CHAPTER 6

Sequences and Series

6.1 Introduction

Sequences also called Progressions, are used to represent ordered lists of numbers. As the members of a sequence are in a definite order, so a correspondence can be established by matching them one by one with the numbers 1, 2, 3, 4,..... For example, if the sequence is 1, 4, 7, 10, ..., *n*th member, then such a correspondence can be set up as shown in the diagram below:

$$\begin{array}{lcl} \text{Position} & & \text{the member of the sequence} \\ 1 & \longrightarrow & 1 \\ 2 & \longrightarrow & 4 \\ 3 & \longrightarrow & 7 \\ 4 & \longrightarrow & 10 \\ \vdots & & \vdots \\ n & \longrightarrow & n\text{th member} \end{array}$$

Thus a sequence is a function whose domain is a subset of the set of natural numbers. A **sequence** is a special type of a function from a subset of N to R or C. Sometimes, the domain of a sequence is taken to be a subset of the set {0, 1, 2, 3,...}, i.e., the set of non-negative integers. If all members of a sequence are real numbers, then it is called a **real sequence**.

Sequences are usually named with letters *a, b, c* etc., and n is used instead of x as a variable. If a natural number n belongs to the domain of a sequence a, the corresponding element in its range is denoted by a_n. For convenience, a special notation a_n is adopted for $a(n)$and the symbol $\{a_n\}$ or $a_1, a_2, a_3, \ldots, a_n, \ldots$is used to represent the sequence a. The elements in the range of the sequence $\{a_n\}$ are called its terms; that is, a_1 is the first term, a_2 the second term and a_n the nth term or the general term.

For example, the terms of the sequence $\{n + (-1)^n\}$ can be written by assigning to n, the values 1, 2, 3 ,... If we denote the sequence by $\{b_n\}$, then

$b_n = n + (-1)^n$ and we have

$b_1 = 1 + (-1)^1 = 1 - 1 = 0$

$b_2 = 2 + (-1)^2 = 2 + 1 = 3$

$b_3 = 3 + (-1)^3 = 3 - 1 = 2$

$b_4 = 4 + (-1)^4 = 4 + 1 = 5$ etc.

If the domain of a sequence is a finite set, then the sequence is called a finite sequence otherwise, an infinite sequence.

Note: An infinite sequence has no last term.

Some examples of sequences are;

i)	1, 4, 9,...,121	ii)	1, 3, 5, 7, 9,...,21	iii)	1, 2, 4,...
iv)	1, 3, 7, 15, 31,...	v)	1, 6, 20, 56,...	vi)	$1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots$

The sequences (i) and (ii) are finite whereas the sequences (iii) to (vi) are infinite.

6.2 Types of sequences

If we are able to find a pattern from the given initial terms of a sequence, then we can deduce a rule or formula for the terms of the sequence:

we can find any term of the given sequence giving corresponding value to n in the nth / general term a_n of a sequence.

Example 1: Write first two, 21st and 26th terms of the sequence whose general term is $(-1)^{n+1}$.

Solution: Given that $a_n = (-1)^{n+1}$. For getting required terms, we put n = 1, 2, 21 and 26.

$a_1 = (-1)^{1+1} = 1$

$a_2 = (-1)^{2+1} = -1$

$a_{21} = (-1)^{21+1} = 1$

$a_{26} = (-1)^{26+1} = -1$

Example 2: Find the sequence if $a_n - a_{n-1} = n + 1$ and $a_4 = 14$

Solution: Putting $n = 2, 3, 4$ in

$a_n - a_{n-1} = n + 1$, we have

$a_2 - a_1 = 3$ (i)

$a_3 - a_2 = 4$ (ii)

$a_4 - a_3 = 5$ (iii)

From (iii), $a_3 = a_4 - 5$

$= 14 - 5 = 9$ $(\because a_4 = 14)$

From (ii), $a_2 = a_3 - 4$

$= 9 - 4 = 5$ $(\because a_3 = 9)$

And from (i), $a_1 = a_2 - 3$

$= 5 - 3 = 2$

Thus the sequence is 2, 5, 9, 14, 20,...

Note: $a_5 - a_4 = 6 \Rightarrow a_5 = a_4 + 6 = 14 + 6 = 20$

Exercise 6.1

1. Write the first four terms of the following sequences, if

i) $a_n = 2n - 3$ ii) $a_n = (-1)^n n^2$ iii) $a_n = (-1)^n (2n - 3)$

iv) $a_n = 3n - 5$ v) $a_n = \dfrac{n}{2n+1}$ vi) $a_n = \dfrac{1}{2^n}$

vii) $a_n - a_{n-1} = n + 2, a_1 = 2$ viii) $a_n = n a_{n-1}, a_1 = 1$

ix) $a_n = (n+1)a_{n-1}, a_1 = 1$ x) $a_n = \dfrac{1}{a + (n-1)d}$

2. Find the indicated terms of the following sequences;

i) $2,6,11,17,\ldots a_7$ ii) $1,3,12,60,\ldots a_6$ iii) $1, \frac{3}{2}, \frac{5}{4}, \frac{7}{8}, \ldots a_7$

iv) $1,1,-3,5,-7,9,\ldots a_8$ v) $1,-3,5,-7,9,-11,\ldots a_8$

3. Find the next two terms of the following sequences;

i) 7,9,12,16,... ii) 1,3,7,15,31,...

iii) –1,2,12,40,... iv) 1,–3,5,–7,9,–11...

6.3 Arithmetic Progression (A.P)

A sequence $\{a_n\}$ is an Arithmetic Sequence or Arithmetic progression (A.P), if $a_n - a_{n-1}$ is the same number for all $n \in N$ and $n > 1$. The difference $a_n - a_{n-1}$ $(n > 1)$ i.e., the difference of two consecutive terms of an A.P., is called the **common difference** and is usually denoted by ***d***.

Rule for the *n*th term of an A.P.:

We know that $a_n - a_{n-1} = d$ $(n > 1)$,

which implies $a_n = a_{n-1} + d$ $(n > 1)$...... (i)

Putting $n = 2, 3, 4,\ldots$in (i) we get

$a_2 = a_1 + d = a_1 + (2 - 1)d$

$a_3 = a_2 + d = (a_1 + d) + d$

$= a_1 + 2d = a_1 + (3 - 1)d$

$a_4 = a_3 + d = (a_1 + 2d) + d$

$= a_1 + 3d = a_1 + (4 - 1)d$

Thus we conclude that

$$a_n = a_1 + (n - 1)d$$

where a_1 is the first term of the sequence.

We have observed that

$a_1 = a_1 + 0d = a_1 + (1 - 1)d$

$a_2 = a_1 + d = a_1 + (2 - 1)d$

$a_3 = a_2 + d = a_1 + (3 - 1)d$

$a_4 = a_3 + d = a_1 + (4 - 1)d$

Thus $a_1, a_1 + d, a_1 + 2d, \ldots, a_1 + (n - 1)d + \ldots$ is a general arithmetic sequence, with a_1, d as the first term and common difference respectively.

Note: $a_n = a_1 + (n - 1)d$ is called the nth term or general term of the A.P.

Example 1: Find the general term and the eleventh term of the A.P. whose first term and the common difference are 2 and –3 respectively. Also write its first four terms.

Solution: Here, $a_1 = 2$, $d = -3$

We know that $a_n = a_1 + (n-1)d$,

so $a_n = 2 + (n-1)(-3) = 2 - 3n + 3$

or $a_n = 5 - 3n$ (i)

Thus the general term of the A.P. is $5 - 3n$.

Putting $n = 11$ in (i), we have

$a_{11} = 5 - 3(11)$

$= 5 - 33 = -28$

We can find a_2, a_3, a_4 by putting $n = 2, 3, 4$ in (i), that is,

$a_2 = 5 - 3(2) = -1$

$a_3 = 5 - 3(3) = -4$

$a_4 = 5 - 3(4) = -7$

Hence the first four terms of the sequence are: 2, –1, –4, –7.

Example 2: If the 5th term of an A.P. is 13 and 17th term is 49, find a_n and a_{13}.

Solution: Given $a_5 = 13$ and $a_{17} = 49$.

Putting $n = 5$ in $a_n = a_1 + (n-1)d$, we have

$a_5 = a_1 + (5-1)d$,

$a_5 = a_1 + 4d$

or $13 = a_1 + 4d$ (i)

Also $a_{17} = a_1 + (17-1)d$

or $49 = a_1 + 16d$

or $49 = (a_1 + 4d) + 12d$

or $49 = 13 + 12d$ (by (i))

$\Rightarrow 12d = 36 \Rightarrow d = 3$

From (i), $a_1 = 13 - 4d = 13 - 4(3) = 1$

Thus $a_{13} = 1 + (13-1)3 = 37$ and

$a_n = 1 + (n-1)3 = 3n - 2$

Example 3: Find the number of terms in the A.P. if; $a_1 = 3$, $d = 7$ and $a_n = 59$.

Solution: Using $a_n = a_1 + (n-1)d$, we have

$59 = 3 + (n-1) \times 7$ $(\because a_n = 59, a_1 = 3 \text{ and } d = 7)$

or $56 = (n-1) \times 7 \Rightarrow (n-1) = 8 \Rightarrow n = 9$

Thus the terms in the A.P. are 9.

Example 4: If $a_{n-2} = 3n - 11$, find the nth term of the sequence.

