CHAPTER

5

Partial Fractions

5.1 Introduction

We have learnt in the previous classes how to add two or more rational fractions into a single rational fraction. For example,

i) $\frac{1}{x-1}+\frac{2}{x+2} = \frac{3x}{(x-1)(x+2)}$

and ii) $\frac{2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x-2} = \frac{5x^2+5x-3}{(x+1)^2(x-2)}$

In this chapter we shall learn how to reverse the order in (i) and (ii) that is to express a single rational function as a sum of two or more single rational functions which are called **Partial Fractions.**

Expressing a rational function as a sum of partial fractions is called **Partial Fraction Resolution**. It is an extremely valuable tool in the study of calculus.

An open sentence formed by using the sign of equality '=' is called an equation. The equations can be divided into the following two kinds:

Conditional equation: It is an equation in which two algebraic expressions are equal for particular value/s of the variable e.g.,

a) $2x = 3$ is a conditional equation and it is true only if $x = \frac{3}{2}$.

b) $x^2 + x - 6 = 0$ is a conditional equation and it is true for $x = 2, -3$ only.

Note: For simplicity, a conditional equation is called an **equation**.

Identity: It is an equation which holds good for all values of the variable e.g.,

a) $(a + b)x = ax + bx$ is an **identity** and its two sides are equal for all values of x.

b) $(x + 3)(x + 4) = x^2 + 7x + 12$ is also an identity which is true for all values of x.

For convenience, the symbol "=" shall be used both for equation and identity.

5.2 Rational Fraction

We know that $\frac{p}{q}$ where $p, q \in Z$ and $q \neq 0$ is called a rational number.

Similarly, the quotient of two polynomials $\frac{P(x)}{Q(x)}$ where $Q(x) \neq 0$, with no common factors, is called a **Rational Fraction**. A rational fraction is of two types:

5.2.1 Proper Rational Fraction

A rational fraction $\frac{P(x)}{Q(x)}$ is called a **Proper Rational Fraction** if the degree of the polynomial $P(x)$ in the numerator is less than the degree of the polynomial $Q(x)$ in the denominator. For example, $\frac{3}{x+1}, \frac{2x-5}{x^2+4}$ and $\frac{9x^2}{x^3-1}$ are proper rational fractions or proper frations.

5.2.2 Improper Rational Fraction

A rational fraction $\frac{P(x)}{Q(x)}$ is called an **Improper Rational Fraction** if the degree of the polynomial $P(x)$ in the numerator is equal to or greater than the degree of the polynomial $Q(x)$ in the denominator.

For example, $\frac{x}{2x-3}, \frac{(x-2)(x+1)}{(x-1)(x+4)}, \frac{x^2-3}{3x+1}$ and $\frac{x^3-x^2+x+1}{x^2+5}$

are improper rational fractions or improper fractions.

Any improper rational fraction can be reduced by division to a mixed form, consisting of the sum of a polynomial and a proper rational fraction.

For example, $\frac{3x^2+1}{x-2}$ is an improper rational fraction. By long division we obtain $\frac{3x^2+1}{x-2} = 3x+6+\frac{13}{x-2}$

i.e., an improper rational fraction has $\frac{3x^2+1}{x-2}$ been reduced to the sum o f a polynomial 3x + 6 and a proper rational fraction $\frac{13}{x-2}$

$$\begin{array}{r|l} & 3x+6 \\ x-2 & 3x+1 \\ & \pm 3x \mp 6x \\ \hline & 6x+1 \\ & \pm 6x \mp 12 \\ \hline & 13 \end{array}$$

When a rational fraction is separated into partial fractions, the result is an identity; i.e., it is true for all values of the variable.

The evaluation of the coefficients of the partial fractions is based on the following theorem:

"If two polynomials are equal for all values o f the variable, then the polynomials have same degree and the coefficients of like powers of the variable in both the polynomials must be equal".

