alpha + 2\beta$

$$= 2(\alpha + \beta) = -\frac{2b}{a}$$

Product of new roots= $2\alpha . 2\beta = 4\alpha\beta = \frac{4c}{a}$

Required equation is given by

y^2 –(Sum of roots) y + Product of roots = 0

$$\Rightarrow \quad y^2 + \frac{2b}{a}y + \frac{4c}{a} = 0 \quad \Rightarrow \quad ay^2 + 2by + 4c = 0$$

Exercise 4.6

1. If α, β are the root of $3x^2 - 2x + 4 = 0$, find the values of

i) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ **ii)** $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ **iii)** $\alpha^4 + \beta^4$

iv) $\alpha^3 + \beta^3$ **v)** $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$ **vi)** $\alpha^2 - \beta^2$

2. If α, β are the root of $x^2 - px - p - c = 0$, prove that

$(1 + \alpha)(1 + \beta) = 1 - c$

3. Find the condition that one root of $x^2 + px + q = 0$ is

i) double the other **ii)** square of the other

iii) additive inverse of the other

iv) multiplicative inverse of the other.

4. If the roots of the equation $x^2 - px + q = 0$ differ by unity, prove that $p^2 = 4q + 1$.

5. Find the condition that $\frac{a}{x-a} + \frac{b}{x-b} = 5$ may have roots equal in magnitude but opposite in signs.

6. If the roots of $px^2 + qx + q = 0$ are α and β then prove that $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{q}{p}} = 0$.

7. If α, β are the roots of the equation $ax^2 + bx + c = 0$, form the equations whose roots are

i) α^2, β^2 **ii)** $\frac{1}{\alpha}, \frac{1}{\beta}$ **iii)** $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$

iv) α^3, β^3 **v)** $\frac{1}{\alpha^3}, \frac{1}{\beta^3}$ **vi)** $\alpha + \frac{1}{\alpha}, \beta + \frac{1}{\beta}$

vii) $(\alpha-\beta)^2, (\alpha+\beta)^2$ **viii)** $-\frac{1}{\alpha^3}, -\frac{1}{\beta^3}$

8. If α, β are the roots of the $5x^2 - x - 2 = 0$, form the equation whose roots are $\frac{3}{\alpha}$ and $\frac{3}{\beta}$.

9. If α, β are the roots of the $x^2 - 3x + 5 = 0$, form the equation whose roots are $\frac{1-\alpha}{1+\alpha}$ and $\frac{1-\beta}{1+\beta}$.

4.10 Nature of the roots of a quadratic equation

We know that the roots of the quadratic equation $ax^2 + bx + c = 0$ are given by the quadratic formula as: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

We see that there are two possible values for x, as discriminated by the part of the formula $\pm\sqrt{b^2-4ac}$.

The nature of the roots of an equation depends on the value of the expression $b^2 - 4ac$, which is called its **Discriminant.**

Case 1: If $b^2 - 4ac = 0$ then the roots will be $-\frac{b}{2a}$ and $-\frac{b}{2a}$ So, the roots are real and repeated equal.

Case 2: If $b^2 - 4ac < 0$ then $\sqrt{b^2-4ac}$ will be imaginary
So, the roots are complex / imaginary and distinct / unequal

Case 3: If $b^2 - 4ac > 0$ then $\sqrt{b^2-4ac}$ will be real.
So, the roots are real and distinct / unequal.

However, If $b^2 - 4ac$ is a perfect square then $\sqrt{b^2-4ac}$ will be rational, and so the roots are rational, otherwise irrational.

Example 1: Discuss the nature of the roots of the following equations:

i) $x^2 + 2x + 3 = 0$ ii) $2x^2 + 5x - 1 = 0$
iii) $2x^2 - 7x + 3 = 0$ iv) $9x^2 - 12x + 4 = 0$

Solution:

i) Comparing $x^2 + 2x + 3 = 0$ with $ax^2 + bx + c = 0$, we have
$a = 1, b = 2, c = 3$
Discriminant (Disc) $= b^2 - 4ac$
$= (2)^2 - 4(1)(3) = 4 - 12 = -8$
$\Rightarrow$ Disc < 0
$\therefore$ The roots are complex / imaginary and distinct / unequal.

ii) Comparing $2x^2 + 5x - 1 = 0$ with $ax^2 + bx + c = 0$, we have
$a = 2, b = 5, c = -1$
Disc $= b^2 - 4ac$
$= (5)^2 - 4(2)(-1)$
$= 25 + 8 = 33$
$\Rightarrow$ Disc > 0 but not a perfect square.
$\therefore$ The roots are irrational and unequal.