Solution: Putting $n = 3, 4, 5$ in $a_{n-2} = 3n - 11$, we have

$a_1 = 3 \times 3 - 11 = -2$

$a_2 = 3 \times 4 - 11 = 1$

$a_3 = 3 \times 5 - 11 = 4$

Thus $a_n = a_1 + (n-1)d = -2 + (n-1) \times 3$ $(\because a_1 = -2, \text{ and } d = 3)$

$= 3n - 5$

Exercise 6.2

1. Write the first four terms of the following arithmetic sequences, if
   i) $a_1 = 5$ and other three consecutive terms are 23, 26, 29
   ii) $a_5 = 17$ and $a_9 = 37$ iii) $3a_7 = 7a_4$ and $a_{10} = 33$
2. If $a_{n-3} = 2n - 5$, find the nth term of the sequence.
3. If the 5th term of an A.P. is 16 and the 20th term is 46, what is its 12th term?
4. Find the 13th term of the sequence $x, 1, 2 - x, 3 - 2x, \ldots$
5. Find the 18th term of the A.P. if its 6th term is 19 and the 9th term is 31.
6. Which term of the A.P. 5, 2, –1,... is –85?
7. Which term of the A.P. –2, 4, 10,...is 148?
8. How many terms are there in the A.P. in which $a_1 = 11$, $a_n = 68$, $d = 3$?
9. If the nth term of the A.P. is $3n - 1$, find the A.P.
10. Determine whether (i) –19, (ii) 2 are the terms of the A.P. 17, 13, 9, ... or not.
11. If l, m, n are the pth, qth and rth terms of an A.P., show that
    i) $l(q - r) + m(r - p) + n(p - q) = 0$
    ii) $p(m - n) + q(n - l) + r(l - m) = 0$

12. Find the *n*th term of the sequence,

$$\left(\frac{4}{3}\right)^2, \left(\frac{7}{3}\right)^2, \left(\frac{10}{3}\right)^2, \ldots$$

13. If $\frac{1}{a}$, $\frac{1}{b}$ and $\frac{1}{c}$ are in A.P., show that $b = \frac{2ac}{a+c}$.

14. If $\frac{1}{a}$, $\frac{1}{b}$ and $\frac{1}{c}$ are in A.P, show that the common difference is $\frac{a-c}{2ac}$.

6.4 Arithmetic Mean (A.M)

A number A is said to be the A.M. between the two numbers a and b if a, A, b are in A.P. If d is the common difference of this A.P., then $A - a = d$ and $b - A = d$.

Thus $A - a = b - A$

or $2A = a + b$

$$\Rightarrow \quad A = \frac{a+b}{2}$$

Note: Middle term of three consecutive terms in A.P. is the A.M. between the extreme terms.

In general ,we can say that a_n is the A.M. between a_{n-1} and a_{n+1}, i.e.,

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}$$

Example 1: Find three A.Ms between $\sqrt{2}$ and $3\sqrt{2}$.

Solution: Let A_1, A_2, A_3 be three A.Ms between $\sqrt{2}$ and $3\sqrt{2}$. Then

$\sqrt{2}, A_1, A_2, A_3, 3\sqrt{2}$ are in A.P.

Here $a_1 = \sqrt{2}$, $a_5 = 3\sqrt{2}$

Using $a_n = a_1 + (n-1)d$, we get

$$a_5 = a_1 + (5-1)d$$

or $3\sqrt{2} = \sqrt{2} + 4d$

$\Rightarrow \quad 3\sqrt{2} - \sqrt{2} = 4d$

$$\Rightarrow \quad d = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

Now $A_1 = a_1 + d = \sqrt{2} + \frac{1}{\sqrt{2}} = \frac{2+1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$,

$$A_2 = A_1 + d = \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$A_3 = A_2 + d = 2\sqrt{2} + \frac{1}{\sqrt{2}} = \frac{4+1}{\sqrt{2}} = \frac{5}{\sqrt{2}}$$

Therefore, $\frac{3}{\sqrt{2}}, 2\sqrt{2}, \frac{5}{\sqrt{2}}$ are three A.Ms between $\sqrt{2}$ and $3\sqrt{2}$.

6.4.1 *n* Arithmetic Means Between two given numbers

The n numbers $A_1, A_2, A_3, \ldots, A_n$ are called n arithmetic means between a and b if $a, A_1, A_2, A_3, \ldots, A_n, b$ are in A.P.

Example 2: Find n A.Ms between a and b.

Solution: Let $A_1, A_2, A_3, \ldots, A_n$ be n arithmetic means between a and b.

Then $a, A_1, A_2, A_3, \ldots, A_n, b$ are in A.P. in which $a_1 = a$ and $a_{n+2} = b$, so

$b = a + ((n+2)-1)d$ (where d is the common difference of the A.P.)

$= a + (n+1)d$

$$\Rightarrow \quad d = \frac{b-a}{n+1}$$

Thus $A_1 = a + d = a + \frac{b-a}{n+1} = \frac{na+b}{n+1}$

$$A_2 = a + 2d = a + 2\left(\frac{b-a}{n+1}\right) = \frac{(n-1)a+2b}{n+1}$$

$$A_3 = a + 3d = a + 3\left(\frac{b-a}{n+1}\right) = \frac{(n-2)a+3b}{n+1}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$
$$A_n = a + nd = a + n\left(\frac{b-a}{n+1}\right) = \frac{a+nb}{n+1}$$

Exercise 6.3

1. Find A.M. between
   i) $3\sqrt{5}$ and $5\sqrt{5}$ ii) $x-3$ and $x+5$
   iii) $1-x+x^2$ and $1+x+x^2$
2. If 5, 8 are two A.Ms between a and b, find a and b.
3. Find 6 A.Ms. between 2 and 5.
4. Find four A.Ms. between $\sqrt{2}$ and $\frac{12}{\sqrt{2}}$.
5. Insert 7 A.Ms. between 4 and 8.
6. Find three A.Ms between 3 and 11.
7. Find n so that $\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ may be the A.M. between a and b.
8. Show that the sum of n A.Ms. between a and b is equal to n times their A.M.

6.5 Series

The sum of an indicated number of terms in a sequence is called a series. For example, the sum of the first seven terms of the sequence $\{n^2\}$ is the series,

1 + 4 + 9 + 16 + 25 + 36 + 49.

The above series is also named as the 7th partial sum of the sequence $\{n^2\}$. If the number of terms in a series is finite, then the series is called a finite series, while a series consisting of an unlimited number of terms is termed as an infinite series.

Sum of first *n* terms of an arithmetic series:

For any sequence $\{a_n\}$, we have,

$S_n = a_1 + a_2 + a_3 + ... + a_n$

If $\{a_n\}$ is an A.P., then S_n can be written with usual notations as:

$$S_n = a_1 + (a_1+d) + (a_1+2d) + ... + (a_n-2d) + (a_n-d) + a_n \qquad \text{(i)}$$

If we write the terms of the series in the reverse order, the sum of n terms remains the same, that is,

$$S_n = a_n + (a_n-d) + (a_n-2d) + ... + (a_1+2d) + (a_1+d) + a_1 \qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$2S_n = (a_1+a_n) + (a_1+a_n) + (a_1+a_n) + ... + (a_1+a_n) + (a_1+a_n) + (a_1+a_n)$$
$$= (a_1+a_n) + (a_1+a_n) + (a_1+a_n) + ...\text{to } n \text{ term}$$
$$= n(a_1+a_n)$$

Thus $\boxed{S_n = \frac{n}{2}(a_1+a_n)}$ (iii)

$$= \frac{n}{2}[a_1 + a_1 + (n-1)d]$$

or $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

Example 1: Find the 19th term and the partial sum of 19 terms of the arithmetic series:

$$2 + \frac{7}{2} + 5 + \frac{13}{2} + ...$$

Solution: Here $a_1 = 2$ and $d = a_2 - a_1 = \frac{3}{2}$

Using $a_n = a_1 + (n-1)d$, we have,

$$a_{19} = 2 + (19-1)\frac{3}{2}$$
$$= 2 + 18\left(\frac{3}{2}\right) = 2 + 27 = 29$$

Using $S_n = \frac{n}{2}(a_1 + a_n)$, we have,

$$S_{19} = \frac{19}{2}(2+29) = \frac{19}{2}(31) = \frac{589}{2}$$

Example 2: Find the arithmetic series if its fifth term is 19 and $S_4 = a_9 + 1$.

Solution: Given that a_5 = 19, that is,

$$a_1 + 4d = 19 \qquad \text{(i)}$$

Using the other given condition, we have,

$$S_4 = \frac{4}{2}[2a_1 + (4-1)d] = a_9 + 1$$

or $\quad 4a_1 + 6d = a_1 + 8d + 1$

$$3a_1 - 1 = 2d \qquad \text{(ii)}$$

Substitution $2d = 3a_1 - 1$ in (i), gives

$$a_1 + 2(3a_1 - 1) = 19$$

or $\quad 7a_1 = 21 \Rightarrow a_1 = 3$

From (i), we have,

$$4d = 19 - a_1 = 19 - 3 = 16$$

$\Rightarrow \quad d = 4$

Thus the series is 3 + 7 +11 +15 +19 + ...

Example 3: How many terms of the series –9 –6 – 3 + 0 + ... amount to 66?

Solution: Here $a_1 = -9$ and $d = 3$ as $-6-(-9) = 3$ and $-3-(-6) = 3$.

Let $S_n = 66$

Using $\quad S_n = \frac{n}{2}[2a_1 + (n-1)d]$, we have,

$$66 = \frac{n}{2}[2(-9) + (n-1)3]$$

or $\quad 132 = n[3n-21] \Rightarrow 44 = n(n-7)$

or $\quad n^2 - 7n - 44 = 0 \Rightarrow n = \frac{7 \pm \sqrt{49+176}}{2} = \frac{7 \pm \sqrt{225}}{2}$

$$= \frac{7 \pm 15}{2} \Rightarrow n = 11, -4$$

But n cannot be negative in this case, so n = 11, that is, the sum of eleven terms amount to 66.