For example,

If $px^3 + qx^2 - ax + b = 2x^3 - 3x^2 - 4x + 5, \quad \forall x$

then p = 2, q = – 3 , a = 4 and b = 5.

5.3 Resolution of a Rational Fraction $\frac{P(x)}{Q(x)}$ into Partial Fractions

Following are the main points of resolving a rational fraction $\frac{P(x)}{Q(x)}$ into partial fractions:

i) The degree of $p(x)$ must be less than that of $Q(x)$. If not, divide and work with the remainder theorem.

ii) Clear the given equation of fractions.

iii) Equate the coefficients of like terms (powers of x).

iv) Solve the resulting equations for the coefficients.

We now discuss the following cases of partial fractions resolution.

Case I: Resolution of $\frac{P(x)}{Q(x)}$ into partial fractions when $Q(x)$ has only non-repeated linear factors:

The polynomial $Q(x)$ may be written as:

$Q(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$, where $a_1 \neq a_2 \neq \ldots \neq a_n$

$$\therefore \quad \frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \ldots\ldots + \frac{A_n}{x-a_n}$$ is an identity.

Where, the coefficients $A_1, A_2, \ldots, A_n$ arenumbers to be found.

The method is explained by the following examples:

Example 1: Resolve, $\frac{7x+25}{(x+3)(x+4)}$ into Partial Fractions.

Solution: Suppose $\frac{7x+25}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}$

Multiplying both sides by $(x + 3)(x + 4)$, we get

$7x + 25 = A(x + 4) + B(x + 3)$

$\Rightarrow \quad 7x + 25 = Ax + 4A + Bx + 3B$

$\Rightarrow \quad 7x + 25 = (A + B)x + 4A + 3B$

This is an identity in x.

So, equating the coefficients of like powers of x we have

$7 = A + B$ and $25 = 4A + 3B$

Solving these equations, we get $\boxed{A=4}$ and $\boxed{B=3}$.

Hence the partial fractions are: $\frac{4}{x+3} + \frac{3}{x+4}$

Alternative Method:

Suppose $\frac{7x+25}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}$

$\Rightarrow \quad 7x + 25 = A(x + 4) + B(x + 3)$

As two sides of the identity are equal for all values of x,
let us put $x = -3$, and $x = -4$ in it.
Putting $x = -3$, we get $-21 + 25 = A(-3 + 4)$

$\Rightarrow \quad \boxed{A = 4}$

Putting $x = -4$, we get $-28 + 25 = B(-4 + 3)$

$\Rightarrow \quad \boxed{B = 3}$

Hence the partial fractions are: $\frac{4}{x+3} + \frac{3}{x+4}$

Example 2: Resolve $\frac{x^2-10x+13}{(x-1)(x^2-5x+6)}$ into Partial Fractions.

Solution: The factor $x^2 - 5x + 6$ in the denominator can be factorized and its factors are $x - 3$ and $x - 2$.

$$\therefore \quad \frac{x^2-10x+13}{(x-1)(x^2-5x+6)} = \frac{x^2-10x+13}{(x-1)(x-2)(x-3)}$$

Suppose $\frac{x^2-10x+13}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$

$\Rightarrow \quad x^2 - 10x + 13 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

which is an identity in x.

