CHAPTER

7

Permutation Combination and Probability

7.1 Introduction

The **factorial notation** was introduced by Christian Kramp (1760 - 1826) in 1808. This notation will be frequently used in this chapter as well as in finding the Binomial Coefficients in later chapter. Let us have an introduction of **factorial notation**.

Let n be a positive integer. Then the product $n(n-1)(n-2)....\ 3\ .\ 2\ .\ 1$ is denoted by $n!$ or $\underline{|n}$ and read as n factorial.

That is, $\boxed{n! = n(n-1)(n-2)\ ....3.2.1}$

For Example,

$1! = 1$

	$2! = 2.1$	$= 2$	$\Rightarrow 2! = 2.1!$
	$3! = 3.2.1$	$= 6$	$\Rightarrow 3! = 3.2!$
	$4! = 4.3.2.1$	$= 24$	$\Rightarrow 4! = 4.3!$
	$5! = 5.4.3.2.1$	$= 120$	$\Rightarrow 5! = 5.4!$
and	$6! = 6.5.4.3.2.1$	$= 720$	$\Rightarrow 6! = 6.5!$

Thus for a positive integer n, we define n factorial as:
$n! = n(n-1)!$ where $0! = 1$

Example 1: Evaluate $\frac{8!}{6!}$

Solution: $\frac{8!}{6!} = \frac{8.7.6.5.4.3.2.1}{6.5.4.3.2.1} = 56$

Example 2: Write 8.7.6.5 in the factorial form

Solution: $8.7.6.5 = \frac{8.7.6.5.4.3.2.1}{4.3.2.1} = \frac{8!}{4!}$

Example 3: Evaluate $\frac{9!}{6!3!}$

Solution: $\frac{9!}{6!3!} = \frac{(9.8.7)6!}{6!(3.2.1)} = 84$

or $\frac{9!}{6!3!} = \frac{9.8.7.6.5.4.3.2.1}{6.5.4.3.2.1.3.2.1} = 84$

Exercise 7.1

1. Evaluate each of the following:

i) $4!$ ii) $6!$ iii) $\frac{8!}{7!}$ iv) $\frac{10!}{7!}$

v) $\frac{11!}{4!7!}$ vi) $\frac{6!}{3!3!}$ vii) $\frac{8!}{4!2!}$ viii) $\frac{11!}{2!4!5!}$

ix) $\frac{9!}{2!(9-2)!}$ x) $\frac{15!}{15!(15-15)!}$ xi) $\frac{3!}{0!}$ xii) $4!.0!.1!$

2. Write each of the following in the factorial form:

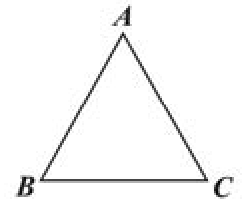
i) 6.5.4. ii) 12.11.10 iii) 20.19.18.17

iv) $\frac{10.9}{2.1}$ v) $\frac{8.7.6}{3.2.1}$ vi) $\frac{52.51.50.49}{4.3.2.1}$

vii) $n(n-1)(n-2)$ viii) $(n+2)(n+1)(n)$ ix) $\frac{(n+1)(n)(n-1)}{3.2.1}$

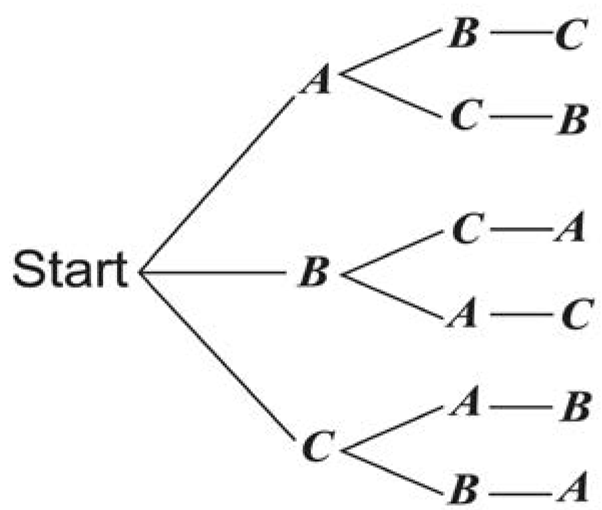
x) $n(n-1)(n-2)....(n-r+1)$

7.2 Permutation



Suppose we like to find the number of different ways to name the triangle with vertices *A*, *B* and *C*.

The various possible ways are obtained by constructing a tree diagram as follows:



To determine the possible ways, we count the paths of the tree, beginning from the start to the end of each branch. So, we get 6 different names of triangle.

ABC, ACB, BCA, BAC, CAB, CBA.

Thus there are six possible ways to write the name of the triangle with vertices *A, B* and *C.*

Explanation: In the figure, we can write any one of the three vertices *A, B, C* at first place. After writing at first place any one of the three vertices, two vertices are left. So, there are two choices to write at second place. After writing the vertices at two places, there is just one vertex left. So, we can write only one vertex at third place.

Another Way of Explanation:

Think of the three places as shown

Since we can write any one of the three vertices at first place, so it is written in 3 different ways as shown. | 3 | | |

Now two vertices are left. So, corresponding to each way of writing at first place, there

are two ways of writing at second place as shown. | 3 | 2 | |

Now just one vertex is left. So, we can write at third place only one vertex in one way as shown. | 3 | 2 | 1 |

The total number of possible ways (arrangements) is the product 3.2.1= 6.

This example illustrates the fundamental principle of counting.

Fundamental Principle of Counting:

Suppose *A* and *B* are two events. The first event *A* can occur in *p* different ways. After *A* has occurred, *B* can occur in *q* different ways. The number of ways that the two events can occur is the product *p.q.*

This principle can be extended to three or more events. For instance, the number of ways that three events *A, B* and *C* can occur is the product *p.q.r*.

