CHAPTER

2

# DIFFERENTIATION

*Animation 2.1: Increasing and Decreasing Functions*
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# 2.1 INTRODUCTION

The ancient Greeks knew the concepts of area, volume and centroids etc. which are related to integral calculus. Later on, in the seventeenth century, Sir Isaac Newton, an English mathematician (1642-1727) and Gottfried Whilhelm Leibniz, a German mathematician, (1646-1716) considered the problem of instantaneous rates of change. They reached independently to the invention of differential calculus. After the development of calculus, mathematics became a powerful tool for dealing with rates of change and describing the physical universe.

**Dependent and Independent Variables**

In differential calculus, we mainly deal with the rate of change of a dependent variable with respect to one or more independent variables. Now, we first explain the terms dependent and independent variables.

We usually write $y = f(x)$ where $f(x)$ is the value of $f$ at $x \in D_f$ (the domain of the function $f$). Let us consider the functional relation $y = f(x) = x^2 + 1$ ....... (A)

For different values of $x \in D_f, f(x)$ or the expression $x^2 + 1$ assumes different values. For example; if $x$ = 1, 1.5, 2 etc., then

$$f(1) = (1)^2 + 1 = 2, f(1.5) = (1.5)^2 + 1 = 2.25 + 1 = 3.25$$

$$f(2) = (2)^2 + 1 = 4 + 1 = 5$$

We see that for the change 1.5 - 1 = 0.5 in the value of $x$, the corresponding change in the value of $y$ or $f(x)$ is given by

$f(1.5) - f(1) = 3.25 - 2 = 1.25$

It is obvious that the change in the value of the expression $x^2 + 1$ (or $f(x)$) depends upon the change in the value of the variable $x$. As $x$ behaves independently, so we call it the independent variable. But the behaviour of $y$ or $f(x)$ depends on the variable $x$, so we call it the dependent variable.

The change in the value of $x$ (positive or negative) is called the increment of $x$ and is denoted by the symbol $\delta x$ (read as delta $x$). The corresponding change in the dependent variable $y$ or $f(x)$ for the change $\delta x$ in the value of $x$ is denoted by $\delta y$ or $\delta f = f(x + \delta x) - f(x)$.





Usually the small changes in the values of the variables are taken as increments of variables.

**Note:** In this Chapter we shall discuss funcions of the form $y = f(x)$ where $x \in D_f$ and is called an independent variable while $y$ is called the dependent variable.

### 2.1.1 AVERAGE RATE OF CHANGE

Suppose a particle (or an object) is moving in a straight line and its positions (from some fixed point) after times $t$ and $t_1$ are given by $s(t)$ and $s(t_1)$, then the distance traveled in the time interval $t_1 - t$ where $t_1 > t$ is $s(t_1) - s(t)$

and the difference quotient $\dfrac{s(t_1) - s(t)}{t_1 - t}$ (i)

represents the average rate of change of distance over the time interval $t_1 - t$.

If $t_1 - t$ is not small, then the average rate of change does not represent an accurate rate of change near t. We can elaborate this idea by a moving particle in a straight line whose position in metres after t seconds is given by

$$s(t) = t^2 + t$$

We construct a table for different values of t as under:

| Interval | Average rate of change (i.e. average speed) |
|---|---|
| $t = 3$ secs to $t = 5$ secs | $\dfrac{s(5) - s(3)}{5 - 3} = \dfrac{(25+5) - (9+3)}{2} = \dfrac{30 - 12}{2} = 9$ |
| $t = 3$ secs to $t = 4$ secs | $\dfrac{s(4) - s(3)}{4 - 3} = \dfrac{(16+4) - 12}{1} = \dfrac{20 - 12}{1} = 8$ |
| $t = 3$ secs to $t = 3.5$ secs | $\dfrac{s(3.5) - s(3)}{3.5 - 3} = \dfrac{\left(\frac{49}{4} + \frac{7}{2}\right) - 12}{0.5} = \dfrac{\frac{15}{4}}{0.5} = 7.5$ |

We see that none of average rates of change approximates to the actual speed of the particle after 3 seconds.

Now we construct a table by taking small intervals.

| Interval | Average rate of change |
|---|---|
| $t=3$ secs to $t=3.1$ secs | $\dfrac{\left((3.1)^2+3.1\right)-12}{3.1-3}=\dfrac{12.71-12}{0.1}=\dfrac{0.71}{0.1}=7.1$ |
| $t=3$ secs to $t=3.01$ secs | $\dfrac{\left((3.01)^2+3.01\right)-12}{3.01-3}=\dfrac{12.0701-12}{0.01}=\dfrac{0.0701}{0.01}=7.01$ |
| $t=3$ secs to $t=3.001$ secs | $\dfrac{\left((3.001)^2+3.001\right)-12}{3.001-3}=\dfrac{12.007001-12}{0.001}=\dfrac{0.007001}{0.001}=7.001$ |

The above table shows that the average rate of change after 3 seconds approximates to 7 metre/sec. as the length of the interval becomes very very small. In other words, we can say that the speed of the particle is 7 metre/sec. after 3 seconds.

If $t_1=t+\delta t$

then the difference quoteint (i) becomes

$$\frac{s(t+\delta t)-s(t)}{\delta t}$$

which represents the average rate of change of distance over the interval $\delta t$ and

$\lim\limits_{\delta t\to 0}\dfrac{s(t+\delta t)-s(t)}{\delta t}$, provided this limit exists, is called the instantaneous rate of change of distance 's' at time $t$.

### 2.1.2 Derivative of a Function

Let $f$ be a real valued function continuous in the interval $(x,x_1)\subseteq D_f$ (the domain of $f$), then

difference quotient $\dfrac{f(x_1)-f(x)}{x_1-x}$ (i)

represents the average rate of change in the value of $f$ with respect to the change $x_1-x$ in the value of independent variable $x$.

If $x_1$, approaches to $x$, then





$$\lim_{x_1\to x}\frac{f(x_1)-f(x)}{x_1-x}$$

provided this limit exists, is called the instantaneous rate of change of $f$ with respect to $x$ at $x$ and is written as $f'(x)$.

If $x_1=x+\delta x$ i.e., $x_1-x=\delta x$, then the expression (i) can be expressed as

$$\frac{f(x+\delta x)-f(x)}{\delta x} \quad \text{(ii)}$$

and

$$\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x} \quad \text{(iii)}$$

provided the limit exists, is defined to be the derivative of $f$ (or **differential coefficient** of $f$) with respect to $x$ at $x$ and is denoted by $f'(x)$ (read as "f-prime of $x$"). The domain of $f'$consists of all $x$ for which the limit exists. If $x\in D_f$ and $f'(x)$ exists, then $f$ is said to be differentiable at $x$. The process of finding $f'$ is called **differentiation**.

**Notation for Derivative**

Several notations are used for derivatives. We have used the functional symbol $f'(x)$, for the derivative of $f$ at $x$. For the function $y=f(x)$.

$$y+\delta y=f(x+\delta x).$$

where $\delta y$ is the increment of $y$ (change in the value of $y$) corresponding to $\delta x$,the change in the value of $x$, then

$$\delta y=f(x+\delta x)-f(x) \quad \text{(iv)}$$

Dividing both the sides of (iv) by $\delta x$, we get

$$\frac{\delta y}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x} \quad \text{(v)}$$

Taking limit of both the sides of (v) as $\delta x\to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \quad \text{(vi)}$$

$\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ is denoted by $\frac{dy}{dx}$, so (vi) is written as $\frac{dy}{dx} = f'(x)$

**Note:** The symbol $\frac{dy}{dx}$ is used for the derivative of $y$ with respect to $x$ and here it is not a quotient of *dy* and *dx*. $\frac{dy}{dx}$ is also denoted by $y'$.

Now we write, in a table the notations for the derivative of $y = f(x)$ used by different mathematicians:

| Name of Mathematician | *Leibniz* | Newton | Lagrange | Cauchy |
|---|---|---|---|---|
| Notation used for derivative | $\frac{dy}{dx}$ or $\frac{df}{dx}$ | $\dot{f}(x)$ | $f'(x)$ | $Df(x)$ |

If we replace $x+\delta x$ by $x$ and $x$ by a, then the expression $f(x+\delta x) - f(x)$ becomes $f(x) - f(a)$. and the change $\delta x$ in the independent variable, in this case, is $x - a$.

So the expression $\frac{f(x+\delta x) - f(x)}{\delta x}$ is written as $\frac{f(x) - f(a)}{x - a}$ (vii)

Taking the limit of the expressiom(vii) when $x \to a$, gives

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a). \text{ Here } f'(a)$$

is called the derivative of $f$ at $x = a$.





## 2.2 FINDING f′(x) FROM DEFINITION OF DERIVATIVE

Given a function $f$, $f'(x)$ if it exists, can be found by the following four steps

Step I Find $f(x+\delta x)$

Step II Simplify $f(x+\delta x) - f(x)$

Step III Divide $f(x+\delta x) - f(x)$ by $\delta x$ to get $\frac{f(x+\delta x) - f(x)}{\delta x}$ and simplify it

Step IV Find $\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$

The method of finding derivatives by this process is called differentiation by definition or by ab-initio or from first principle.

**Example 1:** Find the derivative of the following functions by definition

(a) $f(x) = c$ (b) $f(x) = x^2$

**Solution:** (a) For $f(x) = c$

(i) $f(x+\delta x) = c$

(ii) $f(x+\delta x) - f(x) = c - c = 0$

(iii) $\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{0}{\delta x} = 0$

(iv) $\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} (0) = 0$

Thus $f'(x) = 0$, that is, $\frac{d}{dx}(c) = 0$

(b) For $f(x) = x^2$

(i) $f(x+\delta x) = (x+\delta x)^2$

(ii) $f(x+\delta x) - f(x) = (x+\delta x)^2 - x^2 = x^2 + 2x\delta x + (\delta x)^2 - x^2$

$= 2x\delta x + (\delta x)^2 = (2x + \delta x)\delta x$

(iii) $\dfrac{f(x+\delta x)-f(x)}{\delta x}=\dfrac{(2x+\delta x)\delta x}{\delta x}=2x+\delta x\,,\quad (\delta x\neq 0)$

(iv) $\lim\limits_{\delta x\to 0}\dfrac{f(x+\delta x)-f(x)}{\delta x}=\lim\limits_{\delta x\to 0}(2x+\delta x)=2x$

i.e., $f'(x)=2x$

**Example 2:** Find the derivative of $\sqrt{x}$ at $x=a$ from first principle.

**Solution:** If $f(x)=\sqrt{x}$, then

(i) $f(x+\delta x)=\sqrt{x+\delta x}$ and

(ii) $f(x+\delta x)-f(x)=\sqrt{x+\delta x}-\sqrt{x}$

$$=\frac{\left(\sqrt{x+\delta x}-\sqrt{x}\right)\left(\sqrt{x+\delta x}+\sqrt{x}\right)}{\sqrt{x+\delta x}+\sqrt{x}}\quad\left(\begin{array}{c}\textit{rationalizing the}\\ \textit{numerator}\end{array}\right)$$

$$=\frac{(x+\delta x)-x}{\sqrt{x+\delta x}+\sqrt{x}}$$

i.e., $f(x+\delta x)-f(x)=\dfrac{\delta x}{\sqrt{x+\delta x}+\sqrt{x}}$ **(I)**

(iii) Dividing both sides of(1)by $\delta x$, we have

$$\frac{f(x+\delta x)-f(x)}{\delta x}=\frac{\delta x}{\delta x(\sqrt{x+\delta x}+\sqrt{x})}=\frac{1}{\sqrt{x+\delta x}+\sqrt{x}}(\because \delta x\neq 0)$$

(iv) Taking limit of both the sides as $\delta x\to 0,$ we have

$$\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}=\lim_{\delta x\to 0}\left(\frac{1}{\sqrt{x+\delta x}+\sqrt{x}}\right)$$

i.e., $f'(x)=\dfrac{1}{\sqrt{x}+\sqrt{x}}=\dfrac{1}{2\sqrt{x}}\quad (x>0)$

and $f'(a)=\dfrac{1}{2\sqrt{a}}$





**or**

Putting $x=a$ in $f(x)=\sqrt{x}$, gives $f(a)=\sqrt{a}$

So $f(x)-f(a)=\sqrt{x}-\sqrt{a}$

Using alternative form for the definition of a derivative, we have

$$\frac{f(x)-f(a)}{x-a}=\frac{\sqrt{x}-\sqrt{a}}{x-a}$$

$$=\frac{\left(\sqrt{x}-\sqrt{a}\right)\left(\sqrt{x}+\sqrt{a}\right)}{(x-a)\left(\sqrt{x}+\sqrt{a}\right)}\text{ (rationalizing the numerator)}$$

$$=\frac{x-a}{(x-a)\left(\sqrt{x}+\sqrt{a}\right)}=\frac{1}{\sqrt{x}+\sqrt{a}}\quad (x\neq a)\qquad \textbf{(II)}$$

Taking limit of both the sides of (II)as $x\to a,$ gives

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{1}{\sqrt{x}+\sqrt{a}}=\frac{1}{\sqrt{a}+\sqrt{a}}$$

i.e., $f'(a)=\dfrac{1}{2\sqrt{a}}$

**Example 3:** If $y=\dfrac{1}{x^2}$, then find $\dfrac{dy}{dx}$ at $x=-1$ by ab-initio method.

**Solution:** Here $y=\dfrac{1}{x^2}$, so (i)

$y+\delta y=\dfrac{1}{(x+\delta x)^2}$ (ii)

Subtracting (i) from (ii), we get

$$\delta y=\frac{1}{(x+\delta x)^2}-\frac{1}{x^2}=\frac{x^2-(x+\delta x)^2}{x^2(x+\delta x)^2}$$

$$=\frac{\left(x+(x+\delta x)\right)\left(x-(x+\delta x)\right)}{x^2(x+\delta x)^2}$$

$$=\frac{(2x+\delta x)(-\delta x)}{x^2(x+\delta x)^2}=\frac{-\delta x(2x+\delta x)}{x^2(x+\delta x)^2} \qquad \text{(iii)}$$

Dividing both sides of (iii) by $\delta x$,, we have

$$\frac{\delta y}{\delta x}=\frac{-\delta x(2x+\delta x)}{x^2(x+\delta x)^2\,\delta x}=\frac{-(2x+\delta x)}{x^2(x+\delta x)^2} \qquad (\delta x\neq 0)$$

Taking limit as $\delta x\to 0$,, gives

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{-(2x+\delta x)}{x^2(x+\delta x)^2}$$

$$=\frac{-(2x)}{x^2(x^2)} \qquad \text{(Using quotient theorem of limits)}$$

i.e., $\frac{dy}{dx}=\frac{-2}{x^3}$ and $\frac{dy}{dx}\Big|_{x=-1}=\frac{-2}{(-1)^3}=\frac{-2}{-1}=2$

**Note:** The value of $\frac{dy}{dx}$ at $x=-1$ is written as $\frac{dy}{dx}\Big|_{x=-1}$.

**Example 4:** Find the derivative of $x^{\frac{2}{3}}$ and also calculate the value of derivative at $x$ = 8.

**Solution:** Let $f(x)=x^{\frac{2}{3}}$ .Then

$$f(x+\delta x)=(x+\delta x)^{\frac{2}{3}}$$

and

$$f(x+\delta x)-f(x)=(x+\delta x)^{\frac{2}{3}}-x^{\frac{2}{3}}=\frac{\left((x+\delta x)^{\frac{2}{3}}-x^{\frac{2}{3}}\right)\left[(x+\delta x)^{\frac{4}{3}}+(x+\delta x)^{\frac{2}{3}}.x^{\frac{2}{3}}+x^{\frac{4}{3}}\right]}{(x+\delta x)^{\frac{4}{3}}+(x+\delta x)^{\frac{2}{3}}.x^{\frac{2}{3}}+x^{\frac{4}{3}}}$$





$$=\frac{\left[(x+\delta x)^{\frac{2}{3}}\right]^3-\left(x^{\frac{2}{3}}\right)^3}{(x+\delta x)^{\frac{4}{3}}+(x+\delta x)^{\frac{2}{3}}.x^{\frac{2}{3}}+x^{\frac{4}{3}}}=\frac{(x+\delta x)^2-x^2}{(x+\delta x)^{\frac{4}{3}}+(x+\delta x)^{\frac{2}{3}}.x^{\frac{2}{3}}+x^{\frac{4}{3}}}$$

i.e., $$f(x+\delta x)-f(x)=\frac{\delta x(2x+\delta x)}{(x+\delta x)^{\frac{4}{3}}+(x+\delta x)^{\frac{2}{3}}.x^{\frac{2}{3}}+x^{\frac{4}{3}}} \qquad \text{(i)}$$

Dividing both the sides of (i) by $\delta x$ , we get

$$\frac{f(x+\delta x)-f(x)}{\delta x}=\frac{2x+\delta x}{(x+\delta x)^{\frac{4}{3}}+(x+\delta x)^{\frac{2}{3}}.x^{\frac{2}{3}}+x^{\frac{4}{3}}} \qquad \text{(ii)}$$

Taking limit of both the sides as $\delta x\to 0$, we have

$$f'(x)=\frac{2x}{x^{\frac{4}{3}}+x^{\frac{2}{3}}.x^{\frac{2}{3}}+x^{\frac{4}{3}}}=\frac{2x}{3x^{\frac{4}{3}}}=\frac{2}{3x^{\frac{1}{3}}}$$

and $$f'(8)=\frac{2}{3.(8)^{\frac{1}{3}}}=\frac{1}{3}$$

**Example 5:** Find the derivative of $x^3+2x+3$.

**Solution:** Let $y=x^3+2x+3$. Then

(i) $y+\delta y=(x+\delta x)^3+2(x+\delta x)+3$

(ii) $\delta y=\left[(x+\delta x)^3+2(x+\delta x)+3\right]-\left[x^3+2x+3\right]$

$$=\left[(x+\delta x)^3-x^3\right]+2\left[(x+\delta x)-x\right]+(3-3)$$

$$=\left[(x+\delta x)-x\right]\left[(x+\delta x)^2+(x+\delta x)x+x^2\right]+2\delta x$$

(iii) $$\frac{\delta y}{\delta x}=\frac{\delta x\left[(x+\delta x)^2+(x+\delta x)x+x^2\right]+2\delta x}{\delta x}$$

$$= (x+\delta x)^2 + (x+\delta x)x + x^2 + 2$$

(iv) $\lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x} = \lim\limits_{\delta x \to 0}\left[(x+\delta x)^2 + (x+\delta x)x + x^2 + 2\right]$

$$\frac{dy}{dx} = (x)^2 + (x)x + x^2 + 2$$

i.e., $\dfrac{d}{dx}\left(x^3+2x+3\right) = 3x^2+2$

### 2.2.1 Derivation of $x^n$ where $n \in Z$.

(a) We find the derivative of $x^n$ when n is positive integer.