Exercise 6.4

1. Find the sum of all the integral multiples of 3 between 4 and 97.

2. Sum the series

i) $-3 + (-1) + 1 + 3 + 5 + \dots + a_{16}$

ii) $\frac{3}{\sqrt{2}} + 2\sqrt{2} + \frac{5}{\sqrt{2}} + \dots + a_{13}$

iii) $1.11 + 1.41 + 1.71 + \dots + a_{10}$.

iv) $-8 - 3\frac{1}{2} + 1 + \dots + a_{11}$

v) $(x-a) + (x+a) + (x+3a) + \dots$ to n terms.

vi) $\frac{1}{1-\sqrt{x}} + \frac{1}{1-x} + \frac{1}{1+\sqrt{x}} + \dots$ to n terms.

vii) $\frac{1}{1+\sqrt{x}} + \frac{1}{1-x} + \frac{1}{1-\sqrt{x}} + \dots$ to n terms.

3. How many terms of the series

i) $-7 + (-5) + (-3) + \dots$ amount to 65?

ii) $-7 + (-4) + (-1) + \dots$ amount to 114?

4. Sum the series

i) $3 + 5 - 7 + 9 + 11 - 13 + 15 + 17 - 19 + \dots$ to $3n$ terms.

ii) $1 + 4 - 7 + 10 + 13 - 16 + 19 + 22 - 25 + \dots$ to $3n$ terms.

5. Find the sum of 20 terms of the series whose rth term is $3r + 1$.

6. If $S_n = n(2n-1)$, then find the series.
7. The ratio of the sums of *n* terms of two series in A.P. is $3n+2 : n+1$. Find the ratio of their 8th terms.
8. If S_2, S_3, S_5 are the sums of $2n, 3n, 5n$ terms of an A.P., show that $S_5 = 5(S_3 - S_2)$.
9. Obtain the sum of all integers in the first 1000 integers which are neither divisible by 5 nor by 2.
10. S_8 and S_9 are the sums of the first eight and nine terms of an A.P., find S_9 if $50S_9 = 63S_8$ and $a_1 = 2$ (Hint : $S_9 = S_8 + a_9$)
11. The sum of 9 terms of an A.P. is 171 and its eighth term is 31. Find the series.
12. The sum of S_9 and S_7 is 203 and $S_9 - S_7 = 49$, S_7 and S_9 being the sums of the first 7 and 9 terms of an A.P. respectively. Determine the series.
13. S_7 and S_9 are the sums of the first 7 and 9 terms of an A.P. respectively. If $\dfrac{S_9}{S_7} = \dfrac{18}{11}$ and $a_7 = 20$, find the series.
14. The sum of three numbers in an A.P. is 24 and their product is 440. Find the numbers.
15. Find four numbers in A.P. whose sum is 32 and the sum of whose squares is 276.
16. Find the five numbers in A.P. whose sum is 25 and the sum of whose squares is 135.
17. The sum of the 6th and 8th terms of an A.P. is 40 and the product of 4th and 7th term is 220. Find the A.P.
18. If a^2, b^2 and c^2 are in A.P., show that $\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ are in A.P.

6.6 Word Problems on A.p.

Example 1: Tickets for a certain show were printed bearing numbers from 1 to 100. Odd number tickets were sold by receiving paisas equal to thrice of the number on the ticket while even number tickets were issued by receiving paisas equal to twice of the number on the ticket. How much amount was received by the issuing agency?

Solution: Let S_1 and S_2 be the amounts received for odd number and even number tickets

respectively. Then

$$S_1 = 3[1+3+5+....+99] \quad \text{and} \quad S_2 = 2[2+4+6+...+100]$$

Thus $S_1 + S_2 = 3\times\dfrac{50}{2}(1+99) + 2\times\dfrac{50}{2}(2+100)$, [$\because$ There are 50 terms in each series]

= 7500 + 5100 = 12600

Hence the total amount received by the issuing agency = 12600 paisas = Rs.126

Example 2: A man repays his loan of Rs.1120 by paying Rs.15 in the first installment and then increases the payment by Rs.10 every month. How long will it take to clear his loan?

Solution: It is given that the first installment (in Rs.) is 15 and the monthly increase in payment (in Rs.) is 10.

Here $a_1 = 15$ and $d = 10$

Let the time required (in months) to clear his loan be *n*. Then

$S_n = 1120$, that is,

$$1120 = \frac{n}{2}[2\times15 + (n-1)10] = \frac{n}{2}[30 + (n-1)10]$$

$$= \frac{n}{2}\times10[3 + (n-1)] = 5n(n+2)$$

or $224 = n(n+2) \Rightarrow n^2 + 2n - 224 = 0$

$$\Rightarrow \quad n = \frac{-2\pm\sqrt{4+896}}{2} = \frac{-2\pm\sqrt{900}}{2}$$

$$= \frac{-2\pm30}{2}$$

$$= 14, -16$$

But *n* can not be negative, so *n* = 14, that is, the time required to clear his loan is 14 months.

Example 3: A manufacturer of radio sets produced 625 units in the 4th year and 700 units in the 7th year. Assuming that production uniformly increases by a fixed number every year, find

i) The production in the first year
ii) The total production in 8 years
iii) The production in the 11th year.

Solution: Let a_1 be the number of units produced in the first year and d be the uniform increase in production every year. Then the sequence of products in the successive years is

$$a_1, a_1+d, a_1+2d, \ldots$$

By the given conditions, we have

$$a_4 = 625 \quad \text{and} \quad a_7 = 700, \text{ that is,}$$

$$a_1 + 3d = 625 \qquad \text{(I)}$$

and $$a_1 + 6d = 700 \qquad \text{(II)}$$

Subtracting (I) from (II), we get

$$3d = 75 \Rightarrow d = 25$$

i) From (I), $a_1 + 3(25) = 625 \Rightarrow a_1 = 625 - 75 = 550$

Thus the production in the first year is 550 units.

ii) $$S_8 = \frac{8}{2}[2\times 550 + (8-1)25]$$
$$= 4[1100+175] = 4[1275] = 5100$$

Thus the production in 8 years is 5100 units.

iii) $$a_{11} = a_1 + (11-1)d$$
$$= 550 + 10\times 25 = 550 + 250 = 800$$

Thus the production in the 11th year is 800 units.

Exercise 6.5

1. A man deposits in a bank Rs. 10 in the first month; Rs. 15 in the second month; Rs. 20 in the third month and so on. Find how much he will have deposited in the bank by the 9th month.
2. 378 trees are planted in rows in the shape of an isosceles triangle, the numbers in successive rows decreasing by one from the base to the top. How many trees are there in the row which forms the base of the triangle?
3. A man borrows Rs. 1100 and agree to repay with a total interest of Rs. 230 in 14 installments, each installment being less than the preceding by Rs. 10. What should be his first installment?
4. A clock strikes once when its hour hand is at one, twice when it is at two and so on. How many times does the clock strike in twelve hours ?
5. A student saves Rs.12 at the end of the first week and goes on increasing his saving Rs.4 weekly. After how many weeks will he be able to save Rs.2100?
6. An object falling from rest, falls 9 meters during the first second, 27 meters during the next second, 45 meters during the third second and so on.
   i) How far will it fall during the fifth second?
   ii) How far will it fall up to the fifth second?
7. An investor earned Rs.6000 for year 1980 and Rs. 12000 for year 1990 on the same investment. If his earning have increased by the same amount each year, how much income he has received from the investment over the past eleven years?
8. The sum of interior angles of polygon having sides 3,4,5,...etc. form an A.P. Find the sum of the interior angles for a 16 sided polygon.
9. The prize money Rs. 60,000 will be distributed among the eight teams according to their positions determined in the match-series. The award increases by the same amount for each higher position. If the last place team is given Rs.4000, how much will be awarded to the first place team?
10. An equilateral triangular base is filled by placing eight balls in the first row, 7 balls in the second row and so on with one ball in the last row. After this base layer, second layer is formed by placing 7 balls in its first row, 6 balls in its second row and so on with one ball in its last row. Continuing this process, a pyramid of balls is formed with one ball on top. How many balls are there in the pyramid?

6.7 Geometric Progression (G.P)

A sequence $\{a_n\}$ is a geometric sequence or geometric progression if $\dfrac{a_n}{a_{n-1}}$ is the same non-zero number for all $n \in N$ and $n > 1$. The quotient $\dfrac{a_n}{a_{n-1}}$ is usually denoted by ***r*** and is called **common ratio** of the G.P .It is Clear that *r* is the ratio of any term of the G.P., to its predecessor. The common ratio $r = \dfrac{a_n}{a_{n-1}}$ is defined only if $a_{n-1} \neq 0$, i.e., no term of the geometric sequence is zero.

Rule for *n*th term of a G.P.: Each term after the first term is an *r* multiple of its preceding term. Thus we have,

$$a_2 = a_1 r = a_1 r^{2-1}$$
$$a_3 = a_2 r = (a_1 r) r = a_1 r^2 = a_1 r^{3-1}$$
$$a_4 = a_3 r = (a_1 r^2) r = a_1 r^3 = a_1 r^{4-1}$$
$$\vdots$$

$a_n = a_1 r^{n-1}$ which is the general term of a G.P.

Example 1: Find the 5th term of the G.P., 3,6,12,...

Solution: Here $a_1 = 3,\ a_2 = 6,\ a_3 = 12$, therefore, $r = \dfrac{a_2}{a_1} = \dfrac{6}{3} = 2$

Using $a_n = a_1 r^{n-1}$ for $n = 5$, we have,

$$a_5 = a_1 r^{5-1} = 3.2^{5-1} = 3.2^4 = 48$$

Example 2: Find a_n if $a_4 = \dfrac{8}{27}$ and $a_7 = \dfrac{-64}{729}$ of a G.P.

Solution: To find a_n we have to find a_1 and r.