iii) Comparing $2x^2 - 7x + 3 = 0$ with $ax^2 + bx + c = 0$ we have
$a = 2, b = -7, c = 3$
Disc $= b^2 - 4ac$
$= (-7)^2 - 4(2)(3)$
$= 49 - 24 = 25 = 5^2$
$\Rightarrow$ Disc > 0 and a perfect square.
$\therefore$ The roots are irrational and unequal.

iv) Comparing $9x^2 - 12x + 4 = 0$ with $ax^2 + bx + c = 0$,we have
$a = 9, b = -12, c = 4$
Disc $= b^2 - 4ac$
$= (-12)^2 - 4(9)(4)$
$= 144 - 144 = 0$
$\Rightarrow$ Disc $= 0$
$\therefore$ The roots are real and equal.

Example 2: For what values of m will the following equation have equal root? $(m+1)x^2+2(m+3)x+2m+3=0, m \neq -1$

Solution: Comparing the given equation with $ax^2 + bx + c = 0$

$a=m+1, b=2(m+3), c=2m+3$

Disc $= b^2 - 4ac$

$=[2(m+3)]^2-4(m+1)(2m+3)$

$=4(m^2+6m+9)-4(2m^2+5m+3)$

$= \ 4m^2 \quad 4m \quad 24$

The roots of the given equation will be equal, if Disc. = 0 i.e.,

if $\quad -4m^2+4m+24=0$

$\Rightarrow \quad m^2-m-6=0$

$\Rightarrow \quad (m-3)(m+2)=0 \Rightarrow m=3 \text{ or } m=-2$

Hence if m = 3 or m = –2, the roots of the given equation will be equal.

Example 3: Show that the roots of the following equation are real

$(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$

Also show that the roots will be equal only if $a = b = c$.

Solution: $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$

$\Rightarrow \quad x^2-ax-bx+ab+x^2-bx-cx+bc+x^2-cx-ax+ac=0$

$\Rightarrow \quad 3x^2-2(a+b+c)x+ab+bc+ca=0$

Disc $= b^2 - 4ac$

$=[2(a+b+c)]^2-4(3)(ab+bc+ca)$

$=4(a^2+b^2+c^2+2ab+2bc+2ca-3ab-3bc-3ca)$

$=4(a^2+b^2+c^2-ab-bc-ca)$

$=2(2a^2+2b^2+2c^2-2ab-2bc-2ca)$

$=2[a^2+b^2-2ab+b^2+c^2-2bc+c^2+a^2-2ca]$

$=2[(a-b)^2+(b-c)^2+(c-a)^2]$

= 2(Sum of three squares)

Thus the discriminant cannot be negative.

Hence the roots are real.

The roots will be equal, if the discriminant = 0

This is possible only if $a - b = 0$, $b - c = 0$, $c - a = 0$ i.e., if $a = b = c$.

Exercise 4.7

1. Discuss the nature of the roots of the following equations:

i) $4x^2+6x+1=0$ ii) $x^2-5x+6=0$

iii) $2x^2-5x+1=0$ iv) $25x^2-30x+9=0$

2. Show that the roots of the following equations will be real:

i) $x^2-2\left(m+\dfrac{1}{m}\right)x+3=0; m \bullet 0$

ii) $(b-c)x^2+(c-a)x+(a-b)=0; a,b,c \in Q$

3. Show that the roots of the following equations will be rational:

i) $(p+q)x^2-px-q=0;$ ii) $px^2-(p-q)x-q=0;$

4. For what values of m will the roots of the following equations be equal?

i) $(m+1)x^2+2(m+3)x+m+8 \quad =0$

ii) $x^2-2(1+3m)x+7(3+2m) \quad =0$

iii) $(1+m)x^2-2(1+3m)x+(1+8m) \quad =0$

5. Show that the roots of $x^2+(mx+c)^2=a^2$ will be equal, if $c^2=a^2(1+m^2)$

6. Show that the roots of $(mx + c)^2 = 4ax$ will be equal, if $c=\dfrac{a}{m}; m \neq 0$

7. Prove that $\dfrac{x^2}{a^2}+\dfrac{(mx+c)^2}{b^2}=1$ will have equal roots, if $c^2 = a^2m^2 + b^2$; $a \neq 0$, $b \neq 0$

8. Show that the roots of the equation $(a^2 - bc)x^2 + 2(b^2 - ca)x + c^2 - ab = 0$ will be equal, if either $a^3 + b^3 + c^3 = 3abc$ or $b = 0$.