(a) Let $y = x^n$. Then

$$y + \delta y = (x+\delta x)^n$$

and $\delta y = (x+\delta x)^n - x^n$

Using the binomial theorem, we have

$$\delta y = \left[x^n + nx^{n-1}.\delta x + \frac{n(n-1)}{\lfloor 2}x^{n-2}\left((\delta x^2) + ... + (\delta x)^n\right)\right] - x^n$$

i.e., $\delta y = \delta x\left[nx^{n-1} + \dfrac{n(n-1)}{\lfloor 2}x^{n-2}.\delta x + ... + (\delta x)^{n-1}\right]$ (i)

Dividing both sides of (i) by $\delta x$, gives

$$\frac{\delta y}{\delta x} = nx^{n-1} + \frac{n(n-1)}{\lfloor 2}x^{n-2}.\ \delta x + ... + (\delta x)^{n-1} \quad \text{(ii)}$$

Note that each term on the right hand side of (ii) involves $\delta x$ except the first term, so taking the limit as $\delta x \to 0$, we get $\dfrac{dy}{dx} = nx^{n-1}$

As $y = x^n$, so $\dfrac{d}{dx}\left(x^n\right) = n.x^{n-1}$





**Note:** If $n = 0$, then the formula $\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$ reduces to $\dfrac{d}{dx}\left(x^0\right) = 0x^{0-1} = 0$ i.e., $\dfrac{d}{dx}(1) = 0$ which is correct by example 1 part (a).

(b) Let $y = x^n$ where $n$ is a negative integer.

Let $n = -m$ ($m$ is a positive integer). Then

$$y = x^{-m} = \frac{1}{x^m} \quad \text{(i)}$$

and $y + \delta y = \dfrac{1}{(x+\delta x)^m}$ (ii)

Subtracting (i) from (ii). gives

$$\delta y = \frac{1}{(x+\delta x)^m} - \frac{1}{x^m} = \frac{x^m - (x+\delta x)^m}{x^m(x+\delta x)^m}$$

$$= \frac{x^m - \left(x^m + mx^{m-1}\delta x + \dfrac{m(m-1)}{\lfloor 2}x^{m-2}(\delta x)^2 + ... + (\delta x)^m\right)}{x^m(x+\delta x)^m}$$

(expanding $(x+\delta x)^m$ by binomial theorem)

$$= \frac{-\delta x\left(mx^{m-1} + \dfrac{m(m-1)}{\lfloor 2}x^{m-2}\delta x + ... + (\delta x)^{m-1}\right)}{x^m.(x+\delta x)^m}$$

and $\dfrac{\delta y}{\delta x} = \dfrac{-1}{x^m(x+\delta x)^m}.\left(mx^{m-1} + \dfrac{m(m-1)}{\lfloor 2}x^{m-2}.\delta x + ... + (\delta x)^{m-1}\right)$

Taking limit when $\delta x \to 0$, we get

$$\frac{dy}{dx} = \frac{-1}{x^m.x^m}\left(mx^{m-1}\right) \quad \text{(all terms containing } \delta x \text{ ,vanish)}$$

$$=(-m)x^{m-1}.x^{-2m}=(-m)x^{(-m)-1}=nx^{n-1} \qquad [\because -m=n]$$

or $\quad \dfrac{d}{dx}(x)^n=nx^{n-1}$

So far we have proved that $\dfrac{d}{dx}[x]^n=nx^{n-1}$, if $n\in Z$

The above rule holds if $n\in Q-Z$

For example $\dfrac{d}{dx}\left(x^{\frac{2}{3}}\right)=\dfrac{2}{3}x^{\frac{2}{3}-1}=\dfrac{2}{3x^{\frac{1}{3}}}$

The proof of $\dfrac{d}{dx}\left[x^n\right]=nx^{n-1}$ when $n\in Q-Z$ is left as an exercise.

Note that $\dfrac{d}{dx}\left[x^n\right]=nx^{n-1}$ is called power rule.

## Exercise 2.1

**1.** Find by definition, the derivatives w.r.t 'x' of the following functions defined as:

(i) $2x^2+1$ (ii) $2-\sqrt{x}$ (iii) $\dfrac{1}{\sqrt{x}}$ (iv) $\dfrac{1}{x^3}$ (v) $\dfrac{1}{x-a}$

(vi) $x(x-3)$ (vii) $\dfrac{2}{x^4}$ (viii) $(x+4)^{\frac{1}{3}}$ (ix) $x^{\frac{3}{2}}$ (x) $x^{\frac{5}{2}}$

(xi) $x^m, m\in N$ (xii) $\dfrac{1}{x^m, m\in N}$ (xiii) $x^{40}$ (xiv) $x^{-100}$

**2.** Find $\dfrac{dy}{dx}$ from first principle if

(i) $\sqrt{x+2}$ (ii) $\dfrac{1}{\sqrt{x+a}}$





## 2.2.2 DIFFERENTIATION OF EXPRESSIONS OF THE TYPES:

$$(ax+b)^n \text{ and } \frac{1}{(ax+b)^n}, \qquad n=1,2,3...$$

We find the derivatives of $(ax+b)^n$ and $\dfrac{1}{(ax+b)^n}$ from the first principle when $n\in N$

**Example 1:** Find from definition the differential coefficient of $(ax+b)^n$ w.r.t. '$x$' when n is a positive integer.

**Solution:** Let *y* = (*ax* + *b*)$^n$, (*n* is a positive integer)

Then $y+\delta y=\left[a(x+\delta x)+b\right]^n=\left[(ax+b)+a\delta x\right]^n$

Using the binomial theorem we have

$$y+\delta y=(ax+b)^n+\binom{n}{1}(ax+b)^{n-1}(a\delta x)+\binom{n}{2}(ax+b)^{n-2}(a\delta x)^2+...+(a\delta x)^n$$

$$\delta y=(y+\delta y)-y=\binom{n}{1}(ax+b)^{n-1}(a\delta x)+\binom{n}{2}(ax+b)^{n-2}.a^2(\delta x)^2+...+a^n(\delta x)^n$$

$$=\delta x\left[\binom{n}{1}(ax+b)^{n-1}.a+\binom{n}{2}(ax+b)^{n-2}.a^2\delta x+...+a^n(\delta x)^{n-1}\right]$$

So $\dfrac{\delta y}{\delta x}=\binom{n}{1}(ax+b)^{n-1}a+\binom{n}{2}(ax+b)^{n-2}.a^2\delta x+...+a^n(\delta x)^{n-1}$

Taking limit when $\delta x\to 0$, we have

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[\binom{n}{1}(ax+b)^{n-1}.a+\binom{n}{2}(ax+b)^{n-2}.a^2\delta x+...+a^n(\delta x)^{n-1}\right]$$

Or $\dfrac{dy}{dx}=\binom{n}{1}(ax+b)^{n-1}.a$ [All other terms tends to zero when $\delta x\to 0$]

Thus $\dfrac{d}{dx}(ax+b)^n=n(ax+b)^{n-1}.a$

**Example 2:** **Find from first principle, the derivative of $\frac{1}{(ax+b)^n}$ w.r.t. '$x$',**

**Solution:** Let $y=\frac{1}{(ax+b)^n}$ (when $n$ is a positive integer). Then

$y+\delta y=\frac{1}{[a(x+\delta x)+b]^n}$ and

$\delta y=y+\delta y-y=\frac{1}{[(ax+b)+a\delta x]^n}-\frac{1}{(ax+b)^n}$

or $\delta y=\frac{(ax+b)^n-(ax+b+a\delta x)^n}{[(ax+b)+a\delta x]^n(ax+b)^n}$

or $\delta y=\frac{-1}{[(ax+b)+a\delta x]^n(ax+b)^n}\text{x}[(ax+b)+a\delta x]^n-(ax+b)^n]$ (I)

Using the binomial theorem, we simplify the expression

$[(ax+b)+a\delta x]^n-(ax+b)^n$,That is,

$$[(ax+b)+a\delta x]^n-(ax+b)^n=[(ax+b)^n+\binom{n}{1}(ax+b)^{n-1}(a\delta x)$$

$$+\binom{n}{2}(ax+b)^{n-2}.a^2(\delta x)^2+...+(a\delta x)^n]$$

$$=\binom{n}{1}(ax+b)^{n-1}.a\delta x+\binom{n}{2}(ax+b)^{n-2}.a^2(\delta x)^2+...+a^n(\delta x)^n$$

$$=\delta x\left[\binom{n}{1}(ax+b)^{n-1}.a+\binom{n}{2}(ax+b)^{n-2}a^2\delta x+...+a^n(\delta x)^{n-1}\right]$$

**Now (I) becomes**

$$\delta y=-\frac{\delta x}{[(ax+b)+a\delta x]^n(ax+b)^n}[\binom{n}{1}(ax+b)^{n-1}.a$$





$$+\binom{n}{2}(ax+b)^{n-2}.a^2\delta x+...+\text{a}^n(\delta x)^{n-1}]$$

and $$\frac{\delta y}{\delta x}=-\frac{1}{[(ax+b)+a\delta x]^n(ax+b)^n}[\binom{n}{1}(ax+b)^{n-1}.a$$

$$+\binom{n}{2}(ax+b)^{n-2}.a^2\delta x+...+\text{a}^n(\delta x)^{n-1}]$$

Using the product and sum rules of limits when $\delta x\to 0$, we have

$$\frac{dy}{dx}=-\frac{1}{(ax+b)^n(ax+b)^n}.\binom{n}{1}(ax+b)^{n-1}.a$$

$\left(\because \lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\frac{dy}{dx}\right.$ and all other terms containing $\delta$x vanish$\left.\right)$

or $$\frac{d}{dx}\left[\frac{1}{(ax+b)^n}\right]=\frac{-na}{(ax+b)^{n+1}}=-n(ax+b)^{-(n+1)}.a$$

## Exercise 2.2

**1.** Find from first principles, the derivatives of the following expressions w.r.t. their respective independent variables:

(i) $(ax+b)^3$ (ii) $(2x+3)^5$

(iii) $(3t+2)^{-2}$ (iv) $\frac{1}{(ax+b)^5}$

(v) $\frac{1}{(az-b)^7}$

## 2.3 THEOREMS ON DIFFERENTIATION

We have, so far proved the following two formulas:

1. $\frac{dy}{dx}(c)=0$ i.e.. the derivative of a constant function is zero.

2. $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$ power formula (or rule) when $n$ is any rational number.

Now we will prove other important formulas (or rules) which are used to determine derivatives of different functions efficiently. Henceforth, in all subsequent discussion, *f, g, h* etc. all denote functions differentiable at *x*, unless stated otherwise.

3. Derivative of $y=cf(x)$

**Proof:** Let $y=cf(x)$ . Then

(i) $y+\delta y=cf(x+\delta x)$ and

(ii) $y+\delta y-y=cf(x+\delta x)-cf(x)$

or $\delta y=c\,|\,f(x+\delta x)-f(x)|$ (factoring out c)

(iii) $\frac{\delta y}{\delta x}=c\left(\frac{f(x+\delta x)-f(x)}{\delta x}\right)$

Taking limit when $\delta x\to 0$

(iv) $\lim\limits_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim\limits_{\delta x\to 0}\left[c.\frac{f(x+\delta x)-f(x)}{\delta x}\right]=c.\lim\limits_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$

A constant factor can be taken out from a limit sign.

Thus $\frac{dy}{dx}=c\,f'(x)$ ,that is, $\left[c\,f(x)\right]'=cf'(x)$

or $\frac{dy}{dx}=cf'(x)$ = $\left[c\,f(x)\right]'=cf'(x)$





**Example 1:** Calculate $\frac{d}{dx}\left(3x^{\frac{4}{3}}\right)$

**Solution:** $\frac{d}{dx}\left(3x^{\frac{4}{3}}\right)=3\frac{d}{dx}\left(x^{\frac{4}{3}}\right)$ (Using Formula 3)

$=3x\frac{4}{3}x^{\frac{4}{3}-1}=4x^{\frac{1}{3}}$ (Using power rule)

4. **Derivative of a sum or a Difference of Functions:**

If $f$ and $g$ are differentiable at $x$, then *f* + *g*, *f* – *g* are also differentiable at *x* and $\left[f(x)+g(x)\right]'=f'(x)+g'(x)$, that is, $\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}\left[f(x)\right]+\frac{d}{dx}\left[g(x)\right]$ Also $\left[f(x)-g(x)\right]'=f'(x)-g'(x)$. that is, $\frac{d}{dx}\left[f(x)-g(x)\right]=\frac{d}{dx}\left[f(x)\right]-\frac{d}{dx}\left[g(x)\right]$

**Proof:** Let $\phi(x)=f(x)+g(x)$. Then

(i) $\phi(x+\delta x)=f(x+\delta x)+g(x+\delta x)$ and

(ii) $\phi(x+\delta x)-\phi(x)=f(x+\delta x)+g(x+\delta x)-\left[f(x)+g(x)\right]$

$=\left[f(x+\delta x)-f(x)\right]+\left[g(x+\delta x)-g(x)\right]$ (rearranging the terms)

(iii) $\frac{\phi(x+\delta x)-\phi(x)}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x}+\frac{g(x+\delta x)-g(x)}{\delta x}$

Taking the limit when $\delta x\to 0$

(iv) $\lim\limits_{\delta x\to 0}\frac{\phi(x+\delta x)-\phi(x)}{\delta x}=\lim\limits_{\delta x\to 0}\left[\frac{f(x+\delta x)-f(x)}{\delta x}+\frac{g(x+\delta x)-g(x)}{\delta x}\right]$

$=\lim\limits_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}+\lim\limits_{\delta x\to 0}\frac{g(x+\delta x)-g(x)}{\delta x}$

(The limit of a sum is the sum of the limits)

$\phi'x=f'(x)+g'(x)$, that is $\left[f(x)+g(x)\right]'$ = $f'(x)+g'(x)$

or $\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}\left[f(x)\right]+\frac{d}{dx}\left[g(x)\right]$

The proof for the second part is similar.

**Note:** Sum or difference formula can be extended to find derivative of more than two functions.

**Example 1:** Find the derivative of $y=\frac{3}{4}x^4+\frac{2}{3}x^3+\frac{1}{2}x^2+2x+5$ w.r.t. $x$.

**Solution:** $y=\frac{3}{4}x^4+\frac{2}{3}x^3+\frac{1}{2}x^2+2x+5$

Differentiating with respect to *x*, we have

$$\frac{dy}{dx}\left[\frac{3}{4}x^4+\frac{2}{3}x^3+\frac{1}{2}x^2+2x+5\right]=\frac{d}{dx}\left[\frac{3}{4}x^4\right]+\frac{d}{dx}\left[\frac{2}{3}x^3\right]+\frac{d}{dx}\left[\frac{1}{2}x^2\right]+\frac{d}{dx}(2x)+\frac{d}{dx}(5)$$

(Using formula 4)

$$=\frac{3}{4}\frac{d}{dx}\left(x^4\right)+\frac{2}{3}\frac{d}{dx}\left(x^3\right)+\frac{1}{2}\frac{d}{dx}\left(x^2\right)+2\frac{d}{dx}(x)+0$$ (Using formula 3 and 1)

$$=\frac{3}{4}\left(4x^{4-1}\right)+\frac{2}{3}\left(3x^{3-1}\right)+\frac{1}{2}\left(2x^{2-1}\right)+2\left(1.x^{1-1}\right)$$ (By power formula)

$$=3x^3+2x^2+x+2$$

**Example 2:** Find the derivative of $y=\left(x^2+5\right)\left(x^3+7\right)$ with respect to *x*.

**Solution:** $y=\left(x^2+5\right)\left(x^3+7\right)$ $=x^5+5x^3+7x^2+35$

Differentiating with respect to *x*, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[x^5+5x^3+7x^2+35\right]$$

$$=\frac{d}{dx}\left[x^5\right]+5\frac{d}{dx}\left(x^3\right)+7\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}[35]$$ (Using formulas 3 and 4)

$= 5x^{5-1} + 5 \times 3x^{3-1} + 7 \times 2x^{2-1} + 0$

$= 5x^4 + 15x^2 + 14x$





**Example 3:** Find the derivative of $y=\left(2\sqrt{x}+2\right)\left(x-\sqrt{x}\right)$ with respect to *x*.

**Solution:** $y=\left(2\sqrt{x}+2\right)\left(x-\sqrt{x}\right)$

$$=2\left(\sqrt{x}+1\right).\sqrt{x}\left(\sqrt{x}-1\right)=2\sqrt{x}\left(\sqrt{x}+1\right)\left(\sqrt{x}-1\right)$$

$$=2\sqrt{x}(x+1)=2\left(x^{\frac{3}{2}}-x^{\frac{1}{2}}\right)$$

Differentiating with respect to $x$, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[2\left(x^{\frac{3}{2}}-x^{\frac{1}{2}}\right)\right]$$

$$=2\left[\frac{d}{dx}\left(x^{\frac{3}{2}}\right)-\frac{d}{dx}\left(x^{\frac{1}{2}}\right)\right]=2\left[\frac{3}{2}x^{\frac{3}{2}-1}-\frac{1}{2}x^{\frac{1}{2}-1}\right]$$

$$=3x^{\frac{1}{2}}-x^{\frac{-1}{2}}=3\sqrt{x}-\frac{1}{\sqrt{x}}=\frac{3x-1}{\sqrt{x}}$$

**5. Derivative of a product. (The product Rule)**

If *f* and *g* are differentiable at *x*, then *fg* is also differentiable at *x* and

$\left[f(x)g(x)\right]'=f'(x)g(x)+f(x)g'(x)$, that is,

$$\frac{d}{dx}\left[f(x)g(x)\right]=\left[\frac{d}{dx}\left[f(x)\right]\right]g(x)+f(x)\left[\frac{d}{dx}\left[g(x)\right]\right]$$

**Proof:** Let $\phi(x)=f(x)g(x)$. Then

(i) $\phi(x+\delta x)=f(x+\delta x)g(x+\delta x)$

(ii) $\phi(x+\delta x)-\phi(x)=f(x+\delta x)g(x+\delta x)-f(x)g(x)$

Subtracting and adding $f(x)g(x+\delta x)$ in step (ii), gives

$$\phi(x+\delta x)-\phi(x)=f(x+\delta x)g(x+\delta x)-f(x)g(x+\delta x)+f(x)g(x+\delta x)-f(x)g(x)$$

$$=\left[f(x+\delta x)-f(x)\right]g(x+\delta x)+f(x)\left[g(x+\delta x)-g(x)\right]$$