Putting $x = 1$ in the identity, we get

$(1)^2 - 10(1) + 13 = A(1-2)(1-3) + B(1-1)(1-3) + C(1-1)(1-2)$

$\Rightarrow \quad 1 - 10 + 13 = A(-1)(-2) + B(0)(-2) + C(0)(-1)$

$4 = 2A \quad \boxed{\therefore\ A = 2}$

Putting $x = 2$ in the identity, we get

$(2)^2 - 10(2) + 13 = A(0)(2-3) + B(2-1)(2-3) + C(2-1)(0)$

$\Rightarrow \quad 4 - 20 + 13 = B(1)(-1)$

$\Rightarrow \quad -3 = -B \quad \boxed{\therefore\ B = 3}$

Putting $x = 3$ in the identity, we get

$(3)^2 - 10(3) + 13 = A(3-2)(0) + B(3-1)(0) + C(3-1)(3-2)$

$\Rightarrow \quad 9 - 30 + 13 = C(2)(1)$

$\Rightarrow \quad -8 = 2C \quad \boxed{\therefore\ C = -4}$

Hence partial fractions are: $\frac{2}{x-1} + \frac{3}{x-2} - \frac{4}{x-3}$

Note: In the solution of examples 1 and 2. We observe that the value of the constants have been found by substituting those values of x in the identities which can be got by putting each linear factor of the denominators equal to zero.

In the Example 2

a) the denominator of A is $x - 1$, and the value of A has been found by putting $x - 1 = 0$ i.e ;$x = 1$;

b) the denominator of B is $x - 2$, and the value of B has been found by putting $x - 2 = 0$ i.e., $x = 2$; and

c) the denominator of C is $x - 3$, and the value of C has been found by putting $x - 3 = 0$ i.e.,$x = 3$.

Example 3: Resolve $\frac{2x^3+x^2-x-3}{x(2x+3)(x-1)}$ into Partial Fractions.

Solution:

$\because \quad \frac{2x^3+x^2-x-3}{x(2x+3)(x-1)}$ is an improper fraction so, transform it into mixed from.

Denominator $= x(2x+3)(x-1)$
$= 2x^3 + x^2 - 3x$

$$\begin{array}{r|l} & 1 \\ 2x^3+x^2-3x & 2x^3+x^2-x-3 \\ & \underline{\pm 2x^3 \pm x^2 \mp 3x} \\ & 2x-3 \end{array}$$

$\therefore$ Dividing $2x^3 + x^2 - x - 3$ by $2x^3 + x^2 - 3x$, we have
Quotient = 1 and Remainder = $2x - 3$

$$\therefore \quad \frac{2x^3+x^2-x-3}{x(2x+3)(x-1)} = 1 + \frac{2x-3}{x(2x+3)(x-1)}$$

Suppose $\dfrac{2x-3}{x(2x+3)(x-1)} = \dfrac{A}{x} + \dfrac{B}{2x+3} + \dfrac{c}{x-1}$

$\Rightarrow$ $2x - 3 = A(2x + 3)(x - 1) + B(x)(x - 1) + C(x)(2x + 3)$
which is an identity in x.

Putting $x = 0$ in the identity, we get $\boxed{A=1}$

Putting $2x + 3 = 0 \Rightarrow x = -\dfrac{3}{2}$ in the identity, we get $\boxed{B=-\dfrac{8}{5}}$

Putting $x - 1 = 0 \Rightarrow x = 1$ in the identity, we get $\boxed{C=-\dfrac{1}{5}}$

Hence partial fractions are: $1+\dfrac{1}{x}-\dfrac{8}{5(2x+3)}-\dfrac{1}{5(x-1)}$

Exercise 5.1

Resolve the following into Partial Fractions:

1. $\dfrac{1}{x^2-1}$ **2.** $\dfrac{x^2+1}{(x+1)(x-1)}$

3. $\dfrac{2x+1}{(x-1)(x+2)(x+3)}$ **4.** $\dfrac{3x^2-4x-5}{(x-2)(x^2+7x+10)}$

5. $\dfrac{1}{(x-1)(2x-1)(3x-1)}$ **6.** $\dfrac{x}{(x-a)(x-b)(x-c)}$

7. $\dfrac{6x^3+5x^2-7}{2x^2-x-1}$ **8.** $\dfrac{2x^3+x^2-5x+3}{2x^3+x^2-3x}$

9. $\dfrac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)}$ **10.** $\dfrac{1}{(1-ax)(1-bx)(1-cx)}$

11. $\dfrac{x^2+a^2}{(x^2+b^2)(x^2+c^2)(x^2+d^2)}$

[**Hint:** Put $x^2 = y$ to make factors of the denominator linear]

Case II: when $Q(x)$ has repeated linear factors:

If the polynomial has a factor $(x - a)^n$, $n \geq 2$ and n is a $+ve$ integer, then may be written as the following identity:

$$\therefore \quad \frac{P(x)}{Q(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)^2} + \dots\dots + \frac{A_n}{(x-a_n)^n}$$

where the coefficients $A_1, A_2, \dots, A_n$ are numbers to be found.
The method is explained by the following examples:

Example 1: Resolve, $\dfrac{x^2+x-1}{(x+2)^3}$ into partial fractions.