One important application of the Fundamental Principle of Counting is to determine the number of ways that *n* objects can be arranged in order. An ordering (arrangement) of *n* objects is called a **permutation** of the objects.

A **permutation** of *n* different objects is an ordering (arrangement) of the objects such that one object is first, one is second, one is third and so on.

According to Fundamental Principle of Counting:

i) Three books can be arranged in a row taken all at a time = 3.2.1 = 3! ways

ii) Number of ways of writing the letters of the **WORD** taken all at a time = 4.3.2.1 = 4!

Each arrangement is called a permutation. Now we have the following definition.

A permutation of *n* different objects taken *r* ($\leq$ *n*) at a time is an arrangement of the *r* objects. Generally it is denoted by ${}^nP_r$ or *P*(*n*,*r*).

Prove that: ${}^np_r = n(n-1)(n-2)....(n-r+1) = \frac{n!}{(n-r)!}$

Proof: As there are *n* different objects to fill up *r* places. So, the first place can be filled in *n* ways. Since repetitions are not allowed, the second place can be filled in (*n*–1) ways, the third place is filled in (*n*–2) ways and so on. The *r*th place has *n* – (*r* – 1) = n – *r* + 1 choices to be filled in. Therefore, by the fundamental principle of counting, *r* places can be filled by *n* different objects in *n*(*n* – 1)(*n* – 2) .... (*n* – *r* + 1) ways

$$\therefore \quad p = n(n-1)(n-2)....(n-r+1)$$

$$= \frac{n(n-1)(n-2)....(n-r+1)(n-r)(n-r-1) \;....\; 3.2.1}{(n-r)(n-r-1) \;....\; 3.2.1}$$

$$\Rightarrow {}^nP_r = \frac{n!}{(n-r)!}$$

which completes the proof.

Corollary: If $r = n$, then

$${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

$\Rightarrow$ n different objects can be arranged taken all at a time in $n!$ ways.

Example 1: How many different 4-digit numbers can be formed out of the digits 1, 2, 3, 4, 5, 6, when no digit is repeated?

Solution: The total number of digits = 6

The digits forming each number = 4.

So, the required number of 4-digit numbers is given by:

$${}^6p_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6.5.4.3.2.1}{2.1} = 6.5.4.3 = 360$$

Example 2: How many signals can be made with 4-different flags when any number of them are to be used at a time?

Solution: The number of flags = 4

Number of signals using 1 flag = ${}^4P_1$ = 4

Number of signals using 2 flags = ${}^4P_2$ = 4 . 3 = 1 2

Number of signals using 3 flags = ${}^4P_3$ = 4.3.2 = 24

Number of signals using 4 flags = ${}^4P_4$ = 4.3.2.1 = 24

$\therefore$ Total Number of signals = 4 + 12 + 24 + 24 = 64.

Example 3: In how many ways can a set of 4 different mathematics books and 5 different physics books be placed on a shelf with a space for 9 books, if all books on the same subject are kept together?

Solution: 4 different Mathematics books can be arranged among themselves in 4! ways. 5 different Physics books can be arranged among themselves in 5! ways.To every one way of arranging 4 mathematics books there are 5! ways of arranging 5 physics books. The books in the two subjects can be arranged subject-wise in 2! ways.

So the number of ways of arranging the books as given by.

$4! \times 5! \times 2! = 4 \times 3 \times 2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1 \times 2 \times 1$

$= 5740$

Exercise 7.2

1. Evaluate the following:

i) ${}^{20}P_3$ ii) ${}^{16}P_4$ iii) ${}^{12}P_5$ iv) ${}^{10}P_7$ v) ${}^9P_8$

2. Find the value of n when:

i) ${}^nP_2 = 30$ ii) ${}^{11}P_n = 11.10.9$ iii) ${}^nP_4 : {}^{n-1}P_3 = 9:1$

3. Prove from the first principle that:

i) ${}^nP_r = n.{}^{n-1}P_{r-1}$ ii) ${}^nP_r = {}^{n-1}P_r + r.{}^{n-1}P_{r-1}$

4. How many signals can be given by 5 flags of different colours, using 3 flags at a time?

5. How many signals can be given by 6 flags of different colours when any number of flags can be used at a time?

6. How many words can be formed from the letters of the following words using all letters when no letter is to be repeated:

i) PLANE ii) OBJECT iii) FASTING?

7. How many 3-digit numbers can be formed by using each one of the digits 2, 3, 5, 7, 9 only once?

8. Find the numbers greater than 23000 that can be formed from the digits 1, 2, 3, 5, 6 , without repeating any digit.

HINT: The first two digits on L.H.S. will be 23 etc.

9. Find the number of 5-digit numbers that can be formed from the digits 1, 2, 4, 6, 8 (when no digit is repeated), but
i) the digits 2 and 8 are next to each other;
ii) the digits 2 and 8 are not next to each other.

10. How many 6-digit numbers can be formed, without repeating any digit from the digits 0, 1, 2, 3, 4, 5? In how many of them will 0 be at the tens place?

11. How many 5-digit multiples of 5 can be formed from the digits 2, 3, 5, 7,9, when no digit is repeated.

12. In how many ways can 8 books including 2 on English be arranged on a shelf in such a way that the English books are never together?

13. Find the number of arrangements of 3 books on English and 5 books on Urdu for placing them on a shelf such that the books on the same subject are together.