(iii) $\dfrac{\phi(x+\delta x)-\phi(x)}{\delta x}=\left[\dfrac{f(x+\delta x)-f(x)}{\delta x}\right]g(x+\delta x)+f(x)\left[\dfrac{g(x+\delta x)-g(x)}{\delta x}\right]$

Taking limit when $\delta x \to 0$

(iv) $\lim\limits_{\delta x\to 0}\dfrac{\phi(x+\delta x)-\phi(x)}{\delta x}$

$$= \lim_{\delta x\to 0}\left[\frac{f(x+\delta x)-f(x)}{\delta x}.g(x+\delta x)+f(x).\frac{g(x+\delta x)-g(x)}{\delta x}\right]$$

$$= \lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}.\lim_{\delta x\to 0}g(x+\delta x)+\lim_{\delta x\to 0}f(x).\lim_{\delta x\to 0}\frac{g(x+\delta x)-g(x)}{\delta x}$$

(Using limit theorems)

Thus $\phi'(x)=f'(x)g(x)+f(x)g'(x)$ $\quad\left[\because \lim\limits_{\delta x\to 0}g(x+\delta x)=g(x)\right]$

or $\dfrac{d}{dx}\left[f(x).g(x)\right]=\dfrac{d}{dx}\left[f(x)\right].g(x)+f(x)\left[\dfrac{d}{dx}g(x)\right]$

**Example:** Find derivative of $y=\left(2\sqrt{x}+2\right)\left(x-\sqrt{x}\right)$ with respect to $x$

**Solution:** $y=\left(2\sqrt{x}+2\right)\left(x-\sqrt{x}\right)$

$=2\left(\sqrt{x}+1\right)\left(x-\sqrt{x}\right)$

Differentiating with respect to *x*, we get

$$\frac{dy}{dx}=2\frac{d}{dx}\left[\left(\sqrt{x}+1\right)\left(x-\sqrt{x}\right)\right]$$

$$=2\left[\left(\frac{d}{dx}\left(\sqrt{x}+1\right)\right)\left(x-\sqrt{x}\right)+\left(\sqrt{x}+1\right)\frac{d}{dx}\left(x-\sqrt{x}\right)\right]$$

$$=2\left[\left(\frac{1}{2}x^{\frac{1}{2}-1}+0\right)\left(x-\sqrt{x}\right)+\left(\sqrt{x}+1\right)\times\left(1-\frac{1}{2}x^{\frac{1}{2}-1}\right)\right]$$





$$= 2\left[\frac{1}{2\sqrt{x}}\left(x-\sqrt{x}\right)+\left(\sqrt{x}+1\right)\times\left(1-\frac{1}{2\sqrt{x}}\right)\right]$$

$$= 2\left[\frac{x-\sqrt{x}}{2\sqrt{x}}+\left(\sqrt{x}+1\right)\left(\frac{2\sqrt{x}-1}{2\sqrt{x}}\right)\right]$$

$$= \frac{1}{\sqrt{x}}\left[x-\sqrt{x}+2x-\sqrt{x}+2\sqrt{x}-1\right]$$

$$= \frac{3x-1}{\sqrt{x}}$$

## 6. Derivative of a Quotient (The Quotient Rule)

If *f* and *g* are differentiable at $x$ and $g(x)\neq 0$, for any $x\in D(g)$ then $\dfrac{f}{g}$ is differentiable at $x$ and $\left(\dfrac{f(x)}{g(x)}\right)'=\dfrac{f'(x)g(x)-f(x)g'(x)}{\left[g(x)\right]^2}$

that is, $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{\left[\dfrac{d}{dx}\left[f(x)\right]\right]g(x)-f(x)\left[\dfrac{d}{dx}\left[g(x)\right]\right]}{\left[g(x)\right]^2}$

**Proof:** Let $\phi(x)=\dfrac{f(x)}{g(x)}$ Then

(i) $\phi(x+\delta x)=\dfrac{f(x+\delta x)}{g(x+\delta x)}$

(ii) $\phi(x+\delta x)-\phi(x)=\dfrac{f(x+\delta x)}{g(x+\delta x)}-\dfrac{f(x)}{g(x)}=\dfrac{f(x+\delta x)g(x)-f(x)g(x+\delta x)}{g(x)g(x+\delta x)}$

Subtracting and adding $f(x)g(x)$ in the numerator of step (ii), gives

$$\phi(x+\delta x)-\phi(x)=\frac{f(x+\delta x)g(x)-f(x)g(x)-f(x)g(x+\delta x)+f(x)g(x)}{g(x)g(x+\delta x)}$$

$$=\frac{1}{g(x)g(x+\delta x)}\left[\left(f(x+\delta x)-f(x)\right)g(x)-f(x)\left(g(x+\delta x)-g(x)\right)\right]$$

(iii) $\dfrac{\phi(x+\delta x)-\phi(x)}{\delta x}=\dfrac{1}{g(x)g(x+\delta x)}\left[\dfrac{f(x+\delta x)-f(x)}{\delta x}.g(x)-f(x).\dfrac{g(x+\delta x)-g(x)}{\delta x}\right]$

Taking limit when $\delta x\to 0$

(iv) $\lim\limits_{\delta x\to 0}\dfrac{\phi(x+\delta x)-\phi(x)}{\delta x}$

$$\lim_{x\to 0}\left[\frac{1}{g(x)g(x+\delta x)}\left(\frac{f(x+\delta x)-f(x)}{\delta x}.g(x)-f(x).\frac{g(x+\delta x)-g(x)}{\delta x}\right)\right]$$

Using limit theorems, we have

$$\phi'(x)=\frac{1}{g(x).g(x)}\left[f'(x)g(x)-f(x)g'(x)\right]\quad\left(\because \lim_{\delta x\to 0}g(x+\delta x)=g(x)\right)$$

Thus $\left(\dfrac{f(x)}{g(x)}\right)'=\dfrac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$ or $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)=\dfrac{\left[\frac{d}{dx}[f(x)]\right]g(x)-f(x)\left[\frac{d}{dx}[g(x)]\right]}{[g(x)]^2}$

**First Alternative Proof:**

$\phi(x)=\dfrac{f(x)}{g(x)}$ can be written as $f(x)=\phi(x)g(x)$

Using the procedure used to prove product rule, quotient rule can be proved.

**Second Alternative Proof:** We first prove the reciprocal rule and then use product rule to prove the quotient rule.

**The reciprocal rule.** If $g$ is differentiable at $x$ and $g(x)\neq 0$, then $\dfrac{1}{g}$ is differentiable at $x$ and

$\dfrac{d}{dx}\left[\dfrac{1}{g(x)}\right]=\dfrac{-\frac{d}{dx}[g(x)]}{[g(x)]^2}$ (Proof of reciprocal rule is left as an exercise)





Using the product rule to $f(x).\dfrac{1}{g(x)}$, we have

$$\frac{d}{dx}\left[f(x).\frac{1}{g(x)}\right]=\left(\frac{d}{dx}[f(x)]\right).\frac{1}{g(x)}+f(x).\frac{d}{dx}\left[\frac{1}{g(x)}\right]$$

$$=\frac{\frac{d}{dx}[f(x)]}{g(x)}+f(x)\frac{-\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

i.e., $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{\left[\frac{d}{dx}[f(x)]\right]g(x)-f(x)\left[\frac{d}{dx}[g(x)]\right]}{[g(x)]^2}$

**Example 2:** Find $\dfrac{dy}{dx}$ if $y=\dfrac{(\sqrt{x}+1)\left(x^{\frac{3}{2}}-1\right)}{x^{\frac{1}{2}}-1}$, $(x\neq 1)$

**Solution:** Given that

$$y=\frac{(\sqrt{x}+1)\left(x^{\frac{3}{2}}-1\right)}{x^{\frac{1}{2}}-1}=\frac{(\sqrt{x}+1)\left[(\sqrt{x})^3-(1)^3\right]}{\sqrt{x}-1}$$

$$=\frac{(\sqrt{x}+1)(\sqrt{x}-1)(x+1+\sqrt{x})}{\sqrt{x}-1}=(\sqrt{x}+1)(x+1+\sqrt{x})$$

$$=(\sqrt{x}+1)(\sqrt{x}-1)(x+1+\sqrt{x})=(\sqrt{x}+1)^2+(\sqrt{x}+1)x$$

$$=x+1+2\sqrt{x}+x\sqrt{x}+x=x^{\frac{3}{2}}+2x+2x^{\frac{1}{2}}+1$$

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{\frac{3}{2}}+2x+2x^{\frac{1}{2}}+1\right)=\frac{d}{dx}\left(x^{\frac{3}{2}}\right)+\frac{d}{dx}(2x)+\frac{d}{dx}\left(2x^{\frac{1}{2}}\right)+\frac{d}{dx}(1)$$

$$=\frac{3}{2}x^{\frac{1}{2}}+2(1)+2.\frac{1}{2\sqrt{x}}+0=\frac{3}{2}\sqrt{x}+2+\frac{1}{\sqrt{x}}$$

**Example 3:** Differentiate $\dfrac{\left(\sqrt{x}+1\right)\left(x^{\frac{3}{2}}-1\right)}{x^{\frac{3}{2}}-x^{\frac{1}{2}}}$ with respect to *x*.

**Solution:** Let $y=\dfrac{\left(\sqrt{x}+1\right)\left(x^{\frac{3}{2}}-1\right)}{x^{\frac{3}{2}}-x^{\frac{1}{2}}}$

$$=\frac{\left(\sqrt{x}+1\right)\left[x^{\frac{3}{2}}-1\right]}{\sqrt{x}\left(x-1\right)}$$

$$=\frac{\left(\sqrt{x}+1\right)\left(\sqrt{x}-1\right)\left(x+\sqrt{x}+1\right)}{\sqrt{x}\left(\sqrt{x}-1\right)}=\frac{\left(x-1\right)\left(x+\sqrt{x}+1\right)}{\sqrt{x}\left(\sqrt{x}-1\right)}$$

$$=\frac{x+\sqrt{x}+1}{\sqrt{x}}$$

Differentiating with respect to $x$, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{x+\sqrt{x}+1}{\sqrt{x}}\right]$$

$$=\frac{\sqrt{x}\frac{d}{dx}\left(x+\sqrt{x}+1\right)-\left(x+\sqrt{x}+1\right)\frac{d}{dx}\left(\sqrt{x}\right)}{\left(\sqrt{x}\right)^2}$$

$$=\frac{\sqrt{x}\left(1+\frac{1}{2}x^{-\frac{1}{2}}+0\right)-\left(x+\sqrt{x}+1\right).\left(\frac{1}{2}x^{-\frac{1}{2}}\right)}{x}$$

$$=\frac{\sqrt{x}\left(1+\frac{1}{2\sqrt{x}}\right)-\left(x+\sqrt{x}+1\right)\frac{1}{2\sqrt{x}}}{x}$$





$$=\frac{\sqrt{x}\left(\frac{2\sqrt{x}+1}{2\sqrt{x}}\right)-\frac{x+\sqrt{x}+1}{2\sqrt{x}}}{x}=\frac{2x+\sqrt{x}-x-\sqrt{x}-1}{x.2\sqrt{x}}=\frac{x-1}{2x^{\frac{3}{2}}}$$

**Example 4:** Differentiate $\dfrac{2x^3-3x^2+5}{x^2+1}$ with respect to $x$.

**Solution:** Let $\phi(x)=\dfrac{2x^3-3x^2+5}{x^2+1}$. Then we take

$f(x)=2x^3-3x^2+5$ and $g(x)=x^2+1$

Now $\quad f'(x)=\dfrac{d}{dx}\left[2x^3-3x^2+5\right]=2\left(3x^2\right)-3\left(2x\right)+0=6x^2-6x$

and $\quad g'(x)=\dfrac{d}{dx}\left[x^2+1\right]=2x+0 \;=\; 2x$

Using the quotient formula: $\phi'(x)=\dfrac{f'(x)g(x)-f(x)g'(x)}{\left[g(x)\right]^2}$, we obtain

$$\frac{d}{dx}\left[\frac{2x^3-3x^2+5}{x^2+1}\right]=\frac{\left(6x^2-6x\right)\left(x^2+1\right)-\left(2x^3+3x^2+5\right)\left(2x\right)}{\left(x^2+1\right)^2}$$

$$=\frac{6x^4-6x^3+6x^2-6x-\left(4x^4-6x^3+10x\right)}{\left(x^2+1\right)^2}$$

$$=\frac{6x^4-6x^3+6x^2-6x-4x^4+6x^3-10x}{\left(x^2+1\right)^2}$$

$$=\frac{2x^4+6x^2-16x}{\left(x^2+1\right)^2}$$

## EXERCISE 2.3

Differentiate w.r.t. $x$

**1.** $x^4+2x^3+x^2$ **2.** $x^{-3}+2x^{-3/2}+3$ **3.** $\dfrac{a+x}{a-x}$

**4.** $\dfrac{2x-3}{2x+1}$ **5.** $(x-5)(3-x)$ **6.** $\left(\sqrt{x}-\dfrac{1}{\sqrt{x}}\right)^2$

**7.** $\dfrac{(1+\sqrt{x})\left(x-x^{\frac{3}{2}}\right)}{\sqrt{x}}$ **8.** $\dfrac{(x^2+1)^2}{x^2-1}$ **9.** $\dfrac{x^2+1}{x^2-3}$

**10.** $\dfrac{\sqrt{1+x}}{\sqrt{1-x}}$ **11.** $\dfrac{2x-1}{\sqrt{x^2+1}}$ **12.** $\sqrt{\dfrac{a-x}{a+x}}$

**13.** $\dfrac{\sqrt{x^2+1}}{\sqrt{x^2-1}}$ **14.** $\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$ **15.** $\dfrac{x\sqrt{a+x}}{\sqrt{a-x}}$

**16.** If $y=\sqrt{x}-\dfrac{1}{\sqrt{x}}$, show that $2x\dfrac{dy}{dx}+y=2\sqrt{x}$

**17.** If $y=x^4+2x^2+2$, prove that $\dfrac{dy}{dx}=4x\sqrt{y-1}$

## 2.4 THE CHAIN RULE

The composition *fog* of functions *f* and *g* is the function whose values *f* [*g*(*x*)], are found for each *x* in the domain of *g* for which *g*(*x*) is in the domain of $f.(f[g(x)])$ is read as *f* of *g* of *x*).

**Theorem.** If *g* is differentiable at the point $x$ and *f* is differentiable at the point *g*($x$) then the composition function *fog* is differentiable at the point $x$ and $(fog)'(x)=f'[g(x)].g'(x)$.
The proof of the chain rule is beyond the scope of this book.

If $y=(fog)(x)=f[g(x)]$, then

$$(fog)'(x)=[f[g(x)]]'=\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=f'[g(x)].g'(x) \qquad \text{(i)}$$

Let $u=g(x)$ (ii)

Then $y=f(u)$ (iii)





Differentiating (ii) and (iii) w.r.t $x$ and $u$ respectively, we have.

$$\frac{du}{dx}=\frac{d}{dx}[g(x)]=g'(x)$$

and $\dfrac{dy}{du}=\dfrac{d}{du}[f(u)]=f'u$

Thus (i) can be written in the following forms

(a) $\dfrac{d}{dx}(f(u))=f'(u)\dfrac{du}{dx}$

(b) $\dfrac{dy}{dx}=\dfrac{dy}{du}.\dfrac{du}{dx}$

The proof of the Chain rule is beyond the scope of this book.

**Note: 1.** Let $y=[g(x)]^n$ and $u=g(x)$

Then $y=u^n$ and $\dfrac{dy}{du}=nu^{n-1}$ (power rule)

But $\dfrac{dy}{dx}=\dfrac{dy}{du}.\dfrac{du}{dx}=nu^{n-1}\dfrac{du}{dx}$

or $\dfrac{d}{dx}[g(x)]^n=n[g(x)]^{n-1}.g'(x)$ $\left(\because \dfrac{du}{dx}=g'(x)\right)$

**2.** Reciprocal rule can be written as

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right)=\frac{d}{dx}[g(x)]^{-1}=-1.[g(x)]^{-1-1}.g'(x)$$

$$=(-1)[g(x)]^{-2}.g'(x)$$

**Example 1:** Find the derivative of $(x^3+1)^9$ with respect to

**Solution:** Let $y=(x^3+1)^9$ and $u=x^3+1$ Then $y=u^9$

Now $\frac{du}{dx}=3x^2$ and $\frac{dy}{du}=9u^8$ (Power formula)

Using the formula $\frac{dy}{dx}=9u^8\frac{du}{dx}$ , we have

or $\frac{d}{dx}\left(x^3+1\right)^9=9\left(x^3+1\right)^8\left(3x^2\right)$ $\left(\because u=x^3+1 \text{ and } \frac{du}{dx}=3x^2\right)$

$=27x^2\left(x^3+1\right)^8$

**Example 2:** Differentiate $\sqrt{\frac{a-x}{a+x}}$ ,$(x\neq -a)$ with respect to $x$

**Solution:** Let $y=\sqrt{\frac{a-x}{a+x}}$ and $u=\frac{a-x}{a+x}$. Then $y=u^{\frac{1}{2}}$

Now $\frac{dy}{du}=\frac{1}{2}u^{\frac{1}{2}-1}=\frac{1}{2}u^{-\frac{1}{2}}$

and $\frac{du}{dx}=\frac{d}{dx}\left[\frac{a-x}{a+x}\right]=\frac{\left[\frac{d}{dx}(a-x)\right](a+x)-(a-x)\left[\frac{d}{dx}(a+x)\right]}{(a+x)^2}$

$=\frac{(0-1)(a+x)-(a-x)(0+1)}{(a+x)^2}=\frac{-a-x-a+x}{(a+x)^2}=\frac{-2a}{(a+x)^2}$

Using the formula $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$, we have

$$\frac{d}{dx}\left(\sqrt{\frac{a-x}{a+x}}\right)=\frac{1}{2}u^{-\frac{1}{2}}\left[\frac{-2a}{(a+x)^2}\right]=\frac{1}{2}\left(\frac{a-x}{a+x}\right)^{-\frac{1}{2}}\times\frac{-2a}{(a+x)^2}\left(\because u=\frac{a-x}{a+x}\right)$$

$$=\frac{(a-x)^{-\frac{1}{2}}}{(a+x)^{-\frac{1}{2}}}\times\frac{-a}{(a+x)^2}=\frac{-a}{(a-x)^{\frac{1}{2}}(a+x)^{\frac{3}{2}}}$$