Using $a_n = a_1 r^{n-1}$ (i)

$a_4 = a_1 r^{4-1} = a_1 r^3$, so $a_1 r^3 = \dfrac{8}{27}$ (ii)

And $a_7 = a_1 r^{7-1} = a_1 r^6$, so $a_1 r^6 = \dfrac{-64}{729}$ (iii)

Thus $\dfrac{a_7}{a_4} = \dfrac{-64/729}{8/27} = -\dfrac{8}{27}$ or $r^3 = \left(-\dfrac{2}{3}\right)^3$ $\left(\because \dfrac{a_7}{a_4} = \dfrac{a_1 r^6}{a_1 r^3} = r^3\right)$

$\Rightarrow\ r = -\dfrac{2}{3}$ (taking only real value of *r*)

Put $r^3 = -\dfrac{8}{27}$ in (ii), to obtain a_1 that is,

$$a_1\left(-\frac{8}{27}\right) = \frac{8}{27} \Rightarrow a_1 = -1$$

Now putting $a_1 = -1$ and $r = -\dfrac{2}{3}$ in (i), we get,

$$a_n = (-1)\left(-\frac{2}{3}\right)^{n-1} = (-1)(-1)^{n-1}.\left(\frac{2}{3}\right)^{n-1} = (-1)^n\left(\frac{2}{3}\right)^{n-1} \text{ for } n \geq 1$$

Example 3: If the numbers 1, 4 and 3 are added to there consecutive terms of G.P., the resulting numbers are in A.P. Find the numbers if their sum is 13.

Solution: Let a, ar, ar^2 be three consecutive numbers of the G.P. Then

$a + ar + ar^2 = 13 \Rightarrow a(1 + r + r^2) = 13$ (i)

and $a+1, ar+4, ar^2+3$ are in A.P., according to the given condition.

Thus $ar + 4 = \dfrac{(a+1) + (ar^2 + 3)}{2}$

$$\Rightarrow 2ar + 8 = ar^2 + a + 4$$
$$\Rightarrow a(r^2 - 2r + 1) = 4$$
$$\Rightarrow a(r^2 + r + 1) - 3ar = 4 \quad (\because r^2 - 2r + 1 = (r^2 + r + 1) - 3r)$$
$$\Rightarrow 13 - 3ar = 4 \quad (\because a(1 + r + r^2) = 13)$$

or $3ar = 13 - 4 \Rightarrow ar = 3$ (ii)

Using $a=\frac{3}{r}$, (i) becomes

$$\frac{3}{r}(1+r+r^2)=13$$

or $\quad 3r^2-10r+3=0$

$$\Rightarrow \quad r=\frac{10\pm\sqrt{100-36}}{6}=\frac{10\pm 8}{6}$$

$$r=3 \text{ or } r=\frac{1}{3}$$

If $\quad r=3$ then $a=1$ (using $ar=3$)

and if $r=\frac{1}{3}$ then $a=9$ (using $ar=3$)

Thus the numbers are 1,3,9 or 9,3,1

Exercise 6.6

1. Find the 5th term of the G.P.: 3,6,12,...
2. Find the 11th term of the sequence, $1+i, 2, \frac{4}{1+i}, \ldots$
3. Find the 12th term of $1+i, 2i, -2+2i, \ldots$
4. Find the 11th term of the sequence, $1+i, 2, 2(1-i)$
5. If an automobile depreciates in value 5% every year, at the end of 4 years what is the value of the automobile purchased for Rs.12,000?
6. Which term of the sequence:
7. If *a, b,c,d* are in G.P, prove that
   i) $a-b$, $b-c$, $c-d$ are in G.P.
   ii) a^2-b^2, b^2-c^2, c^2-d^2 are in G.P.
   iii) a^2+b^2, b^2+c^2, c^2+d^2 are in G.P.
8. Show that the reciprocals of the terms of the geometric sequence a_1, a_1r^2, a_1r^4,... form another geometric sequence.

9. Find the *n*th term of the geometric sequence if; $\frac{a_5}{a_3}=\frac{4}{9}$ and $a_2=\frac{4}{9}$
10. Find three, consecutive numbers in G.P whose sum is 26 and their product is 216.
11. If the sum of the four consecutive terms of a G.P is 80 and A.M of the second and the fourth of them is 30. Find the terms.
12. If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in G.P. show that the common ratio is $\pm\sqrt{\frac{a}{c}}$
13. If the numbers 1,4 and 3 are subtracted from three consecutive terms of an A.P., the resulting numbers are in G.P. Find the numbers if their sum is 21.
14. If three consecutive numbers in A.P. are increased by 1, 4, 15 respectively, the resulting numbers are in G.P. Find the original numbers if their sum is 6.

6.8 Geometric Means

A number *G* is said to be a geometric mean (G.M.) between two numbers *a* and *b* if *a*, *G*, *b* are in G.P. Therefore,

$$\frac{G}{a}=\frac{b}{G} \Rightarrow G^2=ab \Rightarrow G=\pm\sqrt{ab}$$

6.8.1 *n* Geometric Means Between two given numbers

The *n* numbers $G_1, G_2, G_3, \ldots, G_n$ are called *n* geometric means between *a* and *b* if $a, G_1, G_2, G_3, \ldots, G_n, b$ are in G.P.

Thus we have, $b=ar^{(n+2)-1}$ where *r* is the common ratio,

or $ar^{n+1}=b$

$$\Rightarrow \quad r=\left(\frac{b}{a}\right)^{1/n+1}$$

Thus $G_1=ar=a\left(\frac{b}{a}\right)^{1/n+1}$

$$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{2/n+1}$$

$$G_3 = ar^3 = a\left(\frac{b}{a}\right)^{3/n+1}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{n/n+1}$$

Note: $G_1.G_2.G_3....G = a^n\left(\frac{b}{a}\right)^{\frac{(1+2+3....+n)}{1}} = a^n\left(\frac{b}{a}\right)^{\frac{n}{2}}$

and $\sqrt[n]{(G_1.G_2.G_3....G_n)} = a\left(\frac{b}{a}\right)^{1/2} = \sqrt{ab}$

= G, the geometric mean between a and b

Example 1: Find the geometric mean between 4 and 16.

Solution: Here a = 4, b = 16, therefore

$$G = \pm\sqrt{ab} \quad = \pm\sqrt{4\times 16}$$
$$= \pm\sqrt{64} \quad = \pm 8$$

Thus the geometric mean may be +8 or –8. Inserting each of two G.Ms. between 4 and 16, we have two geometric sequences 4, 8, 16 and 4, –8, 16. In the first case r = 2 and in the second case r = –2 .

Example 2: Insert three G.Ms. between 2 and $\frac{1}{2}$.

Solution: Let G_1, G_2, G_3 be three G.Ms between 2 and $\frac{1}{2}$. Therefore 2, $G_1, G_2, G_3, \frac{1}{2}$ are in G.P.

Here $a_1 = 2, \quad a_5 = \frac{1}{2}$ and $n = 5$

Using $a_n = a_1r^{n-1}$ we have,

$$a_5 = a_1r^{5-1} \quad \text{i.e.,} \quad a_5 = a_1r^4 \qquad \text{(i)}$$

Now substituting the values of a_5 and a_1 in (i) we have

$$\frac{1}{2} = 2r^4 \quad \text{or} \quad r^4 = \frac{1}{4} \qquad \text{(ii)}$$

Taking square root of (ii), we get,

$$r^2 = \pm\frac{1}{2}$$

So, we have, $r^2 = \frac{1}{2}$ or $r^2 = -\frac{1}{2} = \frac{i^2}{2}$ $\quad (\because\ -1 = i^2)$

$$\Rightarrow \quad r = \pm\frac{1}{\sqrt{2}} \quad \text{or} \quad r = \pm\frac{1}{\sqrt{2}}i$$

when $r = \frac{1}{\sqrt{2}}$, then $G_1 = 2\frac{1}{\sqrt{2}} = \sqrt{2}, G_2 = 2\left(\frac{1}{\sqrt{2}}\right)^2 = 1, G_3 = 2\left(\frac{1}{\sqrt{2}}\right)^3 = \frac{1}{\sqrt{2}}$

when $r = \frac{-1}{\sqrt{2}}$, then $G_1 = 2\left(\frac{-1}{\sqrt{2}}\right) = -\sqrt{2}, G_2 = 2\left(\frac{-1}{\sqrt{2}}\right)^2 = 1, G_3 = 2\left(\frac{-1}{\sqrt{2}}\right)^3 = -\frac{1}{\sqrt{2}}$

when $r = \frac{i}{\sqrt{2}}$, then $G_1 = 2\times\frac{i}{\sqrt{2}} = \sqrt{2}\,i, G_2 = 2\left(\frac{i}{\sqrt{2}}\right)^2 = -1, G_3 = 2\left(\frac{i}{\sqrt{2}}\right)^3 = -\frac{i}{\sqrt{2}}$

when $r = \frac{-i}{\sqrt{2}}$, then $G_1 = 2\left(\frac{-i}{\sqrt{2}}\right) = -\sqrt{2}\,i, G_2 = 2\left(\frac{-i}{\sqrt{2}}\right)^2 = -1, G_3 = 2\left(\frac{-i}{\sqrt{2}}\right)^3 = \frac{i}{\sqrt{2}}$

Note: The real values of r are usually taken but here other cases are considered to widen the out-look of the students.

Example 3: If a, b, c and d are in G.P. show that $a + b$, $b + c$, $c + d$ are in G.P.

Solution: Since a, b, c are in G.P therefore,

$$ac = b^2 \qquad \text{(i)}$$

Also b, c, d are in G.P., so we have

$$bd = c^2 \qquad \text{(ii)}$$

Multiplying both sides, of (ii) by b, we get

$$b^2d = bc^2 \Rightarrow acd = bc^2 \qquad (\because ac = b^2)$$

$$\Rightarrow ad = bc \qquad \text{(iii)}$$

Now $ad + bc = bc + bc \qquad [\because ad = bc]$

i.e., $ad + bc = 2bc \qquad \text{(iv)}$

Adding (i), (ii), and (iv), we have

$$ac + bd + ad + bc = b^2 + c^2 + 2bc$$

or $(a+b)c + (a+b)d = (b+c)^2$

or $(a+b)(c+d) = (b+c)^2$

$\Rightarrow$ $a+b, b+c, c+d$ are in G.P.