Solution: Suppose $\dfrac{x^2+x-1}{(x+2)^3} = \dfrac{A}{x+2}+\dfrac{B}{(x+2)^2}+\dfrac{C}{(x+2)^3}$

$\Rightarrow$ $x^2 + x - 1 = A(x + 2)^2 + B(x + 2) + C$ (i)

$\Rightarrow$ $x^2 + x - 1 = A(x^2 + 4x + 4) + B(x + 2) + C$ (ii)

Putting $x + 2 = 0$ in (i), we get

$(-2)^2 + (-2) - 1 = A(0) + B(0) + C$

$\Rightarrow$ $\boxed{1=C}$

Equating the coefficients of x^2 and x in (ii), we get $\boxed{A=1}$

and $1 = 4A + B$

$\Rightarrow$ $1 = 4 + B$ $\Rightarrow$ $\boxed{B=-3}$

Hence the partial fractions are: $\dfrac{1}{x+2}-\dfrac{3}{(x+2)^2}+\dfrac{1}{(x+2)^3}$

Example 2: Resolve $\dfrac{1}{(x+1)^2(x^2-1)}$ into Partial Fractions.

Solution: Here denominator = $(x+1)^2(x^2-1)$

$= (x+1)^2(x+1)(x-1) = (x+1)^3(x-1)$

$$\therefore \frac{1}{(x+1)^2(x^2-1)} = \frac{1}{(x+1)^3(x-1)}$$

Suppose $\frac{}{(x-1)(x+1)^3} = \frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}+\frac{D}{(x+1)^3}$

$\Rightarrow\ 1 = A(x+1)^3 + B(x+1)^2(x-1) + C(x-1)(x+1) + D(x-1)$ (i)

$\Rightarrow\ 1 = A(x^3+3x^2+3x+1) + B(x^3+x^2-x-1) + C(x^2-1) + D(x-1)$

$\Rightarrow\ 1 = (A+B)x^3 + (3A+B+C)x^2 + (3A-B+D)x + (A-B-C-D)$ (ii)

Putting $x-1=0 \Rightarrow x=1$ in (i), we get,

$$1 = A(2)^3 \quad \Rightarrow \quad \boxed{A = \frac{1}{8}}$$

Putting $x+1=0 \Rightarrow x=-1$ in (i), we get,

$$1 = D(-1-1) \quad \Rightarrow \quad \boxed{D = -\frac{1}{2}}$$

Equating the coefficients of x^3 and x^2 in (ii), we get

$$0 = A+B \quad \Rightarrow B = -A \quad \Rightarrow \quad \boxed{B = -\frac{1}{8}}$$

and $0 = 3A+B+C \Rightarrow 0 = \frac{3}{8}-\frac{1}{8}+C \quad \Rightarrow \quad \boxed{C = -\frac{1}{4}}$

Hence the partial fractions are:

$$\frac{\frac{1}{8}}{x-1}+\frac{-\frac{1}{8}}{x+1}+\frac{-\frac{1}{4}}{(x+1)^2}+\frac{-\frac{1}{2}}{(x+1)^3} = \frac{1}{8(x-1)}-\frac{1}{8(x+1)}-\frac{1}{4(x+1)^2}-\frac{1}{2(x+1)^3}$$

Exercise 5.2

Resolve the following into Partial Fractions:

1. $\frac{2x^2-3x+4}{(x-1)^3}$ **2.** $\frac{5x^2-2x+3}{(x+2)^3}$ **3.** $\frac{4x}{(x+1)^2(x-1)}$

4. $\frac{9}{(x+2)^2(x-1)}$ **5.** $\frac{1}{(x-3)^2(x+1)}$ **6.** $\frac{x^2}{(x-2)(x-1)^2}$

7. $\frac{1}{(x-1)^2(x+1)}$ **8.** $\frac{x^2}{(x-1)^3(x+1)}$ **9.** $\frac{x-1}{(x-2)(x+1)^3}$

10. $\frac{}{(x^2-1)(x+1)^2}$ **11.** $\frac{2x+1}{(x+3)(x-1)(x+2)^2}$ **12.** $\frac{2x^4}{(x-3)(x+2)^2}$

Case III: when $Q(x)$ contains non-repeated irreducible quadratic factor

Definition: A quadratic, factor is irreducible if it cannot be written as the product of two linear factors with real coefficients. For example, x^2+x+1 and x^2+3 are irreducible quadratic factors.

If the polynomial $Q(x)$ contains non-repeated irreducible quadratic factor then $\frac{P(x)}{Q(x)}$ may be written as the identity having partial fractions of the form:

$\frac{Ax+B}{ax^2+bx+c}$ where A and B the numbers to be found.

The method is explained by the following examples:

Example 1: Resolve $\frac{3x-11}{(x^2+1)(x+3)}$ into Partial Fractions.

Solution: Suppose $\frac{3x-11}{(x^2+1)(x+3)} = \frac{Ax+B}{(x^2+1)}+\frac{C}{(x+3)}$

$\Rightarrow \quad 3x - 11 = (Ax + B)(x + 3) + C(x^2 + 1)$ (i)

$\Rightarrow \quad 3x - 11 = (A + C)x^2 + (3A + B)x + (3B + C)$ (ii)

Putting $x + 3 = 0 \Rightarrow x = -3$ in (i), we get

$-9 - 11 = C(9 + 1) \Rightarrow \boxed{C = -2}$

Equating the coefficients of x^2 and x in (ii), we get

$0 = A + C \Rightarrow A = -C \Rightarrow \boxed{A = 2}$

and $3 = 3A + B \Rightarrow B = 3 - 3A \Rightarrow B = 3 - 6 \Rightarrow \boxed{B = -3}$

Hence the partial fraction are: $\dfrac{2x-3}{x^2+1} - \dfrac{2}{x+3}$

Example 2: Resolve $\dfrac{4x^2+8x}{x^4+2x^2+9}$ into Partial Fractions.

Solution: Here, denominator $= x^4 + 2x^2 + 9 = (x^2 + 2x + 3)(x^2 - 2x + 3)$.

$$\therefore \frac{4x^2+8x}{x^4+2x^2+9} = \frac{4x^2+8x}{(x^2+2x+3)(x^2-2x+3)}$$

Suppose

$$\frac{4x^2+8x}{(x^2+2x+3)(x^2-2x+3)} = +\frac{Ax+B}{x^2+2x+3} \quad \frac{Cx+D}{x^2-2x+3}$$

$\Rightarrow \quad 4x^2 + 8x = (Ax + B)(x^2 - 2x + 3) + (Cx + D)(x^2 + 2x + 3)$

$\Rightarrow \quad 4x^2 + 8x = (A + C)x^3 + (-2A + B + 2C + D)x^2$

$+ (3A - 2B + 3C + 2D)x + 3B + 3D$ (I)

which is an identity in x.