**Example 3:** Find $\frac{dy}{dx}$ if $y=\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$ $(x\neq 0)$

**Solution:** $y=\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$

Multiplying the numerator and the denominator by $\sqrt{a+x}-\sqrt{a-x}$, gives

$$y=\frac{\left(\sqrt{a+x}+\sqrt{a-x}\right)\left(\sqrt{a+x}-\sqrt{a-x}\right)}{\left(\sqrt{a+x}-\sqrt{a-x}\right)\left(\sqrt{a+x}-\sqrt{a-x}\right)}$$

$$=\frac{\left(\sqrt{a+x}\right)^2-\left(\sqrt{a-x}\right)^2}{(a+x)+(a-x)-2\sqrt{a^2-x^2}}=\frac{(a+x)-(a-x)}{2a-2\sqrt{a^2-x^2}}=\frac{2x}{2\left(a-\sqrt{a^2-x^2}\right)}$$

that is, $y=\frac{x}{a-\sqrt{a^2-x^2}}$

Let $f(x)=x$ and $g(x)=a-\sqrt{a^2-x^2}$, then

$f(x)'=1$ and $g'(x)=0-\frac{d}{dx}\left(a^2-x^2\right)^{\frac{1}{2}}=-\frac{1}{2}\left(a^2-x^2\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(a^2-x^2\right)$

$=-\frac{1}{2\sqrt{a^2-x^2}}$ x $(-2x)=\frac{x}{\sqrt{a^2-x^2}}$

Using the formula $\frac{dy}{dx}=\frac{f'(x)\,g(x)-f(x)\,g'(x)}{[g(x)]^2}$, we have

$$\frac{dy}{dx}=\frac{1.\left(a-\sqrt{a^2-x^2}\right)-x.\frac{x}{\sqrt{a^2-x^2}}}{\left(a-\sqrt{a^2-x^2}\right)^2}$$

$$=\frac{a\sqrt{a^2-x^2}-\left(a^2-x^2\right)-x^2}{\sqrt{a^2-x^2}\left(a-\sqrt{a^2-x^2}\right)^2}=\frac{a\sqrt{a^2-x^2}-a^2}{\sqrt{a^2-x^2}\left(a-\sqrt{a^2-x^2}\right)^2}$$

$$=\frac{-a\left(a-\sqrt{a^2-x^2}\right)}{\sqrt{a^2-x^2}\left(a-\sqrt{a^2-x^2}\right)^2} \quad = \quad \frac{-a}{\sqrt{a^2-x^2}\left(a-\sqrt{a^2-x^2}\right)^2}$$

**Example 4:** Find $\frac{dy}{dx}$ if $y=\left(1+2\sqrt{x}\right)^3.x^{\frac{3}{2}}$

**Solution:** $y=\left(1+2\sqrt{x}\right)^3.x^{\frac{3}{2}}=\left[\left(1+2\sqrt{x}\right)\left(x^{\frac{1}{2}}\right)\right]^3$

Let $u=\left(1+2\sqrt{x}\right).x^{\frac{1}{2}}$ (i)

Then $y=u^3$ (ii)

Differentiating (ii) with respect to $u$, we have

$$\frac{dy}{dx}=3u^2=3\left[\left(1+2\sqrt{x}\right)x^{\frac{1}{2}}\right]^2=3\left(1+2\sqrt{x}\right)^2.x$$

Differentiating (i) with respect to $x$, gives

$$\frac{du}{dx}=\left(0+2.\frac{1}{2\sqrt{x}}\right)x^{\frac{1}{2}}+\left(1+2\sqrt{x}\right)\frac{1}{2\sqrt{x}}$$

$$=1+\frac{1+2\sqrt{x}}{2\sqrt{x}}=\frac{2\sqrt{x}+1+2\sqrt{x}}{2\sqrt{x}}=\frac{1+4\sqrt{x}}{2\sqrt{x}}$$

Using the formula $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ ,we have

$$\frac{d}{dx}\left[\left(1+2\sqrt{x}\right)^3.x^{\frac{3}{2}}\right]=3\left(1+2\sqrt{x}\right)^2.x \text{ x}\left(\frac{1+4\sqrt{x}}{2\sqrt{x}}\right)$$

$$=\frac{3}{2}\left(1+2\sqrt{x}\right)^2\sqrt{x}\left(1+4\sqrt{x}\right)$$

$$=-\left(1+2\sqrt{x}\right)\left(\sqrt{x}+4x\right)$$

**Example 5:** If $y = (ax + b)^n$ where n is a negative integer, find $\frac{dy}{dx}$ using quotient theorem

**Solution:** Let $n = -m$ where $m$ is a positive integer. Then





$$y=\left(ax+b\right)^n = \left(ax+b\right)^{-m}=\frac{1}{\left(ax+b\right)^m} \qquad \text{(i)}$$

We first find $\frac{d}{dx}\left(ax+b\right)^m$. Let $u=ax+b$. Then

$$\frac{d}{dx}\left(ax+b\right)^m=\frac{d}{dx}\left(u^m\right)=\frac{d}{du}\left(u^m\right)\frac{du}{dx} \qquad \text{(using chain rule)}$$

$$=mu^{m-1} \text{ x a=m}\left(ax+b\right)^{m-1}.a \qquad \left(\because \frac{d}{dx}\left(ax+b\right)=a\right)$$

Now differentiating (i) w.r.t.'$x$', we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{1}{\left(ax+b\right)^m}\right]=\frac{\frac{d}{dx}(1).\left(ax+b\right)^m-1.\frac{d}{dx}\left(ax+b\right)^m}{\left[\left(ax+b\right)^m\right]^2}$$

$$=\frac{0.\left(ax+b\right)^m-1.m\left(ax+b\right)^{m-1}.a}{\left(ax+b\right)^{2m}}$$

$$=\left(-m\left(ax+b\right)^{m-1}.a\right)\text{ x}\left(ax+b\right)^{-2m}=-m\left(ax+b\right)^{m-1-2m}.a$$

$$=(-m)\ (ax+b)^{-m-1}\ .a\ =\ n(ax+b)^{n-1}\ \ .a \qquad (\because -m\ =\ n)$$

**Example 6:** Find $\frac{dy}{dx}$ if $\text{y} = \text{x}^n$ where $\text{n} = \frac{p}{q}$, $\text{q}\neq 0$

**Solution:** Given that $y = x^n$ where $n = \frac{p}{q}$, $q\neq 0$. putting $n = \frac{p}{q}$, *we* have

$$y = x^{\frac{p}{q}} \qquad \text{(i)}$$

Taking qth power of both sides of (i), we get

$$\text{y}^q = x^p \qquad \text{(ii)}$$

Differentiating both sides of (ii) w.r.t. '$x$', gives

$$\frac{d}{dx}(\text{y}^q) = \frac{d}{dx}(\text{x}^p) \text{ or } \frac{d}{dy}(\text{y}^q).\frac{dy}{dx} = \frac{d}{dx}\left(x^p\right) \text{ (Using chain rule)}$$

$$\Rightarrow \text{q y}^{q-1}\frac{dy}{dx}=\text{px}^{p-1} \qquad \text{(iii)}$$

Multiplying both sides of (iii) by *y*, we have

$$q\,.\,y^q\frac{dy}{dx} = py\;x^{p-1} \quad \text{or} \quad q.x^p\;\frac{dy}{dx} = p.\bar{x}\;x^{p-1} \qquad \text{(using (i) and (ii))}$$

$$\Rightarrow \frac{dy}{dx} = \frac{p}{q}\cdot\frac{1}{x^p}\;.x^{\frac{p}{q}}\;x^{p-1} = \frac{p}{q}\times x^{\frac{p}{q}+p-1-p}$$

$$= \frac{p}{q}\,x^{\frac{p}{q}-1} = nx^{n-1} \quad \left[\because \frac{p}{q} = n\right]$$

$$Thus\;\frac{d}{dx}(x^n)\;n\;x^{n-1}\;.$$

## 2.5 DERIVATIVES OF INVERSE FUNCTIONS

If for each $x \in D_f$, $f(x) = y$ and for each $y \in D_g$, $g(x) = x$, then $f$ and $g$ are inverse of each other, that is,

$(g\;o\;f)(x) = g(f(x)) = g(y) = x$ (i)

and $(f\;o\;g)(y) = f(g(y)) = f(x) = y$ (ii)

Using chain rule, we can prove that

$$f'(x).\,g'(y) = 1$$

$$\Rightarrow \quad f'(x) = \frac{1}{g'(y)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad \left(\begin{aligned} &\because \quad f(x) = y \Rightarrow f'(x) = \frac{dy}{dx} \\ &and\;\; g(y) = x \Rightarrow g'(y) = \frac{dx}{dy}\end{aligned}\right)$$

## 2.6 DERIVATIVE OF A FUNCTION GIVEN IN THE FORM OF PARAMETRIC EQUATIONS

The equations $x = at^2$ and $y = 2at$ express $x$ and $y$ as function of $t$. Here the variable $t$ is called a parameter and the equations of $x$ and $y$ in terms of $t$ are called the parametric equations.





Now we explain the method of finding derivatives of functions given in the form of parametric equations by the following examples.

**Example 1:** Find $\frac{dy}{dx}$ if $x = at^2$ and $y = 2at$.

**Solution:** We use the chain rule to find $\frac{dy}{dx}$

$$\text{Here } \frac{dy}{dt} = \frac{d}{dt}(2at) = 2a.1 = 2a$$

$$\text{and } \frac{dx}{dt} = \frac{d}{dt}(at^2) = a\,(2t) = 2at$$

$$\text{so} \quad \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{2a}{y} \qquad (\because 2a = y)$$

$$\text{Eliminating } t\text{, we get } x = a\left(\frac{y}{2a}\right)^2 = a.\frac{y^2}{4a^2} = \frac{y^2}{4a} \Rightarrow y^2 = 4ax \qquad \text{(i)}$$

Differentiating both sides of (i) w.r.t. '$x$' we have

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(4ax)$$

$$\frac{d}{dx}(y^2)\,.\,\frac{dy}{dx} = 4a\frac{d}{dx}(x) \qquad \Rightarrow 2y\frac{dy}{dx} = 4a\;(1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

**Example 2:** Find $\frac{dy}{dx}$ if $x\;1 - t^2$ and $y = 3t^2 - 2t^3$ .

**Solution:** Given that $x = 1 - t^2$ ...... (i) and $y = 3t^2 - 2t^2$ (ii)

Differentiating (i) w.r.t. '$t$' ,we get

$$\frac{dy}{dt}=\frac{d}{dt}\left(1-t^2\right)=\frac{d}{dt}(1)-\frac{d}{dt}\left(t^2\right)=0-2t=-2t$$

Differentiating (ii) w.r.t. '$t$' ,we have

$$\frac{dy}{dt}=\frac{d}{dt}\left(3t^2-2t^2\right)=\frac{d}{dt}\left(3t^2\right)-\frac{d}{dt}\left(2t^3\right)$$

$$=3(2t)-2\left(3t^2\right)=6t-6t^2=6t(1-t)$$

Applying the formula

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$=\frac{6t(1-t)}{-2t}=-3(1-t)=3(t-1)$$

**Example 3:** Find $\frac{dy}{dx}$ if $x=\frac{1-t^2}{1+t^2},\ y=\frac{2t}{1+t}$

**Solution:** Given that $x=\frac{\left(1+t^2\right)}{1+t^2}$ (i) *and* $y=\frac{2t}{1+t^2}$ (ii)

Differentiating (i) w.r.t. '$t$' ,we get

$$\frac{dx}{dt}=\frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)=\frac{\left(\frac{d}{dt}\left(1-t^2\right)\right)\left(1+t^2\right)-\left(1-t^2\right).\frac{d}{dt}\left(1+t^2\right)}{(1+t^2)^2}$$

$$=\frac{(-2t)\left(1+t^2\right)-\left(1-t^2\right)(2t)}{\left(1+t^2\right)^2}=\frac{2t\left(-1-t^2-1+t^2\right)}{\left(1+t^2\right)^2}=\frac{-4t}{\left(1+t^2\right)^2}$$

Differentiating (i) w.r.t. '$t$' ,we have





$$\frac{dy}{dt}=\frac{d}{dt}\left(\frac{2t}{1+t^2}\right)=\frac{\left(\frac{d}{dt}(2t)\right)\left(1+t^2\right)-2t\text{ x }\frac{d}{dt}\left(1+t^2\right)}{\left(1+t^2\right)^2}$$

$$=\frac{2\left(1+t^2\right)-2t(2t)}{\left(1+t^2\right)^2}=\frac{2+2t^2-4t^2}{\left(1+t^2\right)^2}=\frac{2-2t^2}{\left(1+t^2\right)^2}=\frac{2\left(1-t^2\right)}{\left(1+t^2\right)^2}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{\frac{dy}{dt}}{\frac{dy}{dx}}=\frac{\frac{2\left(1-t^2\right)}{\left(1+t^2\right)^2}}{-\frac{4t}{\left(1+t^2\right)^2}}=\frac{2\left(1-t^2\right)}{-4t}=\frac{t^2-1}{2t}$$

## 2.7 Differentiation of Implicit Relations

Sometimes the functional relation is not explicitly expressed in the form $y=f(x)$ but an equation involving $x$ and $y$ is given. To find $\frac{dy}{dx}$ from such an equation, we differentiate each term of the equation and use the chain rule where it is required.The process of finding $\frac{dy}{dx}$ in this way, is called implicit differentiation. We explain the implicit differentiation in the following examples.

**Example 1:** Find $\frac{dy}{dx}$ if $x^2+y^2=4$

**Solution:** Here $x^2+y^2=4$ (i)

Differentiating both sides of (i) w.r.t. '$x$' , we get

$$2x+2y\frac{dy}{dx}=0$$

or $x+y\frac{dy}{dx}=0 \quad \Rightarrow \frac{dy}{dx}=-\frac{x}{y}$

Solving (i) for *y* in terms of *x*, we have

$y=\pm\sqrt{4-x^2}$

$\Rightarrow \quad y=\sqrt{4-x^2}$ (ii)

or $\quad y=-\sqrt{4-x^2}$ (iii)

$\frac{dy}{dx}$ found above represents the derivative of each of functions defined as in *dx* (ii) and (iii)

From (ii) $\frac{dy}{dx}=\frac{1}{2\sqrt{4-x^2}}$ x $(-2x)=-\frac{x}{\sqrt{4-x^2}}$

$=-\frac{x}{y}\left(\because \sqrt{4-x^2}=y\right)$

From (iii) $\frac{dy}{dx}=-\frac{1}{2\sqrt{4-x^2}}$ x $(-2x)=\frac{-x}{-\sqrt{4-x^2}}=-\frac{x}{y}\quad\left(\because -\sqrt{4-x}=y\right)$

**Example 2:** Find $\frac{dy}{dx}$, if $y^2+x^2-4x=5$.

**Solution:** Given that $y^2+x^2-4x=5$ (i)

Differentiating both sides of (i) w.r.t. '$x$', we get

$$\frac{d}{dx}\left[y^2+x^2-4x\right]=\frac{d}{dx}(5)$$

*or* $\quad 2y\frac{dy}{dx}+2x-4=0 \qquad \left[\because \frac{d}{dx}\left(y^2\right)=\frac{d}{dx}\left(y^2\right)\frac{dy}{dx}=2y\frac{dy}{dx}\right]$





$\Rightarrow \quad 2y\frac{dy}{dx}=4-2x \quad \Rightarrow \frac{dy}{dx}=\frac{2(2-x)}{2y}=\frac{2-x}{y}$ (ii)

**Note:** Solving (i) for *y*, we have

$y^2 = 5+4x-x \qquad \Rightarrow \qquad y = \pm\sqrt{5+4x-x^2}$

Thus $y=\sqrt{5+4x-x^2}$ (iii)

or $\quad y=-\sqrt{5+4x-x^2}$ (iv)

Each of these equations (iii) and (iv) defines a function.

Let $\quad y=f_1(x) = \sqrt{5+4x-x^2}$ (v)

and $\quad y=f_1(x) = -\sqrt{5+4x-x^2}$. (vi)

Differentiation (v) w.r.t. '$x$', we get

$$f_1'(x)=\frac{1}{2}\left(5+4x-x^2\right)^{-\frac{1}{2}}\times(4-2x)=\frac{2-x}{\sqrt{5+4x-x^2}}$$

From (v) $,\sqrt{5+4x-x^2}=y, \quad so \quad f_1'(x)=\frac{2-x}{y}$

Also $\quad f_2'(x)=-\frac{1}{2}\left(5+4x-x^2\right)^{-\frac{1}{2}}\times(4-2x)=\frac{2-x}{-\sqrt{5+4x-x^2}}$

From (vi) $\quad -\sqrt{5+4x-x^2}=y, \quad so \quad f_2'(x)=\frac{2-x}{y}$

Thus (ii) represents the derivative of $f_1(x)$ as well as that of $f_2(x)$.