Exercise 6.7

1. Find G.M. between
i) -2 and 8 ii) $-2i$ and $8i$

2. Insert two G.Ms. between
i) 1 and 8 ii) 2 and 16

3. Insert three G.Ms. between
i) 1 and 16 ii) 2 and 32

4. Insert four real geometric means between 3 and 96.

5. If both x and y are positive distinct real numbers, show that the geometric mean between x and y is less than their airthmetic mean.

6. For what value of n, $\dfrac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the positive geometric mean between a and b?

7. The A.M. of two positive integral numbers exceeds their (positive) G.M. by 2 and their sum is 20, find the numbers.

8. The A.M. between two numbers is 5 and their (positive) G.M. is 4. Find the numbers.

6.9 Sum of n terms of a Geometric Series

For any sequence $\{a_n\}$, we have

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n$$

If the sequence $\{a_n\}$ is a geometric sequence, then

$$S_n = a_1 + a_1r + a_1r^2 + \ldots a_1r^{n-1} \qquad \text{(i)}$$

Multiplying both sides of (i) by $1-r$ we get

$$\begin{aligned}(1-r)S_n &= (1-r)\{a_1 + a_1r + a_1r^2 + \ldots + a_1r^{n-1}\}\\ &= (1-r)\{a_1(1 + r + r^2 + \ldots + r^{n-1})\}\\ &= a_1\{(1-r)(1 + r + r^2 + \ldots + r^{n-1})\}\\ &= a_1\{(1 + r + r + \ldots + r^{n-1}) - (r + r^2 + \ldots + r^n)\}\\ &= a_1(1-r^n)\end{aligned}$$

or $S_n = \dfrac{a_1(1-r^n)}{1-r} \qquad (r \neq 1)$

For convenience we use:

$$S_n = \frac{a_1(1-r^n)}{1-r} \qquad \text{if } |r| < 1$$

and $S_n = \dfrac{a_1(r^n - 1)}{r-1} \qquad \text{if } |r| > 1$

Example 1: Find the sum of n terms of the geometric series if $a_n = (-3)\left(\dfrac{2}{5}\right)^n$.

Solution: We can write $(\ 3)\left(-\right)$ as:

$-3\left(\dfrac{2}{5}\right)\left(\dfrac{2}{5}\right)^{n-1} = \left(-\dfrac{6}{5}\right)\left(\dfrac{2}{5}\right)^{n-1}$, that is,

$$a_n = \left(-\frac{6}{5}\right)\left(\frac{2}{5}\right)^{n-1}$$

Identifying $\left(-\dfrac{6}{5}\right)\left(\dfrac{2}{5}\right)^{n-1}$ with a_1r^{n-1},

we have, $a_1 = -\dfrac{6}{5}$ and $r = \dfrac{2}{5} < 1$

Thus $S_n = \dfrac{a_1(1-r^n)}{1-r} = \dfrac{-\frac{6}{5}\left[1-\left(\frac{2}{5}\right)^n\right]}{1-\frac{2}{5}}$

$$= \left(-\frac{6}{5}\right)\left(\frac{5}{3}\right)\left[1-\left(\frac{2}{5}\right)^n\right] = (-2)\left[1-\left(\frac{2}{5}\right)^n\right]$$

Example 2: The growth of a certain plant is 5% of its length monthly. When will the plant be of 4.41 cm if its initial length is 4 cm?

Solution: Let the initial length be l cm.Then at the end of one month, the plant will be of

length $l + \left(1 \times \dfrac{5}{100}\right) = l + \dfrac{l}{20} = \dfrac{21}{20}l$.

The length of the plant at the end of second month $= \dfrac{21}{20}l + \dfrac{21}{20}l \times \dfrac{5}{100}$

$$= \frac{21}{20}l\left(1+\frac{1}{20}\right) = \left(\frac{21}{20}\right)^2 l$$

So, the sequence of lengths at the end of successive months is, $\dfrac{21}{20}l, \left(\dfrac{21}{20}\right)^2 l, \left(\dfrac{21}{20}\right)^3 l, \ldots.$

Here $a_n = \left(\dfrac{21}{20}\right)l \times \left(\dfrac{21}{20}\right)^{n-1} = \left(\dfrac{21}{20}\right)^n l \qquad \left(\because a_1 = \dfrac{21}{20}l, r = \dfrac{21}{20}\right)$

Thus $4.41 = \left(\dfrac{21}{20}\right)^n \times 4 \qquad (\because$ initial length = 4 cm)

or $\left(\dfrac{21}{20}\right)^n = \dfrac{4.41}{4} = \dfrac{441}{400} = \left(\dfrac{21}{20}\right)^2$ which gives n = 2

6.10 The Infinite Geometric Series

Consider the series

$$a_1 + a_1 r + a_1 r^2 + \ldots. + a_1 r^{n-1} + \ldots.,$$

then $\qquad S_n = a_1 + a_1 r + \ldots + a_1 r^{n-1} = \dfrac{a_1(1-r^n)}{1-r} \qquad (r \neq 1)$

But we do not know how to add infinitely many terms of the series.

If $S_n \to a$ limit as $n \to \infty$ then the series is said to be convergent.

If S_n increases indefinitely as n becomes very large then we say that S_n does not exist and the series is said to be divergent.

Case I: If $|r| < 1$,

then r^n can be made as small as we like by taking n sufficiently large, that is,

$r^n \to 0 \quad$ as $\quad n \to \infty$

Obviously $S_n \to \dfrac{a_1}{1-r}$ when $n \to \infty$

In other words we can say that the series converges to the sum $\dfrac{a_1}{1-r}$ that is,

$$S = \lim_{n\to\infty} S_n = \frac{a_1}{1-r}$$

Case II: If $|r| > 1$,

then r^n does not tend to zero when $n \to \infty$

i.e., S_n does not tend to a limit and the series does not converge in this case so the series is divergent.

For example, if we take $a_1 = 1, r = 2$,

then the series, will be

1 + 2 + 4 + 8 + ...

and we have $S_1 = 1, = S_2 = 3, S_3 = 7, S_4 = 15, \ldots., S_n = 2^n - 1$, i.e., $S_1, S_2, S_3, \ldots., S_n$ is a sequence of ever increasing numbers.

In other words we can say that S_n increases indefinitely as $n \to \infty$. Thus the series does not converge.

Case III**:** If $r = 1$, then the series becomes

$$a_1 + a_1 + a_1 + a_1 + \ldots$$

and $S_n = na_1$. In this case S_n does not tend to a limit when $n \to \infty$ and the series does not converge.

Case IV**:** If $r = -1$, then the series becomes

$$a_1 - a_1 + a_1 - a_1 + a_1 - a_1 + \ldots$$

and $S_n = \dfrac{a_1 - (-1)^n a_1}{2}$

i.e., $S_n = a_1$ if n is positive odd integer.

$S_n = 0$ if n is positive even integer.

Thus S_n does not tend to a definite number when $n \to \infty$. In such a case we say that the series is oscillatory.

Example 3: Find the sum of the infinite G.P. $2, \sqrt{2}, 1, \ldots$

Solution: Here $a_1 = 2$

$$r = \frac{a_2}{a_1} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \text{ and}$$

$$S = \frac{2}{1 - \frac{1}{\sqrt{2}}} \qquad \left(\because \frac{1}{\sqrt{2}} < 1\right)$$

$$= \frac{2\sqrt{2}}{\sqrt{2} - 1} = \frac{2\sqrt{2}(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} = \frac{4 + 2\sqrt{2}}{2 - 1} = 4 + 2\sqrt{2}$$

Example 4: Convert the recurring decimal $2.\dot{2}\dot{3}$ into an equivalent common fraction (vulgur fraction).

Solution: $2.\dot{2}\dot{3}$ = 2.232323 ....

= 2 + {.23 + .0023 + .000023 +...}

$$= 2 + \left[\frac{.23}{1 - \frac{1}{100}}\right] \qquad \left(\because \frac{.0023}{.23} = \frac{1}{100}\right)$$

$$= 2 + \frac{100 \times .23}{99} = 2 + \frac{23}{99}$$

$$= \frac{198 + 23}{99} = \frac{221}{99}$$

Example 5: The sum of an infinite geometric series is half the sum of the squares of its terms. If the sum of its first two terms is $\frac{9}{2}$, find the series.

Solution: Let the series be

$$a_1 + a_1 r + a_1 r^2 + \ldots \qquad \text{(i)}$$

Then the series whose terms are the squares of the terms of the above series is

$$a_1^2 + a_1^2 r^2 + a_1^2 r^4 + \ldots \qquad \text{(ii)}$$

Let S_1 and S_2 be the sum of the series (i) and (ii) respectively. Then

$$S_1 = \frac{a_1}{1 - r} \qquad \text{(iii)}$$

and $$S_2 = \frac{a_1^2}{1 - r^2} \qquad \text{(iv)}$$

By the first given condition, we have.

$$S_1 = \frac{1}{2} S_2 \Rightarrow \frac{a_1}{1 - r} = \frac{1}{2}\left(\frac{a_1^2}{1 - r^2}\right)$$

$$\Rightarrow a_1 = 2(1 + r) \qquad \text{(v)}$$

From the other given condition, we get

$$a_1 + a_1 r = \frac{9}{2} \Rightarrow a_1(1+r) = \frac{9}{2} \qquad \text{(vi)}$$

Substituting $a_1 = 2(1+r)$ in (vi), gives

$$2(1+r)\,(1+r) = \frac{9}{2} \Rightarrow (1+r)^2 = \frac{9}{4}$$

$$\Rightarrow 1+r = \pm\frac{3}{2}$$

$$\Rightarrow r = \frac{1}{2}, -\frac{5}{2}$$

For $r = -\frac{5}{2}$, $|r| = \frac{5}{2} > 1$, so we cannot take $r = -\frac{5}{2}$.

if $r = \frac{1}{2}$, then $a_1 = 2(1+\frac{1}{2}) = 3$ $\qquad [\because\ a_1 = 2(1+r)]$

Hence the series is $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \ldots$

Example 6: If $a = 1 - x + x^2 - x^3 + \ldots \qquad |x| < 1$

$b = 1 + x + x^2 + x^3 + \ldots \qquad |x| < 1$

show that 2*ab* = *a* + *b*.