14. In how many ways can 5 boys and 4 girls be seated on a bench so that the girls and the boys occupy alternate seats?

7.2.1 Permutation of Things Not All Different

Suppose we have to find the permutation of the letters of the word BITTER, using all the letters in it. We see that all the letters of the word BITTER are not different and it has 2 Ts in it. Obviously, the interchanging of Ts in any permutation, say BITTER, will not form a new permutation. However, if the two Ts are replaced by T_1 and T_2, we get the following two permutation of BITTER

$$BIT_1T_2ER \text{ and } BIT_2T_1ER$$

Similarly, the replacement of the two Ts by T_1 and T_2 in any other permutation will give rise to 2 permutation.

Now, BIT_1T_2ER consists of 6 different letters which can be permuted among themselves in 6 ! different ways. Hence the number of permutation of the letters of the word BITTER taken all at a time

$$=\frac{6!}{2}=\frac{6.5.4.3.2.1}{2}=360$$

This example guides us to discover the method of finding the permutation of n things all of which are not different. Suppose that out of n things, n_1 are alike of one kind and n_2 are

alike of second kind and the rest of them are all different. Let x be the required number of permutation. Replacing n_1 alike things by n_1 different things and n_2 alike things by n_2 different things, we shall get all the n things distinct from each other which can be permuted among themselves in $n!$ ways. As n_1 different things can be permuted among themselves in $(n_1)!$ ways and n_2 different things can be arranged among themselves in $(n_2)!$ ways, so because of the replacement suggested above, x permutation would increase to $x \times (n_1)! \times (n_2)!$ number of ways.

$$\therefore \quad x \times (n_1)! \times (n_2)! = (n)!$$

Hence $$x=\frac{(n)!}{(n_1)!\times(n_2)!}=\binom{n}{n_1, n_2}$$

Cor. If there are n_1 alike things of one kind, n_2 alike things of second kind and n_3 alike things of third kind, then the number of permutation of n things,taken all at a time is given by:

$$\frac{n!}{(n_1)!\times(n_2)!\times(n_3)!}=\binom{n}{n_1, n_2 n_3}$$

Example 1: In how many ways can be letters of the word MISSISSIPPI be arranged when all the letters are to be used?

Solution: Number of letters in MISSISSIPPI $= 11$

In MISSISSIPPI,
I is repeated 4 times
S is repeated 4 times
P is repeated 2 times
M comes only once.

Required number of permutation $=\binom{11}{4,4,2,1}$

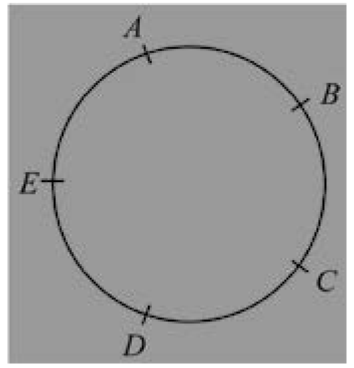
$$=\frac{(11)!}{(4)!\times(4)!\times(2)!\times(1)!}=34650 \text{ ways}$$

7.2.2 Circular Permutation

So far we have been studying permutation of things which can be represented by the points on a straight line. We shall now study the permutation of things which can be represented by the points on a circle. The permutation of things which can be represented by the points on a circle are called **Circular Permutation**.

The method of finding circular permutation is illustrated by the following examples.

Example 2: In how many ways can 5 persons be seated at a round table.



Solution: Let A, B, C, D, E be the 5 persons One of the ways of seating them round a table is shown in the adjoining figure. If each person moves one or two or more places to the left or the right, they will, no doubt, be occupying different seats, but their positions relative to each other will remain the same.

So, when A occupies a certain seat, the remaining 4 persons will be permuting their seats among themselves in 4! ways.

Hence the number of arrangements = 4! = 24

Example 3: In how many ways can a necklace of 8 beads of different colours be made?

Solution: The number of beads = 8

The number of arrangements of 8 beads in the necklace will be like the seating of 8 persons round a table.

$\Rightarrow$ The number of such necklaces (fixing one of the beads) = 7!

Now suppose the beads are a, b, c, d, e , f g , h and the necklace is as shown in Fig. (i) below:

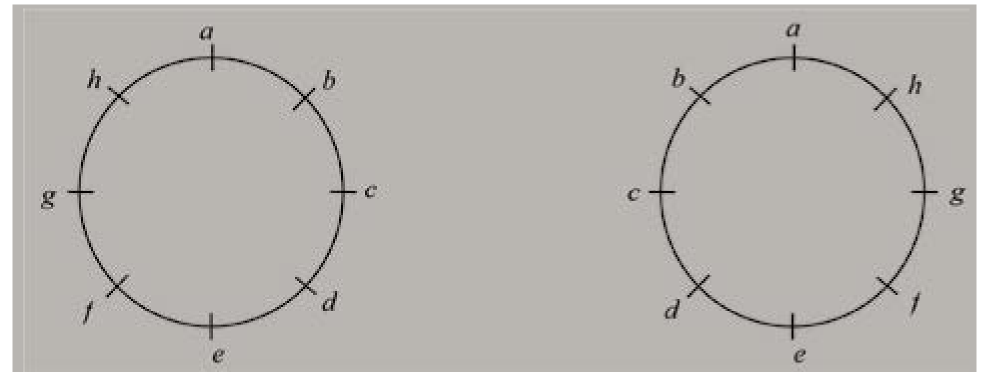


Figure (i) Figure (ii)

By flipping the necklace we get the necklace as shown in figure (ii). We observe that the two arrangements of the beads are actually the same.