**Example 3:** Find $\frac{dy}{dx}$ if $y^2-xy-x^2+4=0$.

**Solution:** Given that $y^2 - xy - x^2 + 4 = 0$ (i)

Differentiating both sides of (i) w.r.t. '$x$', gives

$$\frac{d}{dx}\left[y^2 - xy - x^2 + 4\right] = \frac{d}{dx}(0) \quad = \quad 0$$

or $\quad 2y\frac{dy}{dx} - \left(1.y + x\frac{dy}{dx}\right) - 2x + 0 = 0$

$\Rightarrow \quad (2y - x)\frac{dy}{dx} = 2x + y \qquad \Rightarrow \frac{dy}{dx} = \frac{2x+y}{2y-x}$

**Example 4:** Find $\frac{dy}{dx}$ if $y^3 - 2xy^2 - x^2y + 3x = 0.$

**Solution:** Differentiating both sides of the given equation w.r.t. *'x'* we have

$$\frac{d}{dx}\left[y^3 - 2xy^2 + x^2y + 3x\right] = \frac{d}{dx}(0) \quad = \quad 0$$

or $\quad \frac{d}{dx}(y^3) - \frac{d}{dx}(2xy^2) + \frac{d}{dx}(x^2y) + \frac{d}{dx}(3x) = 0$

$$\frac{d}{dx}(y^3) - 2\left[1.y^2 + x\frac{d}{dx}(y^2)\right] + \left(2xy + x^2\frac{dy}{dx}\right) + 3 = 0$$

Using the chain rule on $\frac{d}{dx}(y^3)$ and $\frac{d}{dx}(y^2)$, we have

$$3y^2\frac{dy}{dx} - 2\left[y^2 + x\left(2y\frac{dy}{dx}\right)\right] + 2xy + x^2\frac{dy}{dx} + 3 = 0$$

or $\quad (3y^2 - 4xy + x^2)\frac{dy}{dx} = 2y^2 - 2xy - 3$

$\Rightarrow \quad \frac{dy}{dx} \quad = \frac{2y^2 - 2xy - 3}{3y^2 - 4xy + x^2}$

**Example 5:** Differentiate $x^2 + \frac{1}{x^2}$ w.r.t. $x - \frac{1}{x}$

**Solution:** Let $y = x^2 + \frac{1}{x^2}$ and $u = x - \frac{1}{x}$. Then





$$\frac{dy}{dx} = 2x + (-2).\frac{1}{x^3} = 2\left(x - \frac{1}{x^3}\right) = \frac{2(x^4-1)}{x^3} = \frac{2(x^2-1)(x^2+1)}{x^3}$$

and $\quad \frac{du}{dx} = 1 - (-1).\frac{1}{x^2} = 1 + \frac{1}{x^2} = \frac{x^2+1}{x^2}$

Thus $\quad \frac{dy}{du} = \frac{dy}{dx}.\frac{dx}{du} = \frac{2(x^2-1)(x^2+1)}{x^3}.\frac{x^2}{x^2+1} = \frac{2(x^2-1)}{x} = 2\left(x - \frac{1}{x}\right)$

## EXERCISE 2.4

**1.** Find $\frac{dy}{dx}$ by making suitable substitutions in the following functions defined as:

(i) $y = \sqrt{\frac{1-x}{1+x}}$ (ii) $y = \sqrt{x + \sqrt{x}}$ (iii) $y = x\sqrt{\frac{a+x}{a-x}}$

(iv) $y = (3x^2 - 2x + 7)^6$ (v) $\sqrt{\frac{a^2+x^2}{a^2-x}}$

**2.** Find $\frac{dy}{dx}$ if:

(i) $3x + 4y + 7 = 0$ (ii) $xy + y^2 = 2$

(iii) $x^2 - 4xy - 5y = 0$ (iv) $4x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

(v) $x\sqrt{1+y} + y\sqrt{1+x} = 0$ (vi) $y(x^2 - 1) = x\sqrt{x^2 + 4}$

**3.** Find $\frac{dy}{dx}$ of the following parametric functions

(i) $x = \theta + \frac{1}{\theta}$ and $y = \theta + 1$ (ii) $x = \frac{a(1-t^2)}{1+t^2}$, $y = \frac{2bt}{1+t^2}$

**4.** Prove that $y\frac{dy}{dx} + x = 0$ if $x = \frac{1-t^2}{1+t^2}$ , $y = \frac{2t}{1+t}$

**5.** Differentiate

(i) $x^2-\frac{1}{x^2}$ w.r.t $x^4$ (ii) $(1+x^2)^n$ w.r.t $x^2$

(iii) $\frac{x^2+1}{x^2-1}$ w.r.t $\frac{x-1}{x+1}$ (iv) $\frac{ax+b}{cx+d}$ w.r.t $\frac{ax^2+b}{ax^2+d}$

(v) $\frac{x^2+1}{x^2-1}$ w.r.t $x^3$

## 2.8 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

While finding derivatives of trigonometric functions, we assume that *x* is measured in radians. The limit theorems $\lim_{x\to0}\frac{\sin x}{x}=1$ and $\lim_{x\to0}\frac{1-\cos x}{x}=0$ are used to find the derivative formulas for sin *x* and cos *x*.

We prove from first principle that

$$\frac{d}{dx}(\sin x)=\cos x \text{ and } \frac{d}{dx}(\cos x)=-\sin x$$

Let $y=\sin x$ Then $y+\delta y = \sin(x+\delta x)$

and $\delta y=\sin(x+\delta x)-\sin x$

$$=2\cos\left(\frac{x+\delta x+x}{2}\right)\sin\left(\frac{x+\delta x-x}{2}\right)=2\cos\left(x+\frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)$$

$$\frac{\delta y}{\delta x}=\frac{2\cos\left(x+\frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}=\cos\left(x+\frac{\delta x}{2}\right)\frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}$$

$$\lim_{\delta x\to0}\frac{\delta y}{\delta x}=\lim_{\delta x\to0}\left[\cos\left(x+\frac{\delta x}{2}\right)\frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}\right]$$





$$=\lim_{\frac{\delta x}{2}\to0}\cos\left(x+\frac{\delta x}{2}\right)\lim_{\frac{\delta x}{2}\to0}\frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}\quad\left(\because \frac{\delta x}{2}\to0 \text{ when } \delta x\to0\right)$$

Thus $\frac{dy}{dx}=\cos x.1.\left(\because \lim_{\delta x/2\to0}\cos\left(x+\frac{\delta x}{2}\right)=\cos x \text{ and } \lim_{\delta x/2\to0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}=1\right)$

Let $y=\cos x$, then $y+\delta y=\cos(x+\delta x)$

and $\delta y=\cos(x+\delta x)-\cos x$

$$=\cos x\cos\delta x-\sin x\sin\delta x-\cos x$$

$$=-\sin x\sin\delta x-\cos x\left(\frac{1-\cos\delta x}{\delta x}\right)$$

$$\frac{\delta y}{\delta x}=(-\sin x).\frac{\sin\delta x}{\delta x}-\cos x\left(\frac{1-\cos\delta x}{\delta x}\right)$$

$$\lim_{\delta x\to0}\frac{\delta y}{\delta x}=\lim_{\delta x\to0}\left[(-\sin x)\frac{\sin\delta x}{\delta x}-\cos x\left(\frac{1-\cos\delta x}{\delta x}\right)\right]$$

$$=\lim_{\delta x\to0}\left[(-\sin x)\frac{\sin\delta x}{\delta x}\right]-\lim_{\delta x\to0}\left[-\cos x\left(\frac{1-\cos\delta x}{\delta x}\right)\right]$$

Thus $\frac{dy}{dx}=(-\sin x).1-(\cos x)(0)\quad\left(\because \lim_{\delta x\to0}\frac{\sin\delta y}{\delta x}=1 \text{ and } \lim_{\delta x\to0}\left(\frac{1-\cos\delta x}{\delta x}\right)=0\right)$

or $\frac{d}{dx}(\cos x)=-\sin x$

Now using $\frac{d}{dx}(\sin x)=\cos x$ and $\frac{d}{dx}(\cos x)=-\sin x$, we prove that

$$\frac{d}{dx}(\sec x)=\sec x\tan x \quad and \quad \frac{d}{dx}(\cot x)=\text{cosec}^2 x$$

Proof of $\frac{d}{dx}(sec\ x) = sec\,x\ \tan x.$

Let $y = sec\,x = \frac{1}{cos\,x}$ (i)

Differentiating (i) w.r.t. '$x$', we have

$$\frac{d}{dx}(y) = \frac{d}{dx}\left[\frac{1}{cos\ x}\right] = \frac{\left[\frac{d}{dx}(1)\right]cos\,x - 1.\frac{d}{dx}(cos\,x)}{(cos\,x)^2} \quad \left(\begin{array}{l}\text{Using}\\ \text{quotient}\\ \text{formula}\end{array}\right)$$

$$= \frac{0.cos\,x - 1.(-sin\,x)}{cos^2 x}$$

$$= \frac{1}{cos\,x} . \frac{sin\,x}{cos\,x} = sec\ x\ \tan x$$

Thus $\frac{d}{dx}(sec\,x) = sec\ x\ \tan x$

Proof of $\frac{d}{dx}(\cot x) = cos\,ec^2\ x$

Let $y = \cot x = \frac{cos\ x}{sin\ x}$ (i)

Differentiating (i) w.r.t. '$x$', we get

$$\frac{d}{dx}(y) = \frac{d}{dx}\left[\frac{cos\ x}{sin\ x}\right] = \frac{\left[\frac{d}{dx}(cos\,x)\right]sin\,x - cos\,x\frac{d}{dx}(sin\,x)}{(sin\,x)^2} \quad \left(\begin{array}{l}\text{Using}\\ \text{quotient}\\ \text{formula}\end{array}\right)$$

$$= \frac{(-sin\ x)sin\,x - cos\,x(cos\,x)}{sin^2 x}$$

$$= \frac{-(sin^2 x + cos^2 x)}{sin^2 x} = -\frac{1}{sin^2 x} = -cos\,ec^2\ x$$

Thus $\frac{d}{dx}(\cot x) = cos\,ec^2\ x$





Now we write the derivatives of six trigonometric functions

(1) $\frac{d}{dx}(sin\,x) = cos\,x$ (2) $\frac{d}{dx}(cos\,x) = sin\,x$

(3) $\frac{d}{dx}(tan\,x) = sec^2\ x$ (4) $\frac{d}{dx}(cot\,x) = -\ cosec^2\ x$

(5) $\frac{d}{dx}(cosec\,x) = -\ cosec\,x\ cot\,x$ (6) $\frac{d}{dx}(sec\,x) = sec\,x\ tan\,x$

**Example 1:** Find the derivative of tan *x* from first principle.

**Solution:** Let $y = tan\,x,$ *then* $y + \delta x = tan\,(x + \delta x)$ *and*

$$\delta y = y + \delta x - y = tan\,(x + \delta x) - tan\,x$$

$$= \frac{sin(x+\delta x)}{cos(x+\delta x)} - \frac{sin\,x}{cos\,x} = \frac{sin(x+\delta x)cos\,x - cos(x+\delta x)sin\,x}{cos(x+\delta x)\ cos\,x}$$

$$= \frac{sin(x+\delta x - x)}{cos(x+\delta x).cos\,x} = \frac{sin\,\delta x}{cos(x+\delta x)\ cos\,x}$$

$$\frac{\delta y}{\delta x} = \frac{1}{cos(x+\delta x).cos\,x} . \frac{sin\,\delta x}{\delta x}$$

or $$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\left(\frac{1}{cos(x+\delta x).cos\,x}\right).\lim_{\delta x \to 0}\left(\frac{sin\ \delta x}{\delta x}\right)$$

Thus $$\frac{dy}{dx} = \frac{1}{(cos\ \ x)(cos\ \ x)}.1 = sec^2\ x \quad \left(\begin{array}{l}\because \lim_{\delta x \to 0} cos(x+\delta x) = cos\ x\\ and \lim_{\delta x \to 0}\frac{sin\ \delta x}{\delta x} = 1\end{array}\right)$$

Thus $\frac{dy}{dx} = sec^2\ x$ or $\frac{d}{dx}(tan\ x) = sec^2\ x$

**Example 2:** Differientiate ab-initio w.r.t. '$x$'

(i) $\cos 2x$ (ii) $\sin\sqrt{x}$ (iii) $\cot^2 x$

**Solution:** (i) Let $y = \cos 2x$, then $y + \delta y = \cos 2(x+\delta x)$

and $\delta y = \cos(2x + 2\delta x) - \cos 2x$

$$= -2\sin\frac{2x+2\delta x+2x}{2}\sin\frac{2x+2\delta x-2x}{2} = -2\sin(2x+\delta x)\sin\delta x$$

Now $\dfrac{\delta y}{\delta x} = -2\sin(2x+\delta x).\dfrac{\sin\delta x}{\delta x}$

Thus $\dfrac{dy}{dx} = \lim_{\delta x\to 0}\left[-2\sin(2x+\delta x).\dfrac{\sin\delta x}{\delta x}\right]$

$$= -2\lim_{\delta x\to 0}(\sin 2x+\delta x).\lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}$$

$$= (-2\sin 2x).1 = -2\sin 2x\left(\because \lim_{\delta x\to 0}\sin(2x+\delta x) = \sin 2x \text{ and } \lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x} = 1\right)$$

(ii) Let $y = \sin\sqrt{x}$, then $y + \delta y = \sin\sqrt{x+\delta x}$

and $\delta y = \sin\sqrt{x+\delta x} - \sin\sqrt{x}$

$$= 2\cos\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)$$

As $\left(\sqrt{x+\delta x}+\sqrt{x}\right)\left(\sqrt{x+\delta x}-\sqrt{x}\right) = (x+\delta x) - x = \delta x$,

So $$\frac{\delta y}{\delta x} = 2\cos\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right).\frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\delta x}$$

$$= \frac{2\cos\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\left(\sqrt{x+\delta x}+\sqrt{x}\right)\left(\sqrt{x+\delta x}-\sqrt{x}\right)}$$





$$\frac{\cos\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)}{\sqrt{x+\delta x}+\sqrt{x}}.\frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}$$

Thus $$\frac{dy}{dx} = \lim_{\delta x\to 0}\left(\frac{\cos\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}}{\sqrt{x+\delta x}+\sqrt{x}}\right).\lim_{\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\to 0}\left(\frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}\right)$$

$$\frac{dy}{dx} = \left(\frac{\cos\frac{\sqrt{x}+\sqrt{x}}{2}}{\sqrt{x}+\sqrt{x}}\right).1 = \frac{\cos\sqrt{x}}{2\sqrt{x}} \qquad \left(\because \frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\to 0 \text{ when } \delta x\to 0\right)$$

(iii) Let $y = \cot^2 x$, *then*

$y + \delta y = \cot^2(x+\delta x)$

$$\delta y = \cot^2(x+\delta x) - \cot^2 x = [\cot(x+\delta x) + \cot x] \times [\cot(x+\delta x) - \cot x]$$

$$= [\cot(x+\delta x) + \cot x].\left(\frac{\cos(x+\delta x)}{\sin(x+\delta x)} - \frac{\cos x}{\sin x}\right)$$

$$= [\cot(x+\delta x) + \cot x] \times \frac{\sin x\cos(x+\delta x) - \cos x\sin(x+\delta x)}{\sin(x+\delta x)\sin x}$$

$$\frac{\delta y}{\delta x} = \left(\frac{\cot(x+\delta x)+\cot x}{\sin(x+\delta x)\sin x}\right).\frac{-\sin\delta x}{\delta x} \quad \left(\begin{array}{l}\sin x\cos(x+\delta x) - \cos x\sin(x+\delta x) \\ = \sin(x-(x+\delta x)) = \sin(-\delta x) = -\sin\delta x\end{array}\right)$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left(\frac{\cot(x+\delta x)+\cot x}{\sin(x+\delta x)\sin x}.(-1)\frac{\sin\delta x}{\delta x}\right)$$

Thus $$\frac{dy}{dx} = \frac{\cot x+\cot x}{\sin x\sin x}.(-1).\,1 \qquad \left(\begin{array}{l}\because \lim_{\delta x\to 0}\cot(x+\delta x) = \cot x \\ \text{and } \lim_{\delta x\to 0}\sin(x+\delta x) = \sin x\end{array}\right)$$

$$= \frac{-2\cot x}{\sin^2 x}.1 = -2\cot x\,\mathrm{cosec}^2 x$$

**Example 3:** Differentiate $\sin^3 x$ w.r.t. $\cos^2 x$

**Solution:** Let $y = \sin^3 x$ and $u = \cos^2 x$

Now $\frac{dy}{dx} = 3\sin^2 x \cos x$ and $\frac{du}{dx} = 2\cos x(-\sin x)$

Thus $\frac{dy}{du} = \frac{dy}{dx}.\frac{dx}{du} = (3\sin^2 x \cos x).\frac{1}{-2\cos x \sin x}$ $\left(\because \frac{dx}{du} = \frac{1}{\frac{dx}{du}}\right)$

$= -\frac{3}{2}\sin x.$

## 2.9 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

Here we want to prove that

1. $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}},$ $x \in (-1,1)$ or $-1 < x < 1$
2. $\frac{d}{dx}[\cos^{-1} x] = -\frac{1}{\sqrt{1-x^2}},$ $x \in (-1,1)$ or $-1 < x < 1$
3. $\frac{d}{dx}[\tan^{-1} x] = -\frac{1}{1+x^2},$ $x \in R$
4. $\frac{d}{dx}[\operatorname{Cosec}^{-1} x] = -\frac{1}{|x|\sqrt{x^2-1}},$ $x \in [-1,1]', [-1,1]' = (-\infty,-1) \cup (1,\infty)$
5. $\frac{d}{dx}[\operatorname{Sec}^{-1} x] = -\frac{1}{|x|\sqrt{x^2-1}},$ $x \in [-1,1]', [-1,1]' = (-\infty,-1) \cup (1,\infty)$
6. $\frac{d}{dx}[\operatorname{Cot}^{-1} x] = -\frac{1}{1+x^2},$ $x \in R$

**Proof of (1).** Let $y = \operatorname{Sin}^{-1} x$ (i).



2. Differentiation eLearn.Punjab

Then $x = \operatorname{Sin} y$ or $x = \sin y$ for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (ii)

Differentiating both sides of (ii) w.r.t. '$x$', we get

$1 = \frac{d}{dx}(\sin y) = \frac{d}{dx}(\sin y)\frac{dy}{dx} = \cos y \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$ for $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$= \frac{1}{\sqrt{1-\sin^2 y}}$ $\left[\because \cos y \text{ is positive for } y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$

Thus $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$

**Proof of (2).** Let $y = \operatorname{Cos}^{-1} x$ (i)

Then $x = \operatorname{Cos} y$ or $x = \cos y$ for $y \in [0, \pi]$ (ii)

Differentiating both sides of (ii) w.r.t. '$x$', gives

$1 = \frac{d}{dx}(\cos y) = \frac{d}{dx}(\cos y)\frac{dy}{dx} = -\sin y \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y}$ for $y \in (0, \pi)$

$= -\frac{1}{\sqrt{1-\cos^2 y}}$ $[\because \sin y \text{ is positive for } y \in (0,\pi)]$

Thus $\frac{d}{dx}(\operatorname{Cos}^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$

**Proof of (3).** Let $y = \operatorname{Tan}^{-1} x$ (i).

Then x=Tan y or $x = \tan y$ for $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (ii)

Differentiating both sides of (ii) w.r.t. '$x$', we have

$$1=\frac{d}{dx}(\tan y)=\frac{d}{dx}(\tan y)\frac{dy}{dx}=\sec^2 y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{\sec^2 y} \quad for \quad y\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2} \quad for \quad x\in R$$

Thus $\frac{d}{dx}\left[Tan^{-1}x\right]=\frac{1}{1+x^2} \quad for \quad x\in R$

**Proof of (4).** Let $y=Co\sec^{-1}x$ (i)

Then $x=Co\sec y \; or \; x=cosec\; y$ for $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$ (ii)

$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$ is also written as $\left[-\frac{\pi}{2}0\right]\cup\left[0,\frac{\pi}{2}\right]$

Differentiating both sides of (ii) w.r.t. '$x$', we get

$$1=\frac{d}{dx}(cosec\, y)=\frac{d}{dx}(cosec\, y)\frac{dy}{dx}$$

$$=(-cosec\, y\cot y)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{1}{cosec\, y\cot y} \quad for \quad y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$$

When $y\in\left(0,\frac{\pi}{2}\right)$, $cosec\, y$ and $\cot y$ are positive.