4.11 System of Two Equations Involving Two Variables

We have, so far, been solving quadratic equations in one variable. Now we shall be solving the equations in two variables, when at least one of them is quadratic. To determine the value of two variables, we need a pair of equations.Such a pair of equations is called a **system of simultaneous equations.**

No general rule for the solution of such equations can be laid down except that some how or the other, one of the variables is eliminated and the resulting equation in one variable is solved.

Case I: One Linear Equation and one Quadratic Equation

If one of the equations is linear, we can find the value of one variable in terms of the other variable from linear equation. Substituting this value of one variable in the quadratic equation, we can solve it. The procedure is illustrated through the following examples:

Example 1: Solve the system of equations:

$x + y = 7$ and $x^2 - xy + y^2 = 13$

Solution: $x + y = 7 \quad \Rightarrow x = 7 - y \qquad$ (i)

Substituting the value of x in the equation $x^2 - xy + y^2 = 13$ we have

$(7 - y)^2 - y(7 - y) + y^2 = 13$

$\Rightarrow \quad 49 - 14y + y^2 - 7y + y^2 + y^2 = 13$

$\Rightarrow \quad 3y^2 - 21y + 36 = 0$

$\Rightarrow \quad y^2 - 7y + 12 = 0$

$\Rightarrow \quad (y - 3)(y - 4) = 0$

$\Rightarrow \quad y = 3$ or $y = 4$

Putting $y = 3$, in (i), we get $x = 7 - 3 = 4$

Putting $y = 4$, in (i), we get $= 7 - 4 = 3$

Hence solution set = {(4, 3), (3, 4)}.

Note: Two quadratic equations in which xy term is missing and the coefficients of x^2 and y^2 are equal, give a linear equation by subtraction.

Example 2: Solve the following equations:

$x^2 + y^2 + 4x = 1$ and $x^2 + (y - 1)^2 = 10$

Solution: The given system of equations is

$$\begin{cases} x^2 + y^2 + 4x = 1 & \text{(i)} \\ x^2 + y^2 - 2y + 1 = 10 & \text{(ii)} \end{cases}$$

Subtraction gives,

$4x + 2y + 8 = 0$

$\Rightarrow 2x + y + 4 = 0$

$\Rightarrow y = -2x - 4 \qquad$ (iii)

Putting the value of y in equation (i),

$x^2 + (-2x - 4)^2 + 4x = 1 \Rightarrow x^2 + 4x^2 + 16x + 16 + 4x = 1$

$\Rightarrow \quad 5x^2 + 20x + 15 = 0 \quad \Rightarrow \quad x + 4x + 3 = 0$

$\Rightarrow \quad (x + 3)(x + 1) = 0 \quad \Rightarrow \quad x = -3$ or $x = -1$

Putting $x = -3$ in (iii), we get; $y = -2(-3) - 4 = 6 - 4 = 2$

Putting $x = -1$ in (iii), we get; $y = -2(-1) - 4 = 2 - 4 = -2$

Hence solution set = $\{(-3,\ 2), (-1,\ -2)\}$.

Exercise 4.8

Solve the following systems of equations:

1. $2x - y = 4; \quad 2x^2 - 4xy - y^2 = 6$
2. $x + y = 5$; $x^2 + 2y^2 = 17$
3. $3x + 2y = 7; \quad 3x^2 = 25 + 2y^2$
4. $x + y = 5; \dfrac{2}{x} + \dfrac{3}{y} = 2, x \neq 0, y \neq 0$
5. $x + y = a + b; \dfrac{a}{x} + \dfrac{b}{y} = 2$
6. $3x + 4y = 25; \dfrac{3}{x} + \dfrac{4}{y} = 2$
7. $(x - 3)^2 + y^2 = 5; \qquad 2x = y + 6$

8. $(x+3)^2+(y-1)^2=5;\quad x^2+y^2+2x=9$

9. $x^2+(y+1)^2=18;\quad (x+2)^2+y^2=21$

10. $x^2+y^2+6x=1\;;\quad x^2+y^2+2(x+y)=3$

Case II: Both the Equations are Quadratic in two Variables

The equations in this case are classified as:

i) Both the equations contain only x^2 and y^2 terms.
ii) One of the equations is homogeneous in x and y.
iii) Both the equations are non-homogeneous.