Hence the required number of necklaces = $= \frac{1}{2} \times (7)! = 2520$

Exercise 7.3

1. How many arrangements of the letters of the following words, taken all together, can be made:
   i) PAKPATTAN ii) PAKISTAN
   ii) MATHEMATICS iv) ASSASSINATION?
2. How many permutation of the letters of the word PANAMA can be made, if P is to be the first letter in each arrangement?
3. How many arrangements of the letters of the word ATTACKED can be made, if each arrangement begins with C and ends with K?
4. How many numbers greater than 1000,000 can be formed from the digits 0, 2, 2,2, 3, 4 ,4 ?
5. How many 6-digit numbers can be formed from the digits 2, 2, 3, 3, 4, 4? How many of them will lie between 400,000 and 430,000?
6. 11 members of a club form 4 committees of 3, 4, 2, 2 members so that no member is a member of more than one committee. Find the number of committees.
7. The D.C.Os of 11 districts meet to discuss the law and order situation in their districts. In how many ways can they be seated at a round table, when two particular D.C.Os insist on sitting together?
8. The Governor of the Punjab calls a meeting of 12 officers. In how many ways can they be seated at a round table?

9. Fatima invites 14 people to a dinner. There are 9 males and 5 females who are seated at two different tables so that guests of one sex sit at one round table and the guests of the other sex at the second table. Find the number of ways in which all gests are seated.

10. Find the number of ways in which 5 men and 5 women can be seated at a round table in such a way that no two persons of the same sex sit together.

11. In how many ways can 4 keys be arranged on a circular key ring?

12. How many necklaces can be made from 6 beads of different colours?

7.3 Combination

While counting the number of possible permutation of a set of objects, the order is important. But there are situations where order is immaterial. For example

i) $ABC, ACB, BAC, BCA, CAB, CBA$ are the six names of the triangle whose vertices are *A, B* and *C*. We notice that inspite of the different arrangements of the vertices of the triangle, they represent one and the same triangle.

ii) The 11 players of a cricket team can be arranged in 11! ways, but they are players of the same single team. So, we are interested in the membership of the committee (group) and not in the way the members are listed (arranged). Therefore, a combination of n different objects taken *r* at a time is a set of *r* objects.

The number of combinations of *n* different objects taken *r* at a time is denoted by ${}^nC_r$ or *C*(*n, r*) or $\binom{n}{r}$ and is given by

$$ {}^nc_r = \frac{n!}{r!(n-r)!} $$

Proof: There are ${}^nC_r$ combinations of *n* different objects taken *r* at a time. Each combination consists of *r* different objects which can be permuted among themselves in *r*! ways. So, each combination will give rise to *r*! permutation. Thus there will be ${}^nC_r$ x *r*! permutation of *n* different objects taken *r* at a time.

$$ {}^nc_r \times r! = {}^np_r $$

$$ \Rightarrow \quad {}^nc_r \times r = \frac{n!}{(n-r)!} \quad \therefore \; {}^nc_r = \frac{n!}{r!(n-r)!} $$

Which completes the proof.

Corollary:

i) If *r* = *n*, then $\quad {}^nc_n = \frac{n!}{n!(n-r)!} = \frac{n!}{\text{n!0!}} = 1$

ii) If *r* = 0, then $\quad {}^nc_0 = \frac{n!}{0!(n-0)!} = \frac{n!}{\text{0!n!}} = 1$

7.3.1 Complementary Combination

Prove that: ${}^nC_r = {}^nC_{n-r}$

Proof: If from *n* different objects, we select *r* objects then (*n* – *r*) objects are left.

Corresponding to every combination of *r* objects, there is a combination of (*n* - *r*) objects and vice versa.

Thus the number of combinations of *n* objects taken *r* at a time is equal to the number of combinations of *n* objects taken (*n* - *r*) at a time.

$$ \therefore \quad {}^nC_r = {}^nC_{n-r} $$

Other wise: $\quad {}^nc_{n-r} = \frac{n!}{(n-r)!(n-n+r)!}$

$$ = \frac{n!}{(n-r)!r!} = \frac{n!}{r!(n-r)!} $$

$$ \Rightarrow {}^nc_{n-r} = {}^nc_r $$

Note: This result will be found useful in evaluating ${}^nC_r$ when $r > \frac{n}{2}$.

e.g $\; {}^{12}C_{10} = {}^{12}C_{12-10} = {}^{12}c_2 = \frac{(12).(11)}{2} =$ (6). (11) =66

Example 1: If ${}^nC_8 = {}^nC_{12}$, find n.

Solution: We know that ${}^nC_r = {}^nC_{n-r}$

$\therefore$ ${}^nC_8 = {}^nC_{n-8}$ (i)

But it is given that ${}^nC_8 = {}^nC_{12}$ (ii)

From (i) and (ii), we conclude that

${}^nC_{n-8} = {}^nC_{12}$

$\Rightarrow$ $n - 8 = 12$

$\therefore$ $n = 20$

Example 2: Find the number of the diagonals of a 6-sided figure.

Solution: A 6-sided figure has 6 vertices. Joining any two vertices we get a line segment.

$\therefore$ Number of line segments ${}^6C_2 = \frac{6!}{2!4!} = 15$

But these line segments include 6 sides of the figure

$\therefore$ Number of diagonals = 1 5 - 6 = 9

Example 3: Prove that: ${}^{n-1}C_r + {}^{n-1}C_{r-1} = {}^nC_r$

Solution: L.H.S. $= {}^{n-1}C_r + {}^{n-1}C_{r-1}$

$$= \frac{\lfloor n-1}{\lfloor r \lfloor n-1-r} + \frac{\lfloor n-1}{\lfloor r-1 \lfloor n-r}$$

$$= \frac{\lfloor n-1}{r\lfloor r-1 \lfloor n-r-1} + \frac{\lfloor n-1}{\lfloor r-1 (n-r) \lfloor n-r-1}$$

$$= \frac{\lfloor n-1}{\lfloor r-1 \lfloor n-r-1}\left[\frac{1}{r} + \frac{1}{n-r}\right] == \frac{\lfloor n-1}{\lfloor r-1 \lfloor n-r-1}\left[\frac{n-r+r}{r(n-r)}\right]$$

$$= \frac{n\lfloor n-1}{r\lfloor r-1 (n-r)\lfloor n-r-1} = \frac{\lfloor n}{\lfloor r \lfloor n-r} = {}^nC_r$$

= R.H.S.