As $cosec\, y=x$, so $x$ is positive in this case

and $\cot y=\sqrt{co\sec^2 y-1}=\sqrt{x^2-1}$ for all $x>1$

Thus $\frac{d}{dx}\left(Co\sec^{-1}x\right)=\frac{-1}{x\sqrt{x^2-1}} \quad for \quad x>1$





When $y\in\left(-\frac{\pi}{2},0\right)$, $cosec\, y$ and $\cot y$ are negative

As $cosec\, y=x$, so $x$ is negative in this case

and $\cot y=-\sqrt{cosec^2 y-1}=-\sqrt{x^2-1}$ when $x<-1$

Thus $\frac{d}{dx}\left[Co\sec^{-1}x\right]=\frac{-1}{x\left(-\sqrt{x^2-1}\right)} \quad (x<-1)$

$$=\frac{-1}{(-x)\sqrt{x^2-1}} \quad (x<-1)$$

$$\frac{d}{dx}\left[cosec^{-1}x\right]=-\frac{1}{|x|\sqrt{x^2-1}} \quad for \quad x\in[-1,1]'$$

**Proof of (5).** is left as an exercise

**Proof of (6).** is similar to that of (4)

**Example 1:** Find $\frac{dy}{dx}$ if $y=x\,Sin^{-1}\left(\frac{x}{a}\right)+\sqrt{a^2+x^2}$

**Solution:** Given that $y=x\,Sin^{-1}\left(\frac{x}{a}\right)+\sqrt{a^2+x^2}$

Differentiating w.r.t. $x$, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[x\,Sin^{-1}\frac{x}{a}+\sqrt{a^2+x^2}\right]=\frac{d}{dx}\left[x\,Sin^{-1}\frac{x}{a}\right]+\frac{d}{dx}\left(a^2+x^2\right)^{1/2}$$

$$=1\,.\,Sin^{-1}\frac{x}{a}+x.\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}.\frac{d}{dx}\left(\frac{x}{a}\right)+\frac{1}{2}.\left(a^2+x^2\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(a^2+x^2\right)$$

$$Sin^{-1}\frac{x}{a}+x\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}.\frac{1}{a}+\frac{1}{2\sqrt{a^2-x^2}}.(-2x)$$

$$Sin^{-1}\frac{x}{a}+x\frac{a}{\sqrt{a^2-x^2}}.\frac{1}{a}-\frac{1}{\sqrt{a^2-x^2}}=Sin^{-1}\frac{x}{a}$$

**Example 2:** If $y=tan\left(2\,Tan^{-1}\frac{x}{2}\right)$, show that $\frac{dy}{dx}=\frac{4(1+y^2)}{4+x^2}$

**Solution:** Let $u=2\,Tan^{-1}\frac{x}{2}$, then

$$y = tan\,u \Rightarrow \frac{dy}{du} = sec^2 u = 1+tan^2 u = 1+y^2$$

and $$\frac{du}{dx} = \frac{d}{dx}\left(2\,Tan^{-1}\frac{x}{2}\right)=2.\frac{1}{1+\left(\frac{x}{2}\right)^2}.\frac{d}{dx}\left(\frac{x}{2}\right)=\frac{2}{1+\frac{x^2}{4}}.\frac{1}{2}=\frac{4}{4+x^2}$$

Thus $$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}=(1+y^2).\frac{4}{4+x^2}=\frac{4(1+y^2)}{4+x^2}$$

## EXERCISE 2.5

**1.** Differentiate the following trigonometric functions from the first principle,

(i) $sin\ x$ (ii) $tan\,3x$ (iii) $sin\,2x+cos\,2x$ (iv) $cos\,x^2$

(v) $tan^2 x$ (vi) $\sqrt{tan\,x}$ (vii) $cos\sqrt{x}$

**2.** Differentiate the following w.r.t. the variable involved

(i) $x^2 sec\,4x$ (ii) $tan^3\theta\,sec^2\theta$

(iii) $(sin\,2\theta-cos\,3\theta)^2$ (iv) $cos\sqrt{x}+\sqrt{sin\,x}$





**3.** Find $\frac{dy}{dx}$ if

(i) $y=x\,cos\,y$ (ii) $x=y\,sin\,y$

**4.** Find the derivative w.r.t. $x$

(i) $cos\sqrt{\frac{1+x}{1+2x}}$ (ii) $sin\sqrt{\frac{1+2x}{1+x}}$

**5.** Differentiate

(i) $sin\,x$ w.r.t. $cot\,x$ (ii) $sin^2 x$ w.r.t. $cos^4 x$

**6.** If $\tan y(1+\tan x)=1-\tan x$, show that $\frac{dy}{dx}=-1$

**7.** If $y=\sqrt{tan\,x+\sqrt{tan\,x+\sqrt{tan\,x}+\ldots\infty}}$, prove that $(2y-1)\frac{dy}{dx}=sec^2 x$.

**8.** If $x=a\,cos^3\theta,\ y=b\,sin^3\theta$, show that $a\frac{dy}{dx}+b\,tan\,\theta=0$

**9.** Find $\frac{dy}{dx}$ if $x=a(cos\,t+sin\,t),\ y=a(sin\,t-t\,cos\,t)$

**10.** Differentiate w.r.t. $x$

(i) $Cos^{-1}\frac{x}{a}$ (ii) $Cot^{-1}\frac{x}{a}$ (iii) $\frac{1}{a}Sin^{-1}\frac{a}{x}$

(iv) $Sin^{-1}\sqrt{1-x^2}$ (v) $Sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$ (vi) $Cot^{-1}\left(\frac{2x}{1-x^2}\right)$

(vii) $Cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

**11.** $\frac{dy}{dx}=\frac{y}{x}$ if $\frac{y}{x}=Tan^{-1}\frac{x}{y}$

**12.** If $y=tan(p\,Tan^{-1}x)$, show that $(1+x^2)y_1-p(1+y^2)=0$

## 2.10 DERIVATIVE OF EXPONENTIAL FUNCTIONS:

A function *f* defined by

$f(x) = a^x$

$a > 0, a \neq 1$ *and* $x$ is any real number.

is called an exponential function

If $a = e$, then $y = a^x$ becomes $y = e^x$. $e^x$ is called the natural exponential function.

Now we find derivatives of $e^x$ and $a^x$ from the first principle:

1. Let $y = e^x$ then

$$y + \delta y = e^{x+\delta x} \text{ and } \delta y = y + \delta y - y = e^{x+\delta x} - e^x = e^x . e^{\delta x} - e^x$$

That is, $\delta y = e^x\left(e^{\delta x} - 1\right)$ *and* $\dfrac{\delta y}{\delta x} = e^x . \left(\dfrac{e^{\delta x} - 1}{\delta x}\right)$

Thus $\lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x} = \lim\limits_{\delta x \to 0} e^x\left(\dfrac{e^{\delta x} - 1}{\delta x}\right) = e^x . \lim\limits_{\delta x \to 0}\left(\dfrac{e^{\delta x} - 1}{\delta x}\right)$

$\left(\because \lim\limits_{\delta x \to 0} e^x = e^x\right)$

$$\frac{dy}{dx} = e^x . 1 \left(\text{Using } \lim_{h \to 0} \frac{e^h - 1}{h} = 1\right)$$

or $\dfrac{d}{dx}\left(e^x\right) = e^x$

2. Let $y = a^x$, then

$$y + \delta y = a^{x+\delta x} \text{ and } \delta y = a^{x+\delta x} - a^x = a^x . a^{\delta x} - a^x = a^x\left(a^{\delta x} - 1\right)$$

Dividing both sides by $\delta x$, we have

$$\frac{\delta y}{\delta x} = a^x\left(\frac{a^{\delta x} - 1}{\delta x}\right)$$

Thus $\dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} a^x\left(\dfrac{a^{\delta x} - 1}{\delta x}\right) = a^x . \lim\limits_{\delta x \to 0}\left(\dfrac{a^{\delta x} - 1}{\delta x}\right)\left(\because \lim\limits_{\delta x \to 0} a^x = a^x\right)$





$$= a^x.(\ln a)\left(\text{Using } \lim_{h \to 0} \frac{a^h - 1}{h} = \log^a{}_e = \ln a\right)$$

*or* $\dfrac{d}{dx}\left(a^x\right) = a^x.(\ln a)$

**Example 1:** Find $\dfrac{dy}{dx}$ if: (i) $y = e^{x^2+1}$ (ii) $y = a^{\sqrt{x}}$

**Solution:** (i) Let $u = x^2 + 1$, then

$$y = e^u \quad ....(A) \text{ and } \frac{du}{dx} = \frac{d}{dx}\left(x^2 + 1\right) = 2x$$

Differentiating both sides of $(A)$ w.r.t. '$x$', we have

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(e^u\right) = \frac{d}{du}\left(e^u\right).\frac{du}{dx} \quad \text{(Using the chain rule)}$$

$$= e^u.\frac{du}{dx} \quad \left(\text{Using } \frac{d}{dx}\left(e^x\right) = e^x\right)$$

Thus $\dfrac{dy}{dx} = e^{x^2+1}.(2x) \quad \left(\because u = x^2 + 1 \text{ and } \dfrac{du}{dx} = 2x\right)$

(ii) Let $u = \sqrt{x}$ Then $y = a^u$ $(A)$

*and* $\dfrac{du}{dx} = \dfrac{d}{dx}\left(x^{1/2}\right) = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$

Differentiating both sides of $(A)$ w.r.t. '$x$', gives

$$\frac{dy}{dx} = \frac{d}{dx}\left(a^u\right) = \frac{d}{du}\left(a^u\right)\frac{du}{dx} \quad \left(\because \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}\right)$$

$$= \left(a^u \ln a\right)\frac{du}{dx} \quad \left(\text{Using } \frac{d}{dx}\left(a^x\right) = a^x \ln a\right)$$

Thus $\dfrac{d}{dx}\left(a^{\sqrt{x}}\right) = \left(a^{\sqrt{x}} \ln a\right).\dfrac{1}{2\sqrt{x}} \quad \left(\because u = \sqrt{x} \text{ and } \dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}\right)$

$$= \frac{\ln a}{2} . a^{\sqrt{x}} . \frac{1}{\sqrt{x}}$$

**Example 2:** Differentiate $y = a^x$ w.r.t. $x$.

**Solution:** Here $y = a^x$

$$= e^{x \ln a}$$

Differentiating w.r.t. '$x$', we have

$$\frac{dy}{dx} = e^{x \ln a} , \frac{d}{dx}(x \ln a)$$

$$= a^x .(\ln a) \quad (\because e^{x \ln a} = a^x)$$

$$= a^x .(\ln a) \quad (\because e^{x \ln a} = a^x)$$

## 2.11 DERIVATIVE OF THE LOGARITHMIC FUNCTION

Logarithmic Function:

If $a > 0$ $a \neq 1$ and $x = a$ , then the function defind by

$$y = log_a^{\ x} \qquad (x > 0)$$

is called the logarithm of $x$ to the base a.

The logarithmic functions $\log_e^{\ x}$ and $\log^x_{10}$ are called natural and common logarithms respectively, $y = \log_e^{\ x}$ is written as $y = \ln x$.

We first find $\frac{d}{dx}(\ln x)$.

Let $y = \ln x$ Then

$$y + \delta y = \ln(x + \delta x) \quad \text{and}$$

$$\delta y = \ln(x + \delta x) - \ln x = \left(\frac{x + \delta x}{x}\right) = \ln\left(1 + \frac{\delta x}{x}\right)$$





Now $\frac{\delta y}{\delta x} = \frac{1}{\delta x} \ln\left(1 + \frac{\delta x}{x}\right)$

$$= \frac{1}{x} . \frac{x}{\delta x} \ln\left(1 + \frac{\delta x}{x}\right) = \frac{1}{x} \ln\left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

Thus $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left(\frac{1}{x} \ln\left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right) = \frac{1}{x} \lim_{\delta x \to 0} \left[\ln\left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right]$

$$\frac{dy}{dx} = \frac{1}{x} . \ln\left[\lim_{\frac{\delta x}{x} \to 0} \left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right]$$

$\left(\because \frac{\delta x}{x} \to 0 \text{ when } \delta x \to 0\right)$

$$= \frac{1}{x} \ln e \qquad \left[\because \lim_{z \to 0} (1 + z)^{\frac{1}{z}} = e\right]$$

$$= \frac{1}{x} . 1 = \frac{1}{x} \qquad (\because log_e^{\ e} = 1)$$

Now we find derivative of the general logarithmic function.

Let $y = log_a^{\ x}$ then

$$y + \delta y = log_a(x + \delta x) \quad \text{and}$$

$$\delta y = \log_a(x + \delta x) - log_a^{\ x} = log\left(\frac{x + \delta x}{x}\right) = log_a\left(1 + \frac{\delta x}{x}\right)$$

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x} log_a\left(1 + \frac{\delta x}{x}\right) = \frac{1}{x} . \frac{x}{\delta x} log_a\left(1 + \frac{\delta x}{x}\right)$$

$$= \frac{1}{x} log_a\left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

Thus $\frac{dy}{dx} = \lim_{\delta x \to 0} \left[\frac{1}{x} log_a\left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right] = \frac{1}{x} \lim_{\delta x \to 0} \left[log_a\left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right]$

$$= \frac{1}{x} log_a\left[\lim_{\frac{\delta x}{x} \to 0} \left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right]$$

$$= \frac{1}{x} \log_a^x \qquad \left( \because \lim_{z \to 0} (1+z)^{\frac{1}{z}} = e \right)$$

$$\text{or } \frac{d}{dx}\left[\log_a^x\right] = \frac{1}{x} \cdot \frac{1}{\ln a} \qquad \left( \because \log_a^e = \frac{1}{\log_e^a} = \frac{1}{\ln a} \right)$$

**Example 1:** Find $\frac{dy}{dx}$ if $y = \log_{10}\left(ax^2 + bx + c\right)$

**Solution:** Let $u = ax^2 + bx + c$ Then

$$y = \log_{10}^u \Rightarrow \frac{dy}{du} = \frac{1}{u} \frac{1}{\ln 10}$$

$$\text{and } \frac{du}{dx} = \frac{d}{dx}(ax + bx + c) = a(2x) + b(1) = 2ax + b$$

$$\text{Thus } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left( \frac{1}{u} \cdot \frac{1}{\ln 10} \right) \frac{du}{dx}$$

$$= \frac{1}{\left(ax^2 + bx + c\right) \ln 10} (2ax + b)$$

$$\text{or } \frac{d}{dx}\left[\log_{10}\left(ax^2 + bx + c\right)\right] = \frac{2ax + b}{(ax^2 + bx + c) \ln 10}$$

**Example 2:** Differentiate $\ln\left(x^2 + 2x\right)$ w.r.t. '$x$'.

**Solution:** Let $y = \ln\left(x^2 + 2x\right)$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left[\ln\left(x^2 + 2x\right)\right] = \frac{1}{\left(x^2 + 2x\right)} \cdot \frac{d}{dx}\left(x^2 + 2x\right) \qquad \text{(Using chain rule)}$$

$$= \frac{1}{x^2 + 2x} \cdot (2x + 2) = \frac{2(x+1)}{x^2 + 2x}$$

$$\text{Thus } \frac{d}{dx}\left[\ln\left(x^2 + 2x\right)\right] = \frac{2(x+1)}{x^2 + 2x}$$





## 2.12 LOGARITHMIC DIFFERENTIATION

Algebraic expressions consisting of product, quotient and powers can be often simplified before differentiation by taking logarithm.

**Example 1:** Differentiate $y = e^{f(x)}$ w.r.t.'$x$'.

**Solution:** Here $y = e^{f(x)}$ (i)

Taking logarithm of both sides of (i), we have

$$\ln y = f(x) . \ln e$$

$$= f(x) \qquad (\because \ln e = 1)$$

Differentiating w.r.t $x$, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = f'(x)$$

$$\text{So } \frac{dy}{dx} = y \times f'(x) = e^{f(x)} \times f'(x)$$

$$\text{or } \frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} \times f'(x)$$

**Example 2:** Find derivative of $\frac{x\sqrt{x^2+3}}{x^2+1}$

**Solution:** Let $y = \frac{x\sqrt{x^2+3}}{\left(x^2+1\right)}$ ......(i)

Taking logarithm of both sides, we have

$$\ln y = \ln\left( \frac{x\sqrt{x^2+3}}{x^2+1} \right) = \ln\left(x\sqrt{x^2+3}\right) - \ln\left(x^2+1\right)$$

$$\text{or } \ln y = \ln x + \frac{1}{2}\ln\left(x^2+3\right) - \ln\left(x^2+1\right) \qquad \text{......(ii)}$$

Differentiating both sides of (ii) w.r.t '$x$',

$$\frac{d}{dx}[In\ y]=\frac{d}{dx}\left[In\ x+\frac{1}{2}In\left(x^2+3\right)-In\left(x^2+1\right)\right]$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{x}+\frac{1}{2}\cdot\frac{1}{x^2+3}\times 2x-\frac{1}{x^2+1}\times 2x$$

$$=\frac{1}{x}+\frac{x}{x^2+3}-\frac{2x}{x^2+1}$$

$$=\frac{\left(x^2+3\right)\left(x^2+1\right)+x.x\left(x^2+1\right)-2x.x\left(x^2+3\right)}{x\left(x^2+3\right)\left(x^2+1\right)}$$

$$=\frac{x^4+4x^2+3+x^4+x^2-2x^4-6x^2}{x\left(x^2+3\right)\left(x^2+1\right)}=\frac{3-x^2}{x\left(x^2+3\right)\left(x^2+1\right)}$$

Thus $\frac{dy}{dx}=\frac{y\left(3-x^2\right)}{x\left(x^2+1\right)\left(x^2+1\right)}=\frac{x\sqrt{x^2+3}}{x^2+1}\cdot\frac{3-x^2}{x\left(x^2+3\right)\left(x^2+1\right)}$

$$=\frac{3-x^2}{\sqrt{x^2+3}.\left(x^2+1\right)^2}$$

**Example 3:** Differentiate $(\ln x)^x$ w.r.t. '$x$'.

**Solution:** Let $y=(\ln x)^x$ (i)

Taking logarithm of both sides of (i) , we have

$$\text{In } y=\text{In}\left[(\text{In } x)^x\right]=x\text{ In}(\text{In } x)$$

Differentiating w.r.t $x$,

$$\frac{1}{y}\frac{dy}{dx}=1.\text{In}(\text{In } x)+x.\frac{1}{\text{In x}}.\frac{d}{dx}(\text{In } x)$$

$$=\text{In}(\text{In } x)+x.\frac{1}{\text{In } x}.\frac{1}{x}=\text{In}(\text{In } x)+\frac{1}{\text{In } x}$$

$$\frac{dy}{dx}=y\left[\text{In}(\text{In } x)+\frac{1}{\text{In } x}\right]=(\text{In } x)^x\left[\text{In}(\text{In } x)+\frac{1}{\text{In } x}\right]$$





## 2.13 DERIVATIVE OF HYPERBOLIC FUNCTIONS

The functions defined by:

$$sinh\ x=\frac{e^x-e^{-x}}{2},x\in R\ ;\ cosh\ x=\frac{e^x+e^{-x}}{2};x\in R$$

$$tanh\ x=\frac{sinh\ x}{cosh\ x}=\frac{e^x-e^{-x}}{e^x+e^{-x}};x\in R$$

are called hyperbolic functions.