Solution: $a = \frac{1}{1-(-x)} \qquad (\because\ r = -x)$

or $\quad a = \frac{1}{1+x} \Rightarrow 1 + x = \frac{1}{a} \qquad$ (i)

and $\quad b = \frac{1}{1-x} \qquad (\because r = x)$

$\Rightarrow \quad 1 - x = \frac{1}{b} \qquad$ (ii)

Adding (i) and (ii), we obtain

$2 = \frac{1}{a} + \frac{1}{b}$, which implies that

$2ab = a + b$

Exercise 6.8

1. Find the sum of first 15 terms of the geometric sequence $1, \frac{1}{3}, \frac{1}{9}, \ldots$

2. Sum to *n* terms, the series

i) .2 + .22 + .222 + ... ii) 3 + 33 + 333 + ...

3. Sum to *n* terms the series

i) $1 + (a+b) + (a^2 + ab + b^2) + (a^3 + a^2b + ab^2 + b^3) + \ldots$

ii) $r + (1+k)r^2 + (1+k+k^2)r^3 + \ldots$

4. Sum the series $2 + (1-i) + \left(\frac{1}{i}\right) + \ldots$ to 8 terms.

5. Find the sums of the following infinite geometric series:

i) $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots$ ii) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ iii) $\frac{9}{4} + \frac{3}{2} + 1 + \frac{2}{3} + \ldots$

iv) 2 + 1 + 0.5 + ... v) $4 + 2\sqrt{2} + 2 + \sqrt{2} + 1 + \ldots$

vi) 0.1 + 0.05 + 0.025 + ...

6. Find vulgar fractions equivalent to the following recurring decimals.

i) $1.\dot{3}\dot{4}$ ii) $0.\dot{7}$ iii) $0.\dot{2}5\dot{9}$

iv) $1.\dot{5}\dot{3}$ v) $0.\dot{1}5\dot{9}$ vi) $1.1\dot{4}\dot{7}$

7. Find the sum to infinity of the series; $r + (1+k)r^2 + (1+k+k^2)r^3 + \ldots$ *r* and *k* being proper fractions.

8. If $y = \frac{x}{2} + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \ldots$ and if $0 < x < 2$, then prove that $x = \frac{2y}{1+y}$

9. If $y = \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \ldots$ and if $0 < x < \frac{3}{2}$, then show that $x = \frac{3y}{2(1+y)}$

10. A ball is dropped from a height of 27 meters and it rebounds two-third of the distance it falls. If it continues to fall in the same way what distance will it travel before coming to rest?

11. What distance will a ball travel before coming to rest if it is dropped from a height of

75 meters and after each fall it rebounds $\frac{2}{5}$ of the distance it fell?

12. If $y = 1 + 2x + 4x^2 + 8x^3 + \ldots$

i) Show that $x = \frac{y-1}{2y}$

ii) Find the interval in which the series is convergent.

13. If $y = 1 + \frac{x}{2} + \frac{x^2}{4} + \ldots$

i) Show that $x = 2\left(\frac{y-1}{y}\right)$

ii) Find the interval in which the serieis is convergent.

14. The sum of an infinite geometric series is 9 and the sum of the squares of its terms is $\frac{81}{5}$. Find the series.

6.11 Word Problems on G.P.

Example 1: A man deposits in a bank Rs. 20 in the first year; Rs. 40 in the second year; Rs. 80 in the third year and so on. Find the amount he will have deposited in the bank by the seventh year.

Solution: The deposits in the succcessive yesrs are

20,40,80,... which is a geometric sequence with

$a_1 = 20$ and $r = 2$

The sum of the seven terms of the above sequence is the total amount deposited in the bank upto the seventh year, so we have to find S_7, that is,

the required deposit in Rs. $= \frac{20(2^7-1)}{2-1} = \frac{20(2^7-1)}{1}$

$= 20(128 - 1) = 20 \times 127$

$= 2540$

Thus the amount deposited in the bank upto the seventh year is Rs. 2540.

Example 2: A person invests Rs.2000/- at 4% interest compounded annually. What the total amount will he get after 5 years?

Solution: Let the principal amtount be *P*. Then

the interest for the first year $= P \times \frac{4}{100} = P \times (.04)$

The total amount at the end of the first year $P + P \times (.04) = P(1+.04)$

The interest for the second year $= [P(1+0.4) \times (.04)]$ and the total amount at the end of second year $= [P(1+0.4)] + [P(1+0.4)] \times (0.4)$

$= P(1+.04)(1+.04) = P(1+.04)^2$

Similarly the total amount at the end of third year $= P(1+.04)^3$

Thus the sequence for total amounts at the end of successive years is

$P(1+.04), P(1+.04)^2, P(1+.04)^3, \ldots$

The amount at the end of the fifth year is the fifth term of the above gemoetric sequence, that is

$a_5 = [P(1+.04)](1+.04)^{5-1}$ $\quad (\because\ a_5 = a_1 r^{5-1}$ and $a_1 = P(1+.04))$

$= P(1+.04)^5$

As *P* = 2000, so the required total amount in rupees = 2000 × $(1+.04)^5$

$\simeq 2000 \times (1.216653) \simeq 2433.31$

Example 3: The population of a big town is 972405 at present and four years before it was 800,000. Find its rate of increase if it increased geometrically.

Solution: Let the rate of increase be *r* % annually. Then the sequence of population is

$$800,000, 800,000\left(1+\frac{r}{100}\right), 800,000 \times \left(1+\frac{r}{100}\right)^2, \ldots$$

and its fifth term = 972405.

In this case we have,

$$a_n = a_1\left(1+\frac{r}{100}\right)^{n-1} \qquad \left(\because \text{ ratio is } \left(1+\frac{r}{100}\right)\right)$$

Thus $972405 = 800,000\left(1+\frac{r}{100}\right)^{5-1}$ $\quad (\because a_5 = 972405 \text{ and } a_1 = 800,000)$

or $\left(1+\frac{r}{100}\right)^4 = \frac{972405}{800,000}$

i.e. $\left(1+\frac{r}{100}\right)^4 = \frac{194481}{160000} \Rightarrow \left(1+\frac{r}{100}\right)^4 = \left(\frac{21}{20}\right)^4 \Rightarrow 1+\frac{r}{100} = \frac{21}{20}$

$$\Rightarrow \frac{r}{100} = \frac{21}{20} - 1 = \frac{1}{20}$$

$$\Rightarrow r = 5$$

Hence the rate of increase is 5%.

Exercise 6.9

1. A man deposits in a bank Rs. 8 in the first year, Rs. 24 in the second year Rs.72 in the third year and so on. Find the amount he will have deposited in the bank by the fifth year.
2. A man borrows Rs. 32760 without interest and agrees to repay the loan in installments, each installment being twice the preceding one. Find the amount of the last installment, if the amount of the first installment is Rs.8.
3. The population of a certain village is 62500. What will be its population after 3 years if it increases geometrically at the rate of 4% annually?
4. The enrollment of a famous school doubled after every eight years from 1970 to 1994. If the enrollment was 6000 in 1994, what was its enrollment in 1970?
5. A singular cholera bacteria produces two complete bacteria in $\frac{1}{2}$ hour. If we start with a colony of a bacteria, how many bacteria will we have in n hours?
6. Joining the mid points of the sides of an equilateral triangle, an equilateral triangle having half the perimeter of the original triangle is obtained. We form a sequence of nested equilateral triangles in the manner described above with the original triangle having

perimeter $\frac{3}{2}$. What will be the total perimeter of all the triangles formed in this way?

6.12 Harmonic Progression (H.P)

A sequence of numbers is called a Harmonic Sequence or Harmonic Progression if the reciprocals of its terms are in arithmetic progression. The sequence $1,\frac{1}{3},\frac{1}{5},\frac{1}{7}$ is a harmonic sequence since their reciprocals 1,3,5,7 are in A.P.

Remember that the reciprocal of zero is not defined, so zero can not be the term of a harmonic sequence.

The general form of a harmonic sequence is taken as:

$$\frac{1}{a_1},\frac{1}{a_1+d},\frac{1}{a_1+2d},\dots \qquad \text{whose } n\text{th term is } \frac{1}{a_1+(n-1)d}$$

Example 1: Find the nth and 8th terms of H.P ; $\frac{1}{2},\frac{1}{5},\frac{1}{8},\dots$

Solution: The reciprocals of the terms of the sequence,

$\frac{1}{2},\frac{1}{5},\frac{1}{8},\dots$ are 2,5,8,...

The numbers 2,5,8,... are A.P., so

$a_1 = 2$ and $d = 5-2 = 3$.

Putting these values in $a_n = a_1+(n-1)d$, we have

$$a_n = 2+(n-1)3$$
$$= 3n-1$$

Thus the nth term of the given sequence $=\frac{1}{a_n} = \frac{1}{3n-1}$

and substituting $n = 8$ in $\frac{1}{3n-1}$, we get the 8th term of the given H.P. which is $\frac{1}{3\times 8-1}=\frac{1}{23}$
.

Alternatively, a_8 of the A.P. $= a_1 + (8-1)d$

$= 2 + (7).3 = 23$

Thus the 8th term of the given H.P. $= \frac{1}{23}$

Example 2: If the 4th term and 7th term of an H.P. are $\frac{2}{13}$ and $\frac{2}{25}$ respectively, find the sequence.

Solution: Since the 4th term of the H.P. $= \frac{2}{13}$ and its 7th term $= \frac{2}{25}$, therefore the 4th and 7th terms of the corresponding A.P. are $\frac{13}{2}$ and $\frac{25}{2}$ respectively.