Hence ${}^{n-1}C_r + {}^{n-1}C_{r-1} = {}^nC_r$

Exercise 7.4

1. Evaluate the following:

i) ${}^{12}C_3$ ii) ${}^{20}C_{17}$ iii) ${}^nC_4$

2. Find the value of n, when

i) ${}^nC_5 = {}^nC_4$ ii) ${}^nC_{10} = \frac{12 \times 11}{2!}$ iii) ${}^nC_{12} = {}^nC_6$

3. Find the values of n and r, when

i) ${}^nC_r = 35$ and ${}^nP_r = 210$

ii) ${}^{n-1}C_{r-1} : {}^nC_r : {}^{n+1}C_{r+1} = 3 : 6 : 1\ 1$

4. How many (a) diagonals and (b) triangles can be formed by joining the vertices of the polygon having:

i) 5 sides ii) 8 sides iii) 12 sides?

5. The members of a club are 12 boys and 8 girls. In how many ways can a committee of 3 boys and 2 girls be formed?

6. How many committees of 5 members can be chosen from a group of 8 persons when each committee must include 2 particular persons?

7. In how many ways can a hockey team of 11 players be selected out of 15 players? How many of them will include a particular player?

8. Show that: ${}^{16}C_{11} + {}^{16}C_{10} = {}^7C_{11}$

9. There are 8 men and 10 women members of a club. How many committees of can be formed, having;

i) 4 women ii) at the most 4 women iii) atleast 4 women?

10. Prove that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

7.4 Probability

We live in an uncertain world where very many events cannot be predicted with complete certainty, e.g.

i) In a cloudy weather, we cannot be sure whether it will or will not rain. However, we can say that there is 1 chance out of 2 that the rain will fall.

ii) There are 6 theorems on circle out of which one theorem is asked in the Secondary School Examination. Evidently, there is **1 chance out of 6** that a particular theorem will be asked in the examination.

In simple situations, we are guided by our experience or intuition. However, we cannot be sure about our predictions. Nevertheless, in more complex situations, we cannot depend upon guess work and we need more powerful tools for analyzing the situations and adopting the safer path for the achievement of our goals.

In order to guide in solving complex problems of everyday life, two French Mathematicians, BLAISE PASCAL (1623-62) and PIERRE DE FERMAT (1601 - 65), introduced **probability theory**. A very simple definition of probability is given below:

Probability is the numerical evaluation of a chance that a particular event would occur.

This definition is too vague to be of any practical use in estimating the chance of the occurrence of a particular event in a given situation. But before giving a comprehensive definition of probability we must understand some terms connected with probability.

Sample Space and Events: The set S consisting of all possible outcomes of a given experiment is called the **sample space**. A particular outcome is called an **event** and usually denoted by *E*. An event *E* is a subset of the sample space *S*. For example,

i) In tossing a fair coin, the possible outcomes are a Head (*H*) or a Tail (*T*) and it is written as: $S = \{H, T\} \Rightarrow n(S) = 2$.

ii) In rolling a die the possible outcomes are 1 dot, 2 dots, 3 dots, 4 dots, 5 dots or 6 dots on the top.

$\therefore \quad S = \{1, 2, 3, 4, 5, 6\} \Rightarrow n(S) = 6$

To get an even number 2, 4 or 6 is such event and is written as:

$E = \{2,4,6\} \Rightarrow n(E) = 3$

Mutually Exclusive Events: If a sample space $S = \{1, 3, 5, 7, 9\}$ and an event $A = \{1, 3, 5\}$ and another event $B = \{9\}$, then *A* and *B* are disjoint sets and they are said to be **mutually exclusive events**. In tossing a coin, the sample space $S = \{H, T\}$. Now, if event $A = \{H\}$ and event $B = \{T\}$, then *A* and *B* are mutually exclusive events.

Equally Likely Events: We know that if a fair coin is tossed, the chance of **head** appearing

on the top is the same as that of the **tail**. We say that these two events are **equally likely**. Similarly, if a die, which is a perfect unloaded cube is rolled, then the face containing 2 dots is as likely to be on the top as the face containing 5 dots. The same will be the case with any other pair of faces. In general, if two events A and B occur in an experiment, then ***A* and *B* are said to be equally likely events if each one of them has equal number of chances of occurrence.**

The following definition of Probability was given by a French Mathematician, P.S. Laplace (1749 - 1827) and it has been accepted as a standard definition by the mathematicians all over the world:

If a random experiment produces *m* different but equally likely out-comes and *n* outcomes out of them are favourable to the occurrence of the event *E*, then the probability of the occurrence of the event *E* is denoted by *P*(*E*) such that

$$P(E) = \frac{n}{m} = \frac{n(E)}{n(S)} = \frac{\text{no.of ways in which event occurs}}{\text{no.the elements of the sample space}}$$

Since the number of outcomes in an event is less than or equal to the number of outcomes in the sample space, the probability of an event must be a number between 0 and 1.

That is, $0 \leq P(E) \leq 1$

i) If P{E) = 0, event E cannot occur and E is called an impossible event.

ii) If P(E) = 1, event E is sure to occur and E is called a certain event.