The reciprocals of these three functions are defined as:

$$cosech\ \ x=\frac{1}{sinh\ x}=\frac{2}{e^x-e^{-x}},x\in R-\{0\};$$

$$sech\ x=\frac{1}{cosh\ x}=\frac{2}{e^x+e^{-x}},x\in R$$

$$coth\ =\frac{1}{tanh\ x}=\frac{e^x+e^{-x}}{e^x-e^{-x}},\ x\in R-\{0\}$$

Derivatives of sin *h x*, cos *h x* and tan *h x* are found as explained below:

$$\frac{d}{dx}(sinh\ x)=\frac{d}{dx}\left[\frac{1}{2}\left(e^x-e^{-x}\right)\right]=\frac{1}{2}\left[e^x-e^{-x}(-1)\right]=\frac{1}{2}\left(e^x+e^{-x}\right)=cosh\ x$$

$$\frac{d}{dx}(cosh\ x)=\frac{d}{dx}\left[\frac{1}{2}\left(e^x+e^{-x}\right)\right]=\frac{1}{2}\left[e^x+e^{-x}.(-1)\right]=\frac{1}{2}\left(e^x-e^{-x}\right)=sinh\ x$$

$$\frac{d}{dx}[tanh\ x]=\frac{d}{dx}\left[\frac{e^x-e^{-x}}{e^x+e^{-x}}\right]=\frac{\left(e^x+e^{-x}\right)\left(e^x+e^{-x}\right)-\left(e^x-e^{-x}\right)\left(e^x-e^{-x}\right)}{\left(e^x+e^{-x}\right)^2}$$

$$=\frac{e^{2x}+e^{-2x}+2-\left(e^{2x}+e^{-2x}-2\right)}{\left(e^x+e^{-x}\right)^2}=\frac{4}{\left(e^x+e^{-x}\right)^2}$$

$$=\left(\frac{2}{e^x+e^{-x}}\right)^2=sech^2\ x.$$

The following results can easily be proved.

$$\frac{d}{dx}\left(cos\,eh\;x\right)=-coth\;\;x\,cos\,ech\;x\,;\;\frac{d}{dx}\left(sec\,h\;x\right)=\;-tanh\;x\;sec\,h\;x$$

$$\frac{d}{dx}\left(coth\;x\right)=\;-cos\,ech^{2}\;x.$$

**Example 1:** Find $\frac{dy}{dx}$ if $y=sinh\;2x$

**Solution:** Let $u=2x$, then

$$y=sinh\;u \qquad \Rightarrow\frac{dy}{du}=cosh\;\;u$$

and $\frac{du}{dx}=\frac{d}{dx}\left(2x\right)=2.$

Thus $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}=cosh\;u\;.\frac{du}{dx}=\left[cosh\left(2x\right)\right].2=2\;cosh\;2x$

or $\frac{d}{dx}\left[sinh\;2x\right]\;=2\;cosh\;2x\;.$

**Example 2:** Find $\frac{dy}{dx}$ *if* $y=tanh\left(x^{2}\right)$

**Solution:** Let $u\;=x^{2}$ , *then* $y=tanh\;\;u\;\Rightarrow\frac{dy}{du}=sec\,h^{2}\;u$

and $\frac{du}{dx}=\frac{d}{dx}\left(x\;\;\right)=2x$

Thus $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}=sec\,h^{2}\;u\;.\frac{du}{dx}=\left[sec\,h^{2}\;\left(x^{2}\right)\right]\times\;2x$

or $\frac{d}{dx}\left[tanh\;x^{2}\right]=\;2x\;sec\,h^{2}\;x^{2}$





## 2.14 DERIVATIVES OF THE INVERSE HYPERBOLIC FUNCTIONS:

The inverse hyperbolic functions are defined by:

1. $y=sinh^{'-1}\;x$ if and if $x=sinh\;y$ ; $x,y\in R$
2. $y=cosh^{-1}\,x$ if and only if $x=cosh\;y$ ; $x\in\left[1,\infty\right),\,y\in\left[0,\infty\right]\left.\right]$
3. $y=tanh^{-1}\,x$ if and only if $x=tanh\;y$ ; $x\in\left(-1,1\right),y\in R$
4. $y=coth^{-1}\,x$ if and only if $x=coth\;y$ ; $x\in\left[-1,1\right]'\,,y\in R-\left\{0\right\}$
5. $y=sec\,h^{-1}\;x$ if and only if x= $sec\,h\;y$ ; $x\in\left(0,1`\right],\,y\in\left[0,\infty\right)$
6. $y=cos\,ech^{-1}x$ if and only if $x=cos\,ech\;y$ ; $x\in R-\left\{0\right\},\,y\in R-\left\{0\right\}$

The following two equations can easily be derived:

(i) $sinh^{-1}\;x\;=\;\text{In}\left(x+\sqrt{x^{2}+1}\right)$ (ii) $cosh^{-1}\;x\;=\;\text{In}\left(x+\sqrt{x^{2}-1}\right)$

Proof of (i).

Let $y=sinh^{-1}\;x$ *for* $x,y\in R$*, then*

$$x=sinh\;y\Rightarrow x=\frac{e^{y}-e^{-y}}{2}$$

$$\Rightarrow 2xe^{y}=e^{2y}-1$$

or $e^{2y}-2xe^{y}-1=0$

Solving the above equation for $e^{y}$ , we have

$$e^{y}=\frac{2x\pm\sqrt{4x^{2}+4}}{2}$$

$$=\frac{2x\pm2\sqrt{x^{2}+1}}{2}=x\pm\sqrt{x^{2}+1}$$

As $e^{y}$ is positive for $y\in R$, so we discard

$$x-\sqrt{x^{2}+1}$$

Thus $e^{y}=x+\sqrt{x^{2}+1}\;\Rightarrow y=In\left(x+\sqrt{x^{2}+1}\right)$

$\Rightarrow sinh^{-1} x = In\left(x+\sqrt{x^2+1}\right)$

Proof of (ii)

Let $y = cosh^{-1} x$ for $x \in [1, \infty), y \in [0, \in)$, then

$$x = cosh\, y \Rightarrow x = \frac{e^y + e^{-y}}{2} \Rightarrow e^{2y} - 2e^y + 1 = 0 \quad ......(I)$$

Solivng (I) gives, $e^y = \dfrac{2x \pm \sqrt{4x^2-4}}{2} = \dfrac{2x \pm 2\sqrt{x^2-1}}{2} = x \pm \sqrt{x^2-1}$.

$e^y = x - \sqrt{x^2-1}$ can be written as y =In $\left(x-\sqrt{x^2-1}\right)$

If $x = 1$, then y = ln $\left(1-\sqrt{1-1}\right) = \ln(1) = 0$ but

ln $\left(x-\sqrt{x^2-1}\right)$ is negative for all x > 1, that is

for each $x \in (1,\infty), y \notin (0,\infty)$,so we discard this value of $e^y$

Thus $e^y = x + \sqrt{x^2+1}$ which give $y = \text{In}\left(x+\sqrt{x^2-1}\right)$, that is

$cosh^{-1} x = In\left(x+\sqrt{x^2-1}\right)$.

**Derivative of $\sinh^{-1} x$:**

Let $y = sinh^{-1} x$ ; $x, y \in R$

Then $x = sinh\, y$

$$\frac{dx}{dy} = cosh\, y \qquad \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y} \qquad \left(\because \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}\right)$$

or $$\frac{dy}{dx} = \frac{1}{cosh\, y} = \frac{1}{\sqrt{1+sinh^2 y}} \qquad (\because cosh\, y > 0)$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(sinh^{-1} x\right) = \frac{1}{\sqrt{1+x^2}} \qquad (x \in R)$$





**Derivative of $\cosh^{-1} x$:**

Let $y = cosh^{-1} x$ ; $x \in [1\infty), y \in [0,\infty)$

Then $x = cosh\, y$

and $$\frac{dx}{dy} = sinh\, y \qquad \Rightarrow \frac{dy}{dx} = \frac{1}{sinh\, y} \qquad \left(\because \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}\right)$$

or $$\frac{dy}{dx} = \frac{1}{sinh\, y} = \frac{1}{\sqrt{cosh^2 y - 1}} \qquad (\because sinh\, y > 0, as\ y > 0)$$

Thus $$\frac{dy}{dx} = \frac{d}{dx}\left(cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}} \qquad (x > 1)$$

As $cosh^{-1} x = \text{In}\left(x+\sqrt{x^2-1}\right)$, so

$$\frac{d}{dx}\left[cosh^{-1} x\right] = \frac{1}{x+\sqrt{x^2-1}}\left(1+\frac{2x}{2\sqrt{x^2-1}}\right) = \frac{1}{x+\sqrt{x^2-1}} \cdot \frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}}$$

**Derivative of $\tanh^{-1} x$ :**

Let $y = \tanh^{-1} x$ ; $x \in (-1, 1), y \in R$

Then $$x = tanh\, y \quad and \quad \frac{dx}{dy} = sec\,h^2 \Rightarrow \frac{dy}{dx} = \frac{1}{sec\,h^2\, y} \qquad \left(\because \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}\right)$$

$$\frac{dy}{dx} = \frac{1}{1-tanh^2 y} = \frac{1}{1-x^2} \qquad (\because sec\,h^2\, y = 1 - tanh^2\, y)$$

Thus $$\frac{d}{dx}\left(tanh^{-1} x\right) = \frac{1}{1-x^2} \qquad ; \quad -1<x<1 \text{ or } |x| < 1$$

The following differentiation formulae can be easily proved.

$$\frac{d}{dx}\left(coth^{-1} x\right) = \frac{1}{1-x^2} \quad or \quad -\frac{1}{x^2-1}; \quad |x| > 1$$

$$\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right)=-\frac{1}{x\sqrt{1-x^2}}; \qquad 0<x<1$$

$$\frac{d}{dx}\left(\operatorname{cosech}^{-1} x\right)=-\frac{1}{x\sqrt{1+x^2}}\ ; x>0$$

$$\text{or } \frac{d}{dx}\left(\operatorname{cosech}^{-1} x\right)=-\frac{1}{|x|\sqrt{1+x^2}}\ ; \qquad x\in R-\{0\}$$

**Example 1:** Find $\frac{dy}{dx}$ *if* $y=\sinh^{-1}(ax+b)$

**Solution:** Let $u=ax+b$, then

$$y = \sinh^{-1} u \qquad \Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1+u^2}}$$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{\sqrt{1+u^2}}\cdot\frac{du}{dx}$$

$$\text{Thus } \frac{d}{dx}\left[\sinh^{-1}(ax+b)\right)=\frac{1}{\sqrt{1+(ax+b)^2}}\cdot a \qquad \left(\because \frac{du}{dx}=\frac{d}{dx}(ax+b)=a\right)$$

**Example 2:** Find $\frac{dy}{dx}$ if $y=\cosh^{-1}(\sec x)$ $\quad 0\le x\le \pi/2$

**Solution:** Let $u=\sec x$, *then*

$$y=\cosh^{-1} u \quad \Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{u^2-1}}$$

$$\text{and } \frac{du}{dx}=\frac{d}{dx}(\sec x) = \sec x \ \tan x$$

$$\text{Thus } \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{\sqrt{u^2-1}}\cdot\frac{du}{dx}$$

$$=\frac{1}{\sqrt{\sec\ x}}(\sec x \tan x)=\frac{1}{\tan x}(\sec x\ \tan x)=\sec\ x$$

$$\text{or } \frac{d}{dx}\left[\cosh^{-1}(\sec x)\right]= \sec x$$





## EXERCISE 2.6

**1.** Find $f'(x)$ if

(i) $f(x) = e^{\sqrt{x}-1}$ (ii) $f(x)=x^3 e^{\frac{1}{x}}\ (x\neq 0)$ (iii) $f(x)=e^x(1+\ln x)$

(iv) $f(x)=\frac{e^x}{e^{-x}+1}$ (v) $\ln\left(e^x+e^{-x}\right)$ (vi) $fx=\frac{e^{ax}-e^{-ax}}{e^{ax}+e^{-ax}}$

(vii) $f(x)=\sqrt{\ln\left(e^{2x}+e^{-2x}\right)}$ (viii) $f(x)=\ln\left(\sqrt{e^{2x}+e^{-2x}}\right)$

**2.** Find $\frac{dy}{dx}$ if

(i) $y=x^2\ln\sqrt{x}$ (ii) $y=x\sqrt{\ln\ x}$ (iii) $y=\frac{x}{\ln x}$

(iv) $y=x^2\ln\frac{1}{x}$ (v) $y=\ln\sqrt{\frac{x^2-1}{x^2+1}}$ (vi) $y=\ln\left(x+\sqrt{x^2+1}\right)$

(vii) $y=\ln\left(9-x^2\right)$ (viii) $y=e^{-2x}\sin 2x$ (ix) $y=e^{-x}\left(x^3+2x^2+1\right)$

(x) $y=x\,e^{\sin x}$ (xi) $y=5e^{3x-4}$ (xii) $y=(x+1)^x$

(xiii) $y=(\ln x)^{\ln x}$ (xiv) $y=\frac{\sqrt{x^2-1}\,(x+1)}{\left(x^3+1\right)^{3/2}}$

**3.** Find $\frac{dy}{dx}$ if

(i) $y=\cosh 2x$ (ii) $y= \sinh 3x$

(iii) $y=\tanh^{-1}(\sin x)\quad -\frac{\pi}{2}<x<\frac{\pi}{2}$ (iv) $y=\sinh^{-1}\left(x^3\right)$

(v) $y=\ln(\tanh x)$ (vi) $y=\sinh^{-1}\left(\frac{x}{2}\right)$

## 2.15 SUCCESSIVE DIFFERENTIATION (OR HIGHER DERIVATIVES):

Sometimes it is useful to find the differential coefficient of a derived function. If we denote $f'$ as the first derivative of $f$, then $(f')'$ is the derivative of $f'$ and is called the second derivative of $f$. For convenience we write it as $f''$.

Similarly $(f'')'$. the derivative of $f''$, is called the third derivative of $f$ and is written as $f'''$.

In general, for $n \geq 4$, the nth derivative of $f$ is written as $f^{(n)}$.

Here we state different notations used for derivatives of higher orders..

| 1st derivative | 2nd derivative | 3rd derivative | *n*th derivative |
|---|---|---|---|
| $y'$ | $y''$ | $y'''$ | $y^{(n)}$ |
| $\frac{dy}{dx}$ | $\frac{d^2y}{dx^2}$ | $\frac{d^3y}{dx^3}$ | $\frac{d^ny}{dx^n}$ |
| $y_1$ | $y_2$ | $y_3$ | $y_n$ |
| $D_y$ | $D^2_y$ | $D^3_y$ | $D^n_y$ |
| $\frac{df}{dx}$ | $\frac{d^2f}{dx^2}$ | $\frac{d^3f}{dx^3}$ | $\frac{d^nf}{dx^n}$ |

**Example 1:** Find higher derivatives of the polynomial

$$f(x)=\frac{1}{12}x^4-\frac{1}{6}x^3+\frac{1}{4}x^2+2x+7$$

**Solution:**

$$f'(x)=\frac{1}{12}\left(4x^3\right)-\frac{1}{6}\left(3x^2\right)+\frac{1}{4}(2x)+2+0=\frac{1}{3}x^3-\frac{1}{2}x^2+\frac{1}{2}x+2$$

$$f''(x)=\frac{1}{3}\left(3x^2\right)-\frac{1}{2}(2x)+\frac{1}{2}(1)+0=x^2-x+\frac{1}{2}$$

$$f'''(x)=2x-1$$

$$f^{iv}(x)=2$$

All other higher derivatives are zero.