Now taking a_1, the first term and *d,* the common difference of the corresponding A.P, we have,

$$a_1 + 3d = \frac{13}{2} \qquad \text{(i)}$$

and $$a_1 + 6d = \frac{25}{2} \qquad \text{(ii)}$$

Subtracting (i) from (ii), gives

$$3d = \frac{25}{2} - \frac{13}{2} = 6 \Rightarrow d = 2$$

From (i), we get

$$a_1 = \frac{13}{2} - 3d = \frac{13}{2} - 6 = \frac{1}{2}$$

Thus a_2 of the A.P. $a_1 + d = \frac{1}{2} + 2 = \frac{5}{2}$

and a_3 of the A.P. $a_1 + 2d = \frac{1}{2} + 2(2)$

$$= \frac{1}{2} + 4 = \frac{9}{2}.$$

Hence the required H.P. is $\frac{2}{1}, \frac{2}{5}, \frac{2}{9}, \frac{2}{13}, \ldots$

6.12.1 Harmonic Mean : A number *H* is said to be the harmonic mean (H.M) between two numbers *a* and *b* if *a*, *H*, *b* are in H.P.

Let *a*, *b* be the two numbers and *H* be their H.M. Then $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.P.

therefore, $\frac{1}{H} = \frac{\frac{1}{a}+\frac{1}{b}}{2} = \frac{\frac{b+a}{ab}}{2} = \frac{a+b}{2ab}$

and $H = \frac{2ab}{a+b}$

For example, H.M. between 3 and 7 is

$$\frac{2\times 3\times 7}{3+7} = \frac{2\times 21}{10} = \frac{21}{5}$$

6.12.2 *n* Harmonic Means between two numbers

$H_1, H_2, H_3 \ldots, H_n$ are called *n* harmonic means (H.Ms) between *a* and *b* if $a, H_1, H_2, H_3, \ldots H_n, b$ are in H.P. If we want to insert *n* H.Ms. between *a* and *b*, we first find *n* A.Ms. $A_1, A_2, \ldots, A_n$ between $\frac{1}{a}$ and $\frac{1}{b}$, then take their reciprocals to get *n* H.Ms between *a* and *b*, that is, $\frac{1}{A_1}, \frac{1}{A_2}, \ldots, \frac{1}{A_n}$ will, be the required *n* H.Ms. between *a* and *b*.

Example 3: Find three harmonic means between $\frac{1}{5}$ and $\frac{1}{17}$.

Solution: Let A_1, A_2, A_3 be three A.Ms. between 5 and 17, that is, $5, A_1, A_2, A_3, 17$ are in A.P.

Using $a_n = a_1 + (n-1)d$, we get,

$17 = 5 + (5-1)d \qquad (\because\ a_5 = 17 \text{ and } a_1 = 5)$

$4d = 12$

$\Rightarrow \quad d = 3$

Thus $A_1 = 5+3 = 8, A_2 = 5+2(3) = 11$ and $A_3 = 5+3(3) = 14$

Hence $\frac{1}{8}, \frac{1}{11}, \frac{1}{14}$ are the required harmonic means.

Example 4: Find *n* H.Ms between *a* and *b*

Solution: Let $A_1, A_2, A_3, \ldots, A_n$, be *n* A.Ms between $\frac{1}{a}$ and $\frac{1}{b}$.

Then $\frac{1}{a}, A_1, A_2, A_3, \ldots, A_n, \frac{1}{b}$ are in A.P.

Using $a_n = a_1 + (n-1)d$, we get,

$$\frac{1}{b} = \frac{1}{a} + (n+2-1)d$$

or $(n+1)d = \frac{1}{b} - \frac{1}{a} \Rightarrow d = \frac{a \quad b}{ab(n \quad 1)}$

Thus $A_1 = \frac{1}{a} + d = \frac{1}{a} + \frac{a-b}{ab(n+1)} = \frac{b(n+1)+(a-b)}{ab(n+1)} = \frac{nb+a}{ab(n+1)}$

$$A_2 = \frac{1}{a} + 2d = \frac{1}{a} + 2\left(\frac{a-b}{ab(n+1)}\right) = \frac{b(n+1)+2(a-b)}{ab(n+1)} = \frac{(n-1)b+2a}{ab(n+1)}$$

$$A_3 = \frac{1}{a} + 3d = \frac{1}{a} + 3\left(\frac{a-b}{ab(n+1)}\right) = \frac{b(n+1)+3(a-b)}{ab(n+1)} = \frac{(n-2)b+3a}{ab(n+1)}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$A_n = \frac{1}{a} + nd = \frac{1}{a} + n\left(\frac{a-b}{ab(n+1)}\right) = \frac{b(n+1)+n(a-b)}{ab(n+1)} = \frac{b+na}{ab(n+1)}$$

Hence *n* H.Ms between *a* and *b* are:

$$\frac{ab(n+1)}{nb+a}, \frac{ab(n+1)}{(n-1)b+2a}, \frac{ab(n+1)}{(n-2)b+3a}, \ldots, \frac{ab(n+1)}{b+na}$$

6.13 Relations between Arithmetic, Geometric and Harmonic Means

We know that for any two numbers *a* and *b*

$$A = \frac{a+b}{2}, G = \pm\sqrt{ab} \quad \text{and} \quad H = \frac{2ab}{a+b}$$

We first find $A \times H$ that is,

$$A \times H = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab$$

$$= G^2$$

Thus *A*, *G*, *H* are in G.P. For example, if

a = –1 and *b* = 5 , then

$$A = \frac{-1+5}{2} = 2, \qquad G = \pm\sqrt{-1\times 5} = \pm\sqrt{5}i$$

$$H = \frac{2(-1).5}{-1+5} = \frac{-10}{4} = \frac{-5}{2}$$

$$A \times H = 2 \times \frac{-5}{2} = -5 \text{ and } G^2 = (\pm\sqrt{5}i)^2 = 5i^2 = -5$$

It follows that $A \times H = G^2$ and *A, G, H* are in G.P.

Note: $G^2 = A \times H$ even if $a, b \in C$

Now we show that $A > H$ for any two distinct positive real numbers.

$$A > H \text{ if } \frac{a+b}{2} > \frac{2ab}{a+b}$$

or $\quad (a+b)^2 > 4ab$

or $\quad (a+b)^2 - 4ab > 0 \Rightarrow (a-b)^2 > 0$

which is true because $a - b$ is a real number and the square of a real number is always positive.

Also $A > G$ if a, b are any two distinct positive real numbers.

$$A > G \text{ if } \frac{a+b}{2} > \pm\sqrt{ab}$$

or $\quad a + b \mp 2\sqrt{ab} > 0$

$\Rightarrow \quad (\sqrt{a} \mp \sqrt{b})^2 > 0$

which is true because $\sqrt{a} \mp \sqrt{b}$ are non zero real numbers and the squares of real numbers are always positive.

Now we prove that

i) $A > G > H$ if a, b are any two distinct positive real numbers and $G = \sqrt{ab}$.

ii) $A < G < H$ if a, b are any two distinct negative real numbers and $G = -\sqrt{ab}$.

To prove (i) we first show that $A > G$, i.e.,

$$A > G \text{ if } \frac{a+b}{2} > \sqrt{ab}$$

$\Rightarrow \quad (\sqrt{a} - \sqrt{b})^2 > 0$

which is true (write the missing steps as given above)

Thus $A > G$ (1)

Again $G > H$,

if $\sqrt{ab} > \dfrac{2ab}{a+b}$

or $a + b > 2\sqrt{ab}$

$\Rightarrow \quad a + b - 2\sqrt{a}\sqrt{b} > 0$

$\Rightarrow \quad (\sqrt{a} - \sqrt{b})^2 > 0$

which is true since $\sqrt{a} - \sqrt{b}$ is a real number.

Thus $G > H$ (2)

From (1) and (2), we have

$A > G > H$

To prove (ii), we show that

$A < G$ if

$$\frac{a+b}{2} < -\sqrt{ab}$$

Let $a = -m$ and $b = -n$ where m and n are positive real numbers. Then

$$\frac{-m-n}{2} < -\sqrt{(-m)(-n)}$$

or $-\dfrac{m+n}{2} < -\sqrt{mn} \Rightarrow \dfrac{m+n}{2} > \sqrt{mn}$

$\Rightarrow (\sqrt{m} - \sqrt{n})^2 > 0$ (See part(i))

which is true, that is,

$A < G$

Similarly, we can prove that

$G < H$

Hence $A < G < H$

Exercise 6.10

1. Find the 9th term of the harmonic sequence

i) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$ ii) $\frac{-1}{5}, \frac{-1}{3}, -1, \ldots$

2. Find the 12th term of the following harmonic sequences

i) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$ ii) $\frac{1}{3}, \frac{2}{9}, \frac{1}{6}, \ldots$

3. Insert five harmonic means between the following given numbers,

i) $\frac{-2}{5}$ and $\frac{2}{13}$ ii) $\frac{1}{4}$ and $\frac{1}{24}$

4. Insert four harmonic means between the following given numbers.

i) $\frac{1}{3}$ and $\frac{1}{23}$ ii) $\frac{7}{3}$ and $\frac{7}{11}$ iii) 4 and 20

5. If the 7th and 10th terms of an H.P. are $\frac{1}{3}$ and $\frac{5}{21}$ respectively, find its 14th term.

6. The first term of an H.P. is $-\frac{1}{3}$ and the fifth term is $\frac{1}{5}$. Find its 9th term.

7. If 5 is the harmonic mean between 2 and *b*, find *b*.

8. If the numbers $\frac{1}{k}, \frac{1}{2k+1}$ and $\frac{1}{4k-1}$ are in harmonic sequence, find k.

9. Find *n* so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be H.M. between *a* and *b*.

10. If a^2, b^2 and c^2 are in A.P. show that $a+b$, $c+a$ and $b+c$ are in H.P.

11. The sum of the first and fifth term of the harmonic sequence is $\frac{4}{7}$, if the first term is $\frac{1}{2}$, find the sequence.