Example 1: A die is rolled. What is the probability that the dots on the top are greater than 4?

Solution: $S = \{1,2,3,4,5,6\} \Rightarrow n(S) = 6$

The event E that the dots on the top are greater than 4 = {5, 6}

$$\Rightarrow n(E) = 2 \qquad \therefore P(E) = \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

Example 2: What is the probability that a slip of numbers divisible by 4 are picked from the slips bearing numbers 1, 2, 3, ...,10

Solution: $S = \{1, 2, 3, ..., 10\} \Rightarrow n(S) = 10$

Let *E* be the event of picking slip with number divisible by 4.

$E = \{4, 8\} \quad \Rightarrow n(E) = 2$

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{2}{10} = \frac{1}{5}$$

7.4.1 Probability that an Event does not Occur

If a sample space S is such that $n(S) = N$ and out of the N equally likely events an event E occurs R times, then, evidently, E does not occur $N - R$ times.

The non-occurrence of the event E is denoted as $\overline{E}$.

Now $P(E) = \frac{n(E)}{n(S)} = \frac{R}{N}$

and $P(\overline{E}) = \frac{n(\overline{E})}{n(S)} = \frac{N-R}{N} = \frac{N}{N} - \frac{R}{N} = 1 - \frac{R}{N}$

$\therefore \quad P(\overline{E}) = 1 - p(E).$

Exercise 7.5

For the following experiments, find the probability in each case:

1. Experiment:
From a box containing orange-flavoured sweets, Bilal takes out one sweet without looking.
Events Happening:
i) the sweet is orange-flavoured
ii) the sweet is lemon-flavoured.

2. Experiment:
Pakistan and India play a cricket match. The result is:
Events Happening: i) Pakistan wins ii) India does not lose.

3. Experiment:
There are 5 green and 3 red balls in a box, one ball is taken out.
Events Happening: i) the ball is green ii) the ball is red.

4. Experiment:
A fair coin is tossed three times. It shows
Events Happening: i) One tail ii) atleast one head.

5. Experiment:
A die is rolled. The top shows
Events Happening: i) 3 or 4 dots ii) dots less than 5.

6. Experiment:
From a box containing slips numbered 1,2, 3, ..., 5 one slip is picked up
Events Happening:
i) the number on the slip is a prime number
ii) the number on the slip is a multiple of 3.

7. Experiment:
Two die, one red and the other blue, are rolled simultaneously. The numbers of dots on the tops are added. The total of the two scores is:
Events Happening: i) 5 ii) 7 iii) 11.

8. Experiment:
A bag contains 40 balls out of which 5 are green, 15 are black and the remaining are yellow. A ball is taken out of the bag.
Events Happening:
i) The ball is black ii) The ball is green iii) The ball is not green.

9. Experiment:
One chit out of 30 containing the names of 30 students of a class of 18 boys and 12 girls is taken out at random, for nomination as the monitor of the class.
Events Happening:
i) the monitor is a boy ii) the monitor is a girl.

10. Experiment:
A coin is tossed four times. The tops show
Events Happening:
i) all heads ii) 2 heads and 2 tails.

7.4.2 Estimating Probability and Tally Marks

We know that $P(E)=\frac{n(E)}{n(S)}$, where E is the event and S is the sample space. The fraction showing the probability is very often such that it is better to find its approximate value. The following examples illustrate the necessity of approximation.

Example 1: The table given below shows the result of rolling a die 100 times. Find the probability in which odd numbers occur.

Event	Tally Marks	Frequency
1	卌 卌 卌 卌 卌	25
2	卌 卌 |||	13
3	卌 卌 ||||	14
4	卌 卌 卌 卌 ||||	24
5	卌 |||	8
6	卌 卌 卌 |	16

Solution: Required probability $=\frac{25+14+8}{100}=\frac{47}{100}=\frac{1}{2}$ (approx.)

Note: In the above experiment, we have written the probability = $\frac{1}{2}$ (approx.)

It may be remembered that the greater the number of trials, the more accurate is the estimate of the probability.

Example 2: The number of rainy days in Murree during the month of July for the past ten years are: 20, 20, 22, 22, 23, 21, 24, 20, 22, 21

Estimate the probability of the rain falling on a particular day of July. Hence find the number of days in which picnic programme can be made by a group of students who wish to spend 20 days in Murree.

Solution: Let E be the event that rain falls on a particular day of a July.

$$P(E)=\frac{20+20+22+22+23+21+24+20+22+21}{31\times 10}$$

$$=\frac{215}{310}=0.7 \quad \text{(approx).}$$

Number of days of raining in 20 days of July = 20 × 0.7 = 14

∴ The number of days fit for picnic = 2 0 – 1 4 = 6

Exercise 7.6

1. A fair coin is tossed 30 times, the result of which is tabulated below. Study the table and answer the questions given below the table:

Event	Tally Marks	Frequency
Head	卌 卌 ||||	14
Tail	卌 卌 卌 |	16

i) How many times does 'head' appear?
ii) How many times does 'tail' appear?
iii) Estimate the probability of the appearance of head?
iv) Estimate the probability of the appearance of tail?

2. A die is tossed 100 times. The result is tabulated below. Study the table and answer the questions given below the table:

Event	Tally Marks	Frequency
1	卌 卌 ||||	14
2	卌 卌 卌 ||	17
3	卌 卌 卌 卌	20
4	卌 卌 卌 |||	18
5	卌 卌 卌	15
6	卌 卌 卌 |	16

i) How many times do 3 dots appear?
ii) How many times do 5 dots appear?
iii) How many times does an even number of dots appear?

iv) How many times does a prime number of dots appear?

v) Find the probability of each one of the above cases.