The methods of solving these types of equations are explained through the following examples:

Example 1: Solve the equations: $\begin{cases} x^2+y^2 = 25 \\ 2x^2+3y^2=6 \end{cases}$

Solution: Let $x^2=u$ and $y^2=v$

By this substitution the given equations become

$u+v=25$ (i)

$2u+3v=66$ (ii)

Multiplying both sides of the equation (i) by 2, we have

$2u+2v=50$ (iii)

Subtraction of (iii) from (ii) gives,

$v=16$

Putting the value of v in (i), we have

$u+16=25 \quad\Rightarrow\quad u=9$

$\therefore\quad x^2=9 \;\Rightarrow\; x=\pm3$ and $y^2=16 \;\Rightarrow\; y=\pm4$

Hence solution set = $\{(\pm3,\ \pm4)\}$.

Example 2: Solve the equations: $x^2-3xy+2y^2=0; 2x^2-3x+y^2=24$

Solution: The given equations are:

$x^2-3xy+2y^2=0$ (i)

$2x^2-3x+y^2=24$ (ii)

Equation $x^2-3xy+2y^2=0$ is homogeneous in x and y

$\Rightarrow (x-y)(x-2y)=0.$ (Factorizing)

$\Rightarrow x-y=0$ or $x-2y=0$

$\Rightarrow x=y$ ...(iii)

Putting the value of x in (ii), we get

$2y^2-3y+y^2=24$

$\Rightarrow y^2-y-8=0$

$\Rightarrow y=\dfrac{1\pm\sqrt{1+32}}{2}$

$\Rightarrow y=\dfrac{1\pm\sqrt{33}}{2}$

when $y=\dfrac{1+\sqrt{33}}{2}$

from (iii) $x=\dfrac{1+\sqrt{33}}{2}$

when $y=\dfrac{1-\sqrt{33}}{2}$

from (iii) $x=\dfrac{1-\sqrt{33}}{2}$

$\Rightarrow x=2y$ (iv)

Putting the value of x in (ii), we get

$2(2y)^2-3(2y)+y^2=24$

$\Rightarrow 8y^2-6y+y^2=24$

$\Rightarrow 3y^2-2y-8=0$

$\Rightarrow (3y+4)(y-2)=0$

$\Rightarrow y=-\dfrac{4}{3},\ 2$

when $y=-\dfrac{4}{3}$,

from (iv) $x=2\left(-\dfrac{4}{3}\right)=\dfrac{8}{3}$

when y = 2,

from (iv) x = 2(2) = 4

Hence following is the solution set.

$\left\{\left(\dfrac{1+\sqrt{33}}{2},\dfrac{1+\sqrt{33}}{2}\right)\left(\dfrac{1-\sqrt{33}}{2},\dfrac{1-\sqrt{33}}{2}\right)\left(-\dfrac{8}{3},-\dfrac{4}{3}\right)(4,2)\right\}$

Example 3: Solve the equations:

$$\begin{cases} x^2-y^2 = 5 \\ 4x^2-3xy=18 \end{cases}$$

Solution Given that $\begin{cases} x^2 - y^2 = 5 & \text{(i)} \\ 4x^2 - 3xy = 18 & \text{(ii)} \end{cases}$

We can get a homogeneous equation in x and y, if we get rid of the constants. For the purpose, we multiply both sides of equation (i) by 18 and both sides of equation (ii) by 5 and get

$$\begin{cases} 18x^2 - 18y^2 = 90 \\ 20x^2 - 15xy = 90 \end{cases}$$

Subtraction gives,

$$2x^2 - 15xy + 18y^2 = 0$$

$$\Rightarrow \quad (x-6y)(2x-3y) = 0$$

$$\Rightarrow \quad x - 6y = 0 \text{ or } 2x - 3y = 0$$

Combining each of these equations with any one of the given equations, we can solve them by the method used in the example 1.