12. If *A*, *G* and *H* are the arithmetic, geometric and harmonic means between *a* and *b* respectively, show that $G^2 = AH$.

13. Find *A, G, H* and show that $G^2 = AH$. if

i) $a=-2, b=-6$ ii) $a=2i, b=4i$ iii) $a=9, b=4$

14. Find *A, G, H* and verify that $A > G > H$ $(G > 0)$, if

i) $a=2, b=8$ ii) $a=\frac{2}{5}, b=\frac{8}{5}$

15. Find *A, G, H* and verify that $A < G < H$ $(G < 0)$, if

i) $a=-2, b=-8$ ii) $a=\frac{-2}{5}, b=\frac{-8}{5}$

16. If the H.M and A.M. between two numbers are 4 and $\frac{9}{2}$ respectively, find the numbers.

17. If the (positive) G.M. and H.M. between two numbers are 4 and $\frac{16}{5}$, find the numbers.

18. If the numbers $\frac{1}{2}, \frac{4}{21}$ and $\frac{1}{36}$ are subtracted from the three consecutive terms of a G.P., the resulting numbers are in H.P. Find the numbers if their product is $\frac{1}{27}$.

6.14 Sigma Notation (or Summation Notation)

The Greek letter $\sum$(sigma) is used to denote sums of different types . For example the notation $\sum_{i=m}^{n} a_i$ is used to express the sum

$$a_m + a_{m+1} + a_{m+2} + \dots + a_n$$ and the sum expression

1 + 3 + 5+ ....to *n* terms.

is written as $\sum_{k=1}^{n}(2k-1)$,

where $(2k-1)$ is the *k*th term of the sum and *k* is called the index of summation. '1' and *n* are called the lower limit and the upper limit of summation respectively. The sum of the first *n* natural numbers, the sum of squares of the first *n* natural numbers and the sum of the cubes of the first *n* natural numbers are expressed in sigma notation as:

$$1+2+3+\dots+n = \sum_{k=1}^{n} k$$

$$1^2+2^2+3^2+\dots+n^2 = \sum_{k=1}^{n} k^2$$

$$1^3+2^3+3^3+\dots+n^3 = \sum_{k=1}^{n} k^3$$

We evaluate $\sum_{k=1}^{n}[k^m-(k-1)^m]$ for any positive integer *m* and shall use this result to find out formulas for three expressions stated above.

$$\sum_{k=1}^{n}[k^m-(k-1)^m]=(1^m-0^m)+(2^m-1^m)+(3^m-2^m)+\dots$$

$$+[(n-1)^m-(n-2)^m]+[n^m-(n-1)^m]=n^m$$

i.e., $\sum_{k=1}^{n}[(k^m-(k-1)^m]=n^m$

If $m=1$,

then $\sum_{k=1}^{n}[(k^1-(k-1)^1]=n^1$

i.e. $\sum_{k=1}^{n}1=n$

6.15 To Find the Formulae for the Sums

i) $\sum_{k=1}^{n}k$ ii) $\sum_{k=1}^{n}k^2$ iii) $\sum_{k=1}^{n}k^3$

i) We know that $k^2-(k-1)^2 = 2k-1$ (A)

Taking summation on both sides of (A) from k = 1 to *n*, we have

$$\sum_{k=1}^{n}[k^2-(k-1)^2]=\sum_{k=1}^{n}(2k-1)$$

i.e., $n^2=2\sum_{k=1}^{n}k-n$ $\quad (\because \sum_{k=1}^{n}1=n)$

or $2\sum_{k=1}^{n}k=n^2+n$

Thus $\sum_{k=1}^{n}k=\dfrac{n(n+1)}{2}$

ii) Consider the identity

$$k^3-(k-1)^3 = 3k^2-3k+1 \qquad \text{(B)}$$

Taking summation of (B) on both sides from $k=1$ to *n*, we get

$$\sum_{k=1}^{n}[k^3-(k-1)^3]=\sum_{k=1}^{n}(3k^2-3k+1)$$

i.e., $n^3=3\sum_{k=1}^{n}k^2-3\sum_{k=1}^{n}k+n$

or $3\sum_{k=1}^{n}k^2=n^3-n+3\sum_{k=1}^{n}k$

$$=n(n+1)(n-1)+3\times\frac{n(n+1)}{2}$$

$$=n(n+1)\left[n-1+\frac{3}{2}\right]=\frac{n(n+1)(2n+1)}{2}$$

Thus $\sum_{k=1}^{n}k^2=\dfrac{n(n+1)(2n+1)}{6}$

iii) We know that $(k-1)^4 = k^4-4k^3+6k^2-4k+1$ and this identity can be written as: (C)

$$k^4-(k-1)^4=4k^3-6k^2+4k-1$$

Taking summation on both sides of (C), from $k=1$ to *n*, we get,

$$\sum_{k=1}^{n}[k^4-(k-1)^4]=\sum_{k=1}^{n}(4k^3-6k^2+4k-1)$$

i.e., $n^4=4\sum_{k=1}^{n}k^3-6\sum_{k=1}^{n}k^2+4\sum_{k=1}^{n}k-n$

or $4\sum_{k=1}^{n}k^3=n^4+n+6\sum_{k=1}^{n}k^2-4\sum_{k=1}^{n}k$

$$=n(n+1)(n^2-n+1)+6\times\frac{n(n+1)(2n+1)}{6}-4\times\frac{n(n+1)}{2}$$

$$=n(n+1)[n^2-n+1+2n+1-2]$$

$$=n(n+1)(n^2+n)=n(n+1).n(n+1)$$

Thus $\sum_{k=1}^{n}k^3=\dfrac{[n(n+1)]^2}{4}=\left[\dfrac{n(n+1)}{2}\right]^2$

Example 1: Find the sum of the series $1^3+3^3+5^3+\ldots \text{to } n \text{ terms}$

Solution: $T_k=(2k-1)^3 \qquad (\because 1+(k-2)2=2k-1)$

$$=8k^3-12k^2+6k-1$$

Let S_n denote the sum of *n* terms of the given series, then

$$S_n=\sum_{k=1}^{n} T_k$$

or $$S_n=\sum_{k=1}^{n}(8k^2-12k^2+6k-1)$$

$$=8\sum_{k=1}^{n}k^3-12\sum_{k=1}^{n}k^2+6\sum_{k=1}^{n}k-\sum_{k=1}^{n}1$$

$$=8\left[\frac{n(n+1)}{2}\right]^2-12\left[\frac{n(n+1)(2n+1)}{6}\right]+6\left[\frac{n(n+1)}{2}\right]-n$$

$$=2n^2(n+1)^2-2n(n+1)(2n+1)+3n(n+1)-n$$

$$=2n^2(n^2+2n+1)-2n(2n^2+3n+1)+n(3n+3)-n$$

$$=2n[(n^3+2n^2+n)-(2n^2+3n+1)]+n(3n+3-1)$$

$$=2n[n^3-2n-1]+n(3n+2)$$

$$=n[2n^3-4n-2+3n+2]$$

$$=n[2n^2-n]=n.[n(2n^2-1)]$$

$$=n^2[2n^2-1]$$

Example 2: Find the sum of *n* terms of series whose *n*th terms is $n^3+\frac{3}{2}n^2+\frac{1}{2}n+1$

Solution: Given that

$$T_n=n^3+\frac{3}{2}n^2+\frac{1}{2}n+1$$

Thus $$T_k=k^3+\frac{3}{2}k^2+\frac{1}{2}k+1$$

and $$S_n=\sum_{k=1}^{n}\left(k^3+\frac{3}{2}k^2+\frac{1}{2}k+1\right)$$

$$=\sum_{k=1}^{n}k^3+\frac{3}{2}\sum_{k=1}^{n}k^2+\frac{1}{2}\sum_{k=1}^{n}k+\sum_{k=1}^{n}1$$

$$=\frac{n^2(n+1)^2}{4}+\frac{3}{2}\times\frac{n(n+1)(2n+1)}{6}+\frac{1}{2}\times\frac{n(n+1)}{2}+n$$

$$=\frac{n}{4}[n(n^2+2n+1)+(2n^2+3n+1)+(n+1)+4]$$

$$=\frac{n}{4}(n^3+2n^2+n+2n^2+3n+1+n+1+4)$$

$$=\frac{n}{4}(n^3+4n^2+5n+6)$$

Exercise 6.11

Sum the following series upto *n* terms.

1. $1\times1+2\times4+3\times7+\ldots$

2. $1\times3+3\times6+5\times9+\ldots$

3. $1\times4+2\times7+3\times10+\ldots$

4. $3\times5+5\times9+7\times13+\ldots$

5. $1^2+3^2+5^2+\ldots$

6. $2^2+5^2+8^2+\ldots$

7. $2\times1^2+4\times2^2+6\times3^2+\ldots$

8. $3\times2^2+5\times3^2+7\times4^2+\ldots$

9. $2\times4\times7+3\times6\times10+4\times8\times13+\ldots$

10.. $1\times4\times6+4\times7\times10+7\times10\times14+\ldots$

11. $1+(1+2)+(1+2+3)+\ldots$

12. $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\ldots$

13. $2+(2+5)+(2+5+8)+\ldots$

14. Sum the series.

i) $1^2-2^2+3^2-4^2+\ldots+(2n-1)^2-(2n)^2$

ii) $1^2-3^2+5^2-7^2+\ldots+(4n-3)^2-(4n-1)^2$

iii) $\frac{1^2}{1}+\frac{1^2+2^2}{2}+\frac{1^2+2^2+3^2}{3}+\ldots \text{ to } n \text{ terms}$

15. Find the sum to *n* terms of the series whose *n*th terms are given.

i) $3n^2+n+1$ ii) n^2+4n+1

16. Given *n*th terms of the series, find the sum to 2*n* terms.

i) $3n^2+2n+1$ ii) n^3+2n+3