Equating the coefficients of x^3, x^2, x, x^0 in I, we have

$0 = A + C$ (i)

$4 = -2A + B + 2C + D$ (ii)

$8 = 3A - 2B + 3C + 2D$ (iii)

$0 = 3B + 2D$ (iv)

Solving (i), (ii), (iii) and (iv), we get

$\boxed{A = 1}$, $\boxed{B = 2}$, $\boxed{C = -1}$ and $\boxed{D = -2}$

Hence the partial fractions are: $\dfrac{x+2}{x^2+2x+3} + \dfrac{-x-2}{x^2-2x+3}$

Exercise 5.3

Resolve the following into Partial Fractions:

1. $\dfrac{9x-7}{(x^2+1)(x+3)}$ **2.** $\dfrac{1}{(x^2+1)(x+1)}$ **3.** $\dfrac{3x+7}{(x^2+4)(x+3)}$

4. $\dfrac{x^2+15}{(x^2+2x+5)(x-1)}$ **5.** $\dfrac{x^2}{(x^2+4)(x+2)}$ **6.** $\dfrac{x^2+1}{x^3+1}$

7. $\dfrac{x^2+2x+2}{(x^2+3)(x+1)(x-1)}$ **8.** $\dfrac{1}{(x-1)^2(x^2+2)}$ **9.** $\dfrac{x^4}{1-x^4}$

10. $\dfrac{x^2-2x+3}{x^4+x^2+1}$

Case IV: when $Q(x)$ has repeated irreducible quadratic factors

If the polynomial $Q(x)$ contains a repeated irreducible quadratic factors $(a_n x^2 + bx + c)^n$, $n \geq 2$ and n is a +ve integer, then $\dfrac{P(x)}{Q(x)}$ may be written as the following identity:

$$\frac{P(x)}{Q(x)} = \frac{A_1x + B_1}{a_1x^2 + bx + c} \quad \frac{A_2x + B_2}{(a_2x^2 + bx + c)^2} \quad \ldots\ldots \quad \frac{A_nx + B_n}{(a_nx^2 + bx + c)^n}$$

where $A_1, B_1, A_2, B_2, \ldots A_n, B_n$ are numbers to be found. The method is explained through the following example:

Example 1: Resolve $\frac{4x^2}{(x^2+1)^2(x-1)}$ into partial fractions.

Solution: Let $\frac{4x^2}{(x^2+1)^2(x-1)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x-1}$

$\Rightarrow \quad 4x^2 = (Ax+B)(x^2+1)(x-1) + (Cx+D)(x-1) + E(x^2+1)^2 \qquad$ (i)

$\Rightarrow \quad 4x^2 = (A+E)x^4 + (-A+B)x^3 + (A-B+C+2E)x^2$
$\qquad +(-A+B-C+D)x + (-B-D+E) \qquad$ (ii)

Putting $x-1=0 \Rightarrow x=1$ in (i), we get

$4 = E(1+1)^2 \qquad \Rightarrow \boxed{E=1}$

Equating the coefficients of x^4, x^3, x^2, x, in (ii), we get

$0 = A+E \quad \Rightarrow \quad A = -E \qquad \Rightarrow \boxed{A=-1}$

$0 = -A+B \quad \Rightarrow \quad B = A \qquad \Rightarrow \boxed{B=-1}$

$4 = A-B+C+2E$

$\Rightarrow \quad C = 4-A+B-2E = 4+1-1-2 \qquad \Rightarrow \boxed{C=2}$

$0 = -A+B-C+D$

$\Rightarrow \quad D = A-B+C = -1+1+2 = 2 \qquad \Rightarrow \boxed{D=2}$

Hence partial fractions are: $\frac{-x-1}{x^2+1} + \frac{2x+2}{(x^2+1)^2} + \frac{1}{x-1}$

Exercise 5.4

Resolve into Partial Fractions.

1. $\frac{x^3+2x+2}{(x^2+x+1)^2}$ **2.** $\frac{x^2}{(x^2+1)^2(x-1)}$

3. $\frac{2x-5}{(x^2+2)^2(x-2)}$ **4.** $\frac{8x^2}{(x^2+1)^2(1-x^2)}$

5. $\frac{4x^4+3x^3+6x^2+5x}{(x-1)(x^2+x+1)^2}$ **6.** $\frac{2x^4-3x^3-4x}{(x^2+2)^2(x+1)^2}$