$$x - 6y = 0$$

$$\Rightarrow \quad x = 6y$$

$$\because \quad x^2 - y^2 = 5 \qquad \text{from (i)}$$

$$\therefore \quad (6y)^2 - y^2 = 5$$

$$\Rightarrow \quad 35y^2 = 5$$

$$\Rightarrow \quad y^2 = \frac{1}{7}$$

$$\Rightarrow \quad y = \pm\frac{1}{\sqrt{7}}$$

when $y = \frac{1}{\sqrt{7}}$,

when $\cdot y = \frac{1}{\sqrt{7}} x \quad 6\left(\frac{-1}{\sqrt{7}}\right) \quad \frac{-6}{\sqrt{7}}$

or

$$2x - 3y = 0$$

$$\Rightarrow \quad 2x = 3y \quad \Rightarrow \quad x = \frac{3}{2}y$$

$$\because \quad x^2 - y^2 = 5 \qquad \text{from (i)}$$

$$\therefore \quad \left(\frac{3}{2}y\right)^2 - y^2 = 5$$

$$\Rightarrow \quad 9y^2 - 4y^2 = 20$$

$$\Rightarrow \quad 5y^2 = 20$$

$$\Rightarrow \quad y^2 = 4$$

$$\Rightarrow \quad y = \pm 2$$

when $y = 2,$

when $y = -2$

$$x = \frac{3}{2}(-2) = -3$$

Hence Solution set $= \left\{\left(\frac{6}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right), \left(\ \frac{6}{\sqrt{7}}, \ \frac{1}{\sqrt{7}}\right), (\ 3,\ \ 2), (3,2)\right\}$

Exercise 4.9

Solve the following systems of Equations:

1. $2x^2 = 6 + 3y^2$; $3x^2 - 5y^2 = 7$
2. $8x^2 = y^2$; $x^2 + 2y^2 = 19$
3. $2x^2 - 8 = 5y^2$; $x^2 - 13 = \ 2y^2$
4. $x^2 - 5xy + 6y^2 = 0$; $x^2 + y^2 = 45$
5. $12x^2 - 25xy + 12y^2 = 0$; $4x^2 + 7y^2 = 148$
6. $12x^2 - 11xy + 2y^2 = 0$; $2x^2 + 7xy = 60$
7. $x^2 - y^2 = 16$; $xy = 15$
8. $x^2 + xy = 9$; $x^2 - y^2 = 2$
9. $y^2 - 7 = 2xy$; $2x^2 + 3 = xy$
10. $x^2 + y^2 = 5$; $xy = 2$

4.12 Problems on Quadratic Equations

We shall now proceed to solve the problems which, when expressed symbolically, lead to quadratic equations in one or two variables.

In order to solve such problems, we must:

1) Suppose the unknown quantities to be x or y etc.
2) Translate the problem into symbols and form the equations satisfying the given conditions.

Translation into symbolic expression is the main feature of solving problems leading to equations. So, it is always helpful to proceed from concrete to abstract e.g. we may say that:

i) 5 is greater than 3 by 2 = 5 – 3 **ii)** x is greater than 3 by $x-3$
iii) 5 is greater than y by $5-y$ **iv)** x is greater than y by $x-y$.

The method of solving the problems will be illustrated through the following examples:

Example 1: Divide 12 into two parts such that the sum of their squares is greater than twice their product by 4.

Solution: Suppose one part = x

$\therefore$ The other part $= 12-x$

Sum of the squares of the parts $= x^2+(12-x)^2$

twice the product of the parts $= 2(x)(12-x)$

By the condition of the question,

$$x^2+(12-x)^2-2x(12-x)=4$$

$$\Rightarrow \quad x^2+144-24x+x^2-24x+2x^2=4$$

$$\Rightarrow \quad 4x^2-48x+140=0 \qquad \Rightarrow \quad x^2-12x+35=0$$

$$\Rightarrow \quad (x-5)(x-7)=0 \qquad \Rightarrow \quad x=5 \text{ or } x=7$$

If one part is 5, then the other part = 12 – 5 = 7,
and if one part is 7, then the other part = 12 – 7 = 5

Here both values of x are admissible.
Hence required parts are 5 and 7.

Example 2: A man distributed Rs.1000 equally among destitutes of his street. Had there been 5 more destitutes each one would have received Rs. 10 less. Find the number of destitutes.

Solution: Suppose number of destitutes = x

Total sum = 1000 Rs.

$\therefore$ Each desitute gets = $\frac{1000}{x}$ Rs.

For 5 more destitutes, the number of destitutes would have been $x+5$

$\therefore$ Each destitute would have got = $\frac{1000}{x+5}$ Rs.

This sum would have been Rs. 10 less than the share of each destitute in the previous case.

$$\therefore \quad \frac{1000}{x+5}=\frac{1000}{x}-10$$

$$\Rightarrow \quad 1000x = 1000(x+5) - 10(x+5)(x)$$

$$\Rightarrow \quad x^2+5x-500=0$$

$$\Rightarrow \quad (x+25)(x-20)=0$$

$$\Rightarrow \quad x=-25 \text{ or } x=20$$

The number of destitutes cannot be negative. So, –25 is not admissible.
Hence the number of destitutes is 20.