3. The eggs supplied by a poultry farm during a week broke during transit as follows:

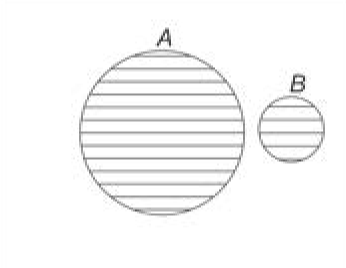
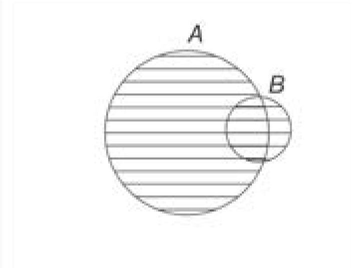
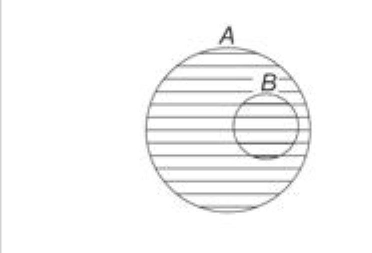
$$1\%,\ 2\%,\ 1\frac{1}{2}\%,\ \frac{1}{2}\%,\ 1\%,\ 2\%,\ 1\%$$

Find the probability of the eggs that broke in a day. Calculate the number of eggs that will be broken in transiting the following number of eggs:

i) 7,000   ii) 8,400   iii) 10,500

7.4.3 Addition of Probabilities

We have learnt in chapter 1, that if A and B are two sets, then the shaded parts in the following diagram represent $A \cup B$.

If A and B are disjoint	If A and B are overlapping	If $A \subset B$
		

The above diagrams help us in understanding the formulas about the sum of two probabilities.

We know that:

$P(E)$ is the probability of the occurrence of an event E.

If A and B are two events, then

$P(A)$ = the probability of the occurrence of event A;

$P(B)$ = the probability of the occurrence of event B;

$P(A \cup B)$ = the probability of the occurrence of $A \cup B$;

$P(A \cap B)$ = the probability of the occurrence of $A \cap B$;

The formulas for the addition of probabilities are:

i) $P(A \cup B) = P(A) + P(B)$, when A and B are disjoint

ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

when A and B are overlapping or $B \subseteq A$.

Let us now learn the application of these formulas in solving problems involving the addition of two probabilities.

Example 1: There are 20 chits marked 1, 2, 3,....,20 in a bag. Find the probability of picking a chit, the number written on which is a multiple of 4 or a multiple of 7.

Solution: Here $S = \{1, 2, 3, ..., 20\} \Rightarrow n(S) = 20$

Let A be the event of getting multiples of 4.

$\therefore\ A = \{4, 8, 12, 16, 20\} \Rightarrow n(A) = 5$

$$\therefore P(A) = \frac{5}{20} = \frac{1}{4}$$

Let B be the event of getting multiples of 7

$\therefore\ B = \{7, 14\} \Rightarrow n(B) = 2$

$$\therefore P(B) = \frac{2}{20} = \frac{1}{10}$$

As A and B are disjoint sets

$$\therefore\quad P(A \cup B) = p(A) + p(B) = \frac{1}{4} + \frac{1}{10} = \frac{7}{20}$$

Example 2: A die is thrown. Find the probability that the dots on the top are prime numbers or odd numbers.

Solution: Here $S = \{1, 2, 3, 4, 5, 6\} \Rightarrow n(S) = 6$

Let A = Set of prime numbers = $\{2,3,5\} \Rightarrow n(A) = 3$

Let B = Set of odd numbers = $\{1,3,5\} \Rightarrow n(B) = 3$

$\therefore A \cap B = \{2,3,5\} \cap \{1,3,5\} = \{3,5\} \Rightarrow n(A \cap B) = 2$

Now $P(A) = \frac{3}{6} = \frac{1}{2}$, $P(B) = \frac{3}{6} = \frac{1}{2}$, $P(A \cap B) = \frac{2}{6} = \frac{1}{3}$

Since A and B are overlapping sets.

$$P(A\cup B) = P(A)+P(B)-P(A\cap B)$$

$$=\frac{1}{2}+\frac{1}{2}-\frac{1}{3}=\frac{2}{3}$$

Exercise 7.7

1. If sample space= {1, 2, 3 , 9 } , Event A = {2, 4, 6, 8} and Event B = {1, 3, 5}, find $P(A\cup B)$.

2. A box contains 10 red, 30 white and 20 black marbles. A marble is drawn at random. Find the probability that it is either red or white.

3. A natural number is chosen out of the first fifty natural numbers. What is the probability that the chosen number is a multiple of 3 or of 5?

4. A card is drawn from a deck of 52 playing cards. What is the probability that it is a diamond card or an ace?

5. A die is thrown twice. What is the probability that the sum of the number of dots shown is 3 or 11?

6. Two dice are thrown. What is the probability that the sum of the numbers of dots appearing on them is 4 or 6 ?

7. Two dice are thrown simultaneously. If the event A is that the sum of the numbers of dots shown is an odd number and the event B is that the number of dots shown on at least one die is 3. Find $P(A\cup B)$.

8. There are 10 girls and 20 boys in a class. Half of the boys and half of the girls have blue eyes. Find the probability that one student chosen as monitor is either a girl or has blue eyes.

7.4.4 Multiplication of Probabilities

We can multiply probabilities of **dependent** as well as **independent** events. But, in this section, we shall find the multiplication of independent events only. Before learning the formula of the multiplication of the probabilities of independent events, it is necessary to understand that what is meant by independent events.