Example 3: The length of a room is 3 meters greater than its breadth. If the area of the room is 180 square meters, find length and the breadth of the room.

Solution: Let the breadth of room = x meters

and the length of room = $x + 3$ meters

$\therefore$ Area of the room = $x(x + 3)$ square meters

By the condition of the question

$x(x + 3) = 180$ (i)

$\Rightarrow$ $x^2 + 3x - 180 = 0$ (ii)

$\Rightarrow$ $(x + 15)(x - 12) = 0$

$\therefore$ $x = -15$ or $x = 12$

As breadth cannot be negative so $x = -15$ is not admissible

$\therefore$ when $x = 12$, we get length $x + 3 = 12 + 3 = 15$

$\therefore$ breadth of the room = 12 meter and length of the room = 15 meter

Example 4: A number consists of two digits whose product is 8. If the digits are interchanged, the resulting number will exceed the original one by 18. Find the number.

Solution : Suppose tens digit = x

and units digit = y

$\therefore$ The number = $10x + y$

By interchanging the digits, the new number = $10y + x$

Product of the digits = xy

By the condition of question;

$xy = 8$ (i)

and $10y + x = 10x + y + 18$ (ii)

Solving (i) and (ii) ;we get

$x = -4$ or $x = 2$.

when $x = -4$, $y = -2$ and when $x = 2$, $y = 4$

Rejecting negative values of the digits,

Tens digit = 2

and Units digit = 4

Hence the required number = 24

Exercise 4.10

1. The product of one less than a certain positive number and two less than three times the number is 14. Find the number.
2. The sum of a positive number and its square is 380. Find the number.
3. Divide 40 into two parts such that the sum of their squares is greater than 2 times their product by 100.
4. The sum of a positive number and its reciprocal is $\frac{26}{5}$ .Find the number.
5. A number exceeds its square root by 56. Find the number.
6. Find two consecutive numbers, whose product is 132. (**Hint:** Suppose the numbers are x and $x + 1$).
7. The difference between the cubes of two consecutive even

numbers is 296. Find them.
(**Hint:** Let two consecutive even numbers be x and $x + 2$)

8. A farmer bought some sheep for Rs. 9000. If he had paid Rs. 100 less for each, he would have got 3 sheep more for the same money. How many sheep did he buy, when the rate in each case is uniform?

9. A man sold his stock of eggs for Rs. 240. If he had 2 dozen more, he would have got the same money by selling the whole for Rs. 0.50 per dozen cheaper. How many dozen eggs did he sell?

10. A cyclist travelled 48 km at a uniform speed. Had he travelled 2 km/hour slower, he would have taken 2 hours more to perform the journey. How long did he take to cover 48 km?

11. The area of a rectangular field is 297 square meters. Had it been 3 meters longer and one meter shorter, the area would have been 3 square meters more. Find its length and breadth.

12. The length of a rectangular piece of paper exceeds its breadth by 5 cm. If a strip 0.5 cm wide be cut all around the piece of paper, the area of the remaining part would be 500 square cms. Find its original dimensions.

13. A number consists of two digits whose product is 18. If the digits are interchanged, the new number becomes 27 less than the original number. Find the number.

14. A number consists of two digits whose product is 14. If the digits are interchanged, the resulting number will exceed the original number by 45. Find the number.

15. The area of a right triangle is 210 square meters. If its hypoteneuse is 37 meters long. Find the length of the base and the altitude.

16. The area of a rectangle is 1680 square meters. If its diagonal is 58 meters long, find the length and the breadth of the rectangle.

17. To do a piece of work, A takes 10 days more than B. Together they finish the work in 12 days. How long would B take to finish it alone?
Hint: If some one takes x days to finish a work. The one day's work will be $\frac{1}{x}$.

18. To complete a job, A and B take 4 days working together. A alone takes twice as long as B alone to finish the same job. How long would each one alone take to do the job?

19. An open box is to be made from a square piece of tin by cutting a piece 2 dm square

from each corner and then folding the sides of the remaining piece. If the capacity of the box is to be finish 128 c.dm, find the length of the side of the piece.

20. A man invests Rs. 100,000 in two companies. His total profit is Rs. 3080. If he receives Rs. 1980 from one company and at the rate 1% more from the other, find the amount of each investment.