Two events A and B are said to be independent; if the occurrence of any one of

them does not influence the occurrence of the other event. In other words, regardless of whether event A has or has not occurred, if the probability of the event B remains the same, then A and B are independent events.

Suppose a bag contains 12 balls. If 4 balls are drawn from it twice in such a way that:

i) the balls of the first draw are not replaced before the second draw;

ii) the balls of the first draw are replaced before the second draw.

In the case (i), the second draw will be out of (12 – 4 = 8) balls which means that the out-comes of the second draw will depend upon the events of the first draw and the two events will not be independent. However, in case (ii), the number of balls in the bag will be the same for the second draw as has been the case at the time of first draw i.e. the first draw will not influence the probability of the event of second draw. So the two events in this case will be independent.

Theorem: If A and B are two independent events, the probability that both of them occur is equal to the probability of the occurrence of A multiplied by the probability of the occurrence of B. Symbolically, it is denoted as:

$$P(A\cap B) = P(A).P(B)$$

Proof: Let event A belong to the sample space S_1 such that

$$n(S_1) = n_1 \quad \text{and} \quad n(A) = m_1 \quad \Rightarrow P(A)=\frac{m_1}{n_1}$$

Let event B belong to the sample space S_2 such that

$$n(S_2) = n_2 \quad \text{and} \quad n(B) = m_2 \quad \Rightarrow p(B)=\frac{m_2}{n_2}$$

$\because$ A and B are independent events

$\therefore$ Total number of combined outcomes of A and B = $n_1 n_2$

and total number of favourable outcomes = $m_1 m_2$

$$\therefore \quad P(A\cap B) = \frac{m_1m_2}{n_1n_2} = \frac{m_1}{n_1}.\frac{m_2}{n_2}=P(A).P(B)$$

Note: The above proof of the formula holds good even if the sample spaces of *A* and *B* are the same. The formula $P(A \cap B) = P(A).P(B)$ can be generalized as:

$$P(A_1 \cap A_2 \cap A_3 \cap ..... \cap A_n) = P(A_1).P(A_2).P(A_3)........P(A_n)$$

where $A_1, A_2, A_3, ..., A_n$ are independent events.

Example 1 The probabilities that a man and his wife will be alive in the next 20 years are 0.8 and 0.75 respectively. Find the probability that both of them will be alive in the next 20 years.

Solution: If *P*(*A*) is the probability that the man will be alive in 20 years and *P*(*B*) is the probability that his wife be alive in 20 years.

$\because$ The two events are independent:

$\therefore$ $P(A) = 0.8$ $P(B) = 0.75$

The probability that both man and wife will be alive in 20 years is given by:

$$P(A \cap B) = 0.8 \times 0.75 = 0.6$$

Example 2: Two dice are thrown. E_1 is the event that the sum of their dots is an odd number and E_2 is the event that 1 is the dot on the top of the first die. Show that $P(E_1 \cap E_2) = P(E_1).P(E_2)$

Solution: E_1 = {(1,2), (1 ,4), (1, 6), (2, 3), (2, 5), (3,4), (3 ,6), (4, 3) (4, 5) (5, 6), (2, 1), (4, 1), (6,1), (3, 2), (5, 2), (6, 3), (5, 4), (6, 5)}

$\Rightarrow$ $n(E_1) = 18$

E_2 = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)}

$\Rightarrow$ $n(E_2) = 6$

$\because$ $n(S) = 6 \times 6 = 36$

$$\therefore \quad P(E_1) = \frac{18}{36} = \frac{1}{2} \text{ and } P(E_2) = \frac{6}{36} = \frac{1}{6}$$

$\because$ E_1 and E_2 are independent

$$\therefore \quad P(E_1).P(E_2) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$$

Now $E_1 \cap E_2$ = {(1,2), (1,4), (1,6)}

$\Rightarrow$ $n(E_1 \cap E_2) = 3$

$$\therefore \quad P(E_1 \cap E_2) = \frac{3}{36} = \frac{1}{12}$$

Hence $P(E_1 \cap E_2) = P(E_1).P(E_2)$

Exercise 7.8

1. The probability that a person A will be alive 15 years hence is $\frac{5}{7}$ and the probability that another person B will be alive 15 years hence is $\frac{7}{9}$. Find the probability that both will be alive 15 years hence.
2. A die is rolled twice: Event E_1 is the appearance of even number of dots and event E_2 is the appearance of more than 4 dots. Prove that: $P(E_1 \cap E_2) = P(E_1).P(E_2)$
3. Determine the probability of getting 2 heads in two successive tosses of a balanced coin.
4. Two coins are tossed twice each. Find the probability that the head appears on the first toss and the same faces appear in the two tosses.
5. Two cards are drawn from a deck of 52 playing cards. If one card is drawn and replaced before drawing the second card, find the probability that both the cards are aces.
6. Two cards from a deck of 52 playing cards are drawn in such a way that the card is replaced after the first draw. Find the probabilities in the following cases:
   i) first card is king and the second is a queen.
   ii) both the cards are faced cards i.e. king, queen, jack.
7. Two dice are thrown twice. What is probability that sum of the dots shown in the first throw is 7 and that of the second throw is 11 ?
8. Find the probability that the sum of dots appearing in two successive throws of two dice is every time 7.

9. A fair die is thrown twice. Find the probability that a prime number of dots appear in the first throw and the number of dots in the second throw is less than 5.

10. A bag contains 8 red, 5 white and 7 black balls, 3 balls are drawn from the bag. What is the probability that the first ball is red, the second ball is white and the third ball is black, when every time the ball is replaced?

HINT: $\left(\frac{8}{20}\right)\left(\frac{5}{20}\right)\left(\frac{7}{20}\right)$ is the probability.