**Example 2:** Find $\frac{d^3y}{dx^3}$ if $y=ln\left(x+\sqrt{x^2+a^2}\right)$

**Solution:** Give that $y=ln\left(x+\sqrt{x^2+a^2}\right)$ (i)

Differentiating both sides of (i) w.r.t. '$x$', we have

$$\frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+a^2}}\frac{d}{dx}\left(x+\sqrt{x^2+a^2}\right)$$

$$=\frac{1}{x+\sqrt{x^2+a^2}}\cdot\left[1+\frac{1\times 2x}{2\sqrt{x^2+a^2}}\right]$$

$$=\frac{1}{x+\sqrt{x^2+a^2}}\times\left(\frac{\sqrt{x^2+a^2}+x}{2\sqrt{x^2+a^2}}\right)$$

That is, $\frac{dy}{dx}=\frac{1}{\sqrt{x^2+a^2}}$ (ii)

Differentiating (ii) w.r.t. '$x$', we have

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\left(x^2+a^2\right)^{-1/2}\right]=-\frac{1}{2}\left(x^2+a^2\right)^{-3/2}\times 2x$$

or $\frac{d^2y}{dx^2}=-\frac{x}{\left(x^2+a^2\right)^{3/2}}$ (iii)

Differentiating (iii) w.r.t. '$x$', we get

$$\frac{d^3y}{dx^3}=-\frac{1.\left(x^2+a^2\right)^{3/2}-x.\frac{3}{2}\left(x^2+a^2\right)^{1/2}.2x}{\left(x^2+a^2\right)^{3/2}\big]}$$

$$=-\frac{\left(x^2+a^2\right)^{1/2}\left[\left(x^2+a^2\right)-3x^2\right]}{\left(x^2+a^2\right)^3}=-\frac{a^2-2x^2}{\left(x^2+a^2\right)^{5/2}}$$

$$\frac{d^3y}{dx^3}=\frac{2x^2-a^2}{\left(x^2+a^2\right)^{5/2}}$$

**Example 3:** Find $\frac{d^2y}{dx^2}$ if $y^3+3ax^2+x^3=0$

**Solution:** Given that $y^3+3ax^2+x^3=0$ (i)

Differentiating both sides of (i) w.r.t. '$x$', gives

$$\frac{d}{dx}\left[y^3+3ax^2+x^3\right]=\frac{d}{dx}(0)\quad=0$$

$$3y^2\frac{dy}{dx}+3a(2x)+3x^2=0\qquad\Rightarrow y^2\frac{dy}{dx}=-\left(2ax+x^2\right)$$

$$\Rightarrow\frac{dy}{dx}=-\frac{2ax+x^2}{y^2}\qquad\text{(ii)}$$

Differentiating both sides of (ii) w.r.t. '$x$', gives

$$\frac{d^2y}{dx^2}=(-1)\frac{d}{dx}\left[\frac{2ax+x^2}{y^2}\right]=-\frac{(2a+2x)y^2-\left(2ax+x^2\right)\left(2y\frac{dy}{dx}\right)}{\left(y^2\right)^2}$$

$$=-\frac{2(a+x)y^2-\left(2ax+x^2\right).2y\times\left(-\frac{2ax+x^2}{y^2}\right)}{y^4}$$

$$=-\frac{2\left[(a+x)y^2+\frac{\left(2ax+x^2\right)\left(2ax+x^2\right)}{y}\right]}{y^4}$$

$$=-\frac{2\left[(a+x)y^3+\left(2ax+x^2\right)^2\right]}{y^4.y}$$

$$=-\frac{2\left[(a+x)\left(-3ax^2-x^3\right)+x^2(2a+x)^2\right]}{y^5}\qquad\left(\because y^3=-3ax^2-x^3\right)$$

$$=-\frac{2x^2\left[-(a+x)(3a+x)+\left(4a^2+x^2+4ax\right)\right]}{y^5}$$

$$=-\frac{2x^2\left[-\left(3a^2+4ax+x^2\right)+4a^2+x^2+4ax\right]}{y^5}$$

$$=-\frac{2x^2\left[a^2\right]}{y^5}\quad=\quad\frac{-2a^2x^2}{y^5}$$





**Example 1:** If $x=a(\theta-\sin\theta)$, $y=a(1+\cos\theta)$. Then

show that $y^2\frac{d^2y}{dx^2}+a=0$

**Solution:** Given that $x=a(\theta+\sin\theta)$ (i)

and $y=a(1+cos\,\theta)$ (ii)

Differentiating (i) and (ii) w.r.t '$\theta$', we get

$$\frac{dx}{d\theta}=a(1+cos\,\theta)\qquad\text{(iii)}$$

$$\text{and}\quad\frac{dy}{d\theta}=a(-\sin\theta)\qquad\text{(iv)}$$

Using $\frac{dy}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ we have

$$=\frac{-a\sin\theta}{a(1+cos\,\theta)}=\frac{-\sin\theta}{1+\cos\theta}$$

That is, $\frac{dy}{dx}=-\frac{\sin\theta}{1+\cos\theta}$ (v)

Differentiating (v) w.r.t. '$x$'

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(-\frac{\sin\theta}{1+\cos\theta}\right)=-\frac{d}{d\theta}\left(\frac{\sin\theta}{1+\cos\theta}\right)\times\frac{d\theta}{dx}$$

$$=-\frac{\cos\theta(1+\cos\theta)-\sin\theta(-\sin\theta)}{(1+\cos\theta)^2}\cdot\frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2}=-\frac{\cos\theta+\cos^2\theta+\sin^2\theta}{(1+\cos\theta)^2}\cdot\frac{d\theta}{dx}$$

$$=-\frac{1+\cos\theta}{(1+\cos\theta)^2}\times\frac{1}{a(1+\cos\theta)}\qquad\left(\because\frac{dx}{d\theta}=a(1+\cos\theta)\right)$$

$$=-\frac{1}{a}\cdot\frac{1}{(1+\cos\theta)^2}=-\frac{1}{a}\cdot\frac{1}{\left(\frac{y}{a}\right)^2}\qquad\left(\because 1+\cos\theta=\frac{y}{a}\right)$$

$$=-\frac{1}{a}\times\frac{a^2}{y^2}=-\frac{a}{y^2}$$

or $y^2\frac{d^2y}{dx^2}=-a \qquad \Rightarrow y^2\frac{d^2y}{dx^2}+a=0$

**Example 5:** Find the first four derivatives of $\cos(ax+b)$.

**Solution:** Let $y=\cos(ax+b)$, then

$$y_1=\frac{d}{dx}[\cos(ax+b)]=-\sin(ax+b).\frac{d}{dx}(ax+b)$$

$$=-\sin(ax+b)\times(a+0)=-a\sin(ax+b)$$

$$y_2=-a\frac{d}{dx}[\sin(ax+b)]=(-a)[\cos(ax+b)\times(a+0)]$$

$$=-a^2\cos(ax+b)$$

$$y_3=-a^2\frac{d}{dx}[\cos(ax+b)]=(-a^2)[-\sin(ax+b)\times(a+0)]$$

$$=a^3\sin(ax+b)$$

$$y_4=a^3\frac{d}{dx}[\sin(ax+b)]=a^3\times[\cos(ax+b)]\times a=a^4\cos(ax+b)$$

**Example 6:** If $y=e^{-ax}$, then show that $\frac{d^3y}{dx^3}+a^3y=0$

**Solution:** As $y=e^{-ax}$, so $\frac{dy}{dx}=\frac{d}{dx}(e^{-ax})=e^{-ax}.\frac{d}{dx}(-ax)=e^{-ax}.(-a)$

That is $\frac{dy}{dx}=-ay \qquad (\because e^{-ax}=y)$





Now $\frac{dy}{dx}\left[\frac{dy}{dx}\right]=\frac{d}{dx}[-ay]\Rightarrow\frac{d^2y}{dx^2}=-a\frac{dy}{dx}=(-a)(-ay)\quad\left(\because\frac{dy}{dx}=-ay\right)$

or $\frac{d^2y}{dx^2}=a^2y \qquad$ (i)

Differentiating (i) w.r.t. '$x$' we get

$$\frac{d}{dx}\left[\frac{d^2y}{dx^2}\right]=\frac{d}{dx}[a^2y]\Rightarrow\frac{d^3y}{dx^3}=a^2\frac{dy}{dx}=a^2(-ay)=-a^3y$$

Thus $\frac{d^3y}{dx^3}+a^3y=0$

**Example 7:** If $y=Sin^{-1}\frac{x}{a}$, then show that $y_2=x(a^2-x^2)$

**Solution:** $y=\sin^{-1}\frac{x}{a}$, so

$$y_1=\frac{dy}{dx}=\frac{d}{dx}\left[Sin^{-1}\frac{x}{a}\right]=\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\times\frac{d}{dx}\left(\frac{x}{a}\right)$$

$$=\frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}}.\frac{1}{a}=\frac{a}{\sqrt{a^2-x^2}}.\frac{1}{a}=(a^2-x^2)^{-1/2}$$

$$y_2=\frac{d}{dx}\left[(a^2-x^2)^{-1/2}\right]=-\frac{1}{2}(a^2-x^2)^{-3/2}\times(-2x)=x(a^2-x^2)^{-3/2}$$

## EXERCISE 2.7

**1.** Find $y_2$ if

(i) $y = 2x^5 - 3x^4 + 4x^3 + x - 2$ (ii) $y = (2x+5)^{3/2}$ (iii) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

**2.** Find $y_2$ if

(i) $y = x^2 . e^{-x}$ (ii) $y = \ln\left(\frac{2x+3}{3x+2}\right)$

**3.** Find $y_2$ if

(i) $x^2 + y^2 = a^2$ (ii) $x^3 - y^3 = a$ (iii) $x = a\cos\theta, y = a\sin\theta$

(iv) $x = at^2,\ y = bt^4$ (v) $x^2 + y^2 + 2gx + 2fy + c = 0$

**4.** Find $y_4$ if

(i) $y = \sin 3x$ (ii) $y = \cos^3 x$ (iii) $y = \ln(x^2 - 9)$

**5.** If $x = \text{Sin}\ \theta, y = \text{Sin}\ m\theta$, Show that $(1-x^2)y_2 - xy_1 + m^2 y = 0$

**6.** If $y = e^x \sin x$, show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

**7.** If $y = e^{ax}\sin bx$, show that $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+b^2)y = 0$

**8.** If $y = (\text{Cos}^{-1} x)^2$, prove that $(1-x^2)y_2 - xy_1 - 2 = 0$

**9.** If *y* = *a* cos (*ln x*) + *b* sin (*ln x*), prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$.

## 2.16 SERIES EXPANSIONS OF FUNCTIONS

A series of the form $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + ...... + a_nx^n + .....$ is called a power series expansion of a function $f(x)$ where $a_0, a_1, a_2, ... a_n, ...$ are constants and x is a variable.

We determine the coefficient $a_0, a_1, a_2, ..., a_n, ...$ to specify power series by finding successive derivatives of the power series and evaluating them at $x = 0$. That is,





$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + ...... + a_nx^n + ..... \quad f(0) = a_0$$
$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + ...... + na\ x^{n\ 1} + ..... \quad f'(0) = a_1$$
$$f''(x) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + ... + n(n-1)a_nx^{n-2} + ... \quad f''(0) = 2a_2$$
$$f'''(x) = 6a_3 + 24a_4x + 60a_5x^2 + .... \quad f'''(0) = 6a_3$$
$$f^{(4)}(x) = 24a_4 + 120a_5x ........ \quad f^{(4)}(0) = 24a_4$$

So we have $a_0 = f(0),\ a_1 = f'(0), a_2 = \frac{f''(0)}{2!}, a_3 = \frac{f'''(0)}{3!}, a_4 = \frac{f^{(4)}(0)}{4!}$

Following the above pattern, we can write $a_n = \frac{f^n(0)}{n!}$

Thus substituting these values in the power series, we have

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + .... + \frac{f^n(0)}{n!}x^n + ....$$

This expansion of $f(x)$ is called the **Maclaurin series** expansion.
The above expansion is also named as **Maclaurin's Theorem** and can be stated as:

If $f(x)$ is expanded in ascending powers of $x$ as an infinite series, then

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + .... + \frac{f^n(0)}{n!}x^n + ....$$

Note that a function *f* can be expanded in the Maclaurin series if the function is defined in the interval containing 0 and its derivatives exist at $x = 0$.

The expansion is only valid if it is convergent.

**Example 1:** Expand $f(x) = \frac{1}{1+x}$ in the Maclaurin series.

**Solution:** *f* is defined at $x = 0$ that is, $f(0) = 1$. Now we find successive derivatives of *f* and their values at $x = 0$.

$$f'(x) = (-1)(1+x)^{-2} \text{ and } f'(0) = -1,$$
$$f''(x) = (-1)(-2)(1+x)^{-3} \text{ and } f''(0) = (-1)^2 \underline{|2}$$
$$f'''(x) = (-1)(-2)(-3)(1+x)^{-4} \text{ and } f'''(0) = (-1)^3 \underline{|3}$$

$$f^{(4)}(x) = (-1)(-2)(-3)(-4)(1+x)^{-5} \text{ and } f^{(4)}(0) = (-1)^4 \underline{|4}$$

Following the pattern, we can write $f^{(n)}(0) = (-1)^n \underline{|n}$

Now substituting $f(0)=1,\ f'(0)=-1, f''(0)=(-1)^2 \underline{|2}$.

$f'''(0)=(-1)^3 \underline{|3}, f^{(4)}(0)=(-1)^4 \underline{|4}, \ldots f^{(n)}(0)=(-1)^n \underline{|n}$ in the formula.

$$f(x) = f(0)+f'(0)x+\frac{f''(0)}{\underline{|2}}x^2+\frac{f'''(0)}{\underline{|3}}x^3+\frac{f^{(4)}(0)}{\underline{|4}}x^4 = \ldots + \frac{f^{(n)}(0)}{\underline{|n}}x^x + ,\ldots$$

$$+\frac{f^{(n)}(0)}{\underline{|n}}x^x+,\ldots \text{ we have}$$

$$\frac{1}{1+x} = 1+(-1)x+(-1)^2\frac{\underline{|2}}{\underline{|2}}x^2+(-1)^3\frac{\underline{|3}}{\underline{|3}}x^3+(-1)^4\frac{\underline{|4}}{\underline{|4}}x^4+\ldots+\frac{(-1)^n\underline{|n}}{\underline{|n}}x^n+\ldots$$

Thus, the Maclaurin series for $\frac{1}{1+x}$ is the geometric series with the first term 1 and common ratio –x.

**Note:** Applying the formula $S = \frac{a_1}{1-r}$, we have

$$1-x+x_2-x_3+\ldots = \frac{1}{1-(-x)} = \frac{1}{1+x}$$

**Example 2:** Find the Maclaurin series for $\sin x$

**Solution:** Let $f(x)=\sin x$. Then $f(0)=\sin 0 = 0$.

$f'(x)=\cos x$ and $f'(0)=\cos 0 = 1$ ; $f''(x) = -\sin x$ and $f''(0)=-\sin 0 = 0$;

$f'''(x)=-\cos x$ and $f'''(0)=-\cos 0=-1$ ; $f^{(4)}(x)=-(-\sin x)=-\sin x$

and $f^{(4)}(0)=\sin(0)=0$.

$f^{(5)}=(x) = \cos x$ and $f^{(5)}(0)=\cos 0 = 1, f^{(6)}(x)=-\sin x$

and $f^{(6)}(0)=0$ ; $f^{(7)} = -\cos x$ and $f^{(7)}(0)=-1$

Putting these values in the formula





$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{\underline{|2}}x^2+\frac{f'''(0)}{\underline{|3}}x^3+\frac{f^{(4)}(0)}{\underline{|4}}x^4+\frac{f^{(5)}(0)}{\underline{|5}}+\ldots, \text{we have}$$

$$\sin x = 0+1.x+\frac{0}{\underline{|2}}x^2+\frac{-1}{\underline{|3}}x^3+\frac{0}{\underline{|4}}x^4+\frac{1}{\underline{|5}}x^5+\frac{0}{\underline{|6}}x^6++\frac{-1}{\underline{|7}}x^7+\ldots$$

$$= x-\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}-\frac{x^7}{\underline{|7}}+\ldots\ldots$$

**Example 3:** **Expand $a^x$ in the Maclaurin series.**

**Solution:** Let $f(x)=a^x$, then

$f'(x)=a^x \ln a,\ f''(x)=a^x(\ln a)^2, f'''(x)=a^x(\ln a)^3$

$f^{(4)}(x)=a^x(\ln a)^4, \ldots, f^{(n)}(x) = a^x(\ln a)^{(n)}$.

Putting $x=0$ in $f(x), f'(x), f''(x), f'''(x), f^{(4)}(x), \ldots f^{(n)}(x)$, we get

$f(0)=a^0=1, f'(0)=a^0 \ln a = \ln a, f''(0)=(\ln a)^2, f'''(0)=(\ln a)^3$

$f^{(4)}(0)=(\ln a)^4, \ldots, f^{(n)}(0)=(\ln a)^n$.

Substituting these values in the formula

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{\underline{|2}}x^2+\frac{f'''(0)}{\underline{|3}}x^3+\ldots+\frac{f^{(n)}(0)}{\underline{|n}}x^n+\ldots, \text{we have}$$

$$a^x = 1+(\ln a).x+\frac{(\ln a)^2}{\underline{|2}}x^2+\frac{(\ln a)^3}{\underline{|3}}x^3+\ldots+\frac{(\ln a)^n}{\underline{|n}}x^n+\ldots$$

**Note:** If we put *a* = *e* in the above expansion, we get

$$e^x = 1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots+\frac{x^n}{\underline{|n}}+\ldots \qquad (\because \ln e = 1)$$

Replacing $x$ by 1, we have

$$e = 1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots+\frac{1}{\underline{|n}}$$

**Example 4:** **Expand $(1+x)^n$ in the Maclaurin series.**

**Solution:** Let $f(x)=(1+x)^n$, then

$$f'(x)=n(1+x)^{n-1}, \qquad f''(x)=n(n-1)(1+x)^{n-2}$$

$$f'''(x)=n(n-1)(n-2)(1+x)^{n-3},\ f^{(4)}(x)=n(n-1)(n-2)(n-3)(1+x)^{n-4}$$

Putting $x=0$, we get

$$f(0)=(1+0)^n=1,\ f'(0)=n(1+0)^{n-1}=n,$$

$$f''(0)=n(n-1)(1+0)^{n-2}=n(n-1)$$

$$f'''(0)=n(n-1)(n-2)(1+0)^{n-3}=n(n-1)(n-2),$$

$$f^{(4)}(0)=n(n-1)(n-2)(n-3)(1+0)^{n-4}=n(n-1)(n-3)$$

Substituting these values in the formula

$$f(x)=f(0)+f'(0).x+\frac{f''(0)}{\underline{|2}}x^2+\frac{f'''(0)}{\underline{|3}}x^3+\ldots, \text{ we have}$$

$$(1+x)^n=1+n.x+\frac{n(n-1)}{\underline{|2}}x^2+\frac{n(n-1)(n-2)}{\underline{|3}}x^3+\ldots$$

## 2.17 TAILOR SERIES EXPANSIONS OF FUNCTIONS:

If $f$ is defined in the interval containing $'a'$ and its derivatives of all orders exist at $x=a$, then we can expand $f(x)$ as

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{\underline{|2}}(x-a)^2+\frac{f'''(a)}{\underline{|3}}(x-a)^3$$

$$+\frac{f^{(4)}(a)}{\underline{|4}}(x-a)^4+\ldots+\frac{f^{(n)}(a)}{\underline{|n}}(x-a)^n+\ldots$$

Let $f(x)=a_0+a_1(x-a)+a_2(x-a)^2+a_3(x-a)^3+a_4(x-a)^4+\ldots$

$$+a_n(x-a)^n+\ldots$$

Obviously $f(a)=a_0$ ( $\because$ putting $x=a$, all other terms vanish )





$$f'(x)=a_1+2a_2(x-a)+3a_3(x-a)^2+4a_4(x-a)^3+\ldots+na_n(x-a)^{n-1}+\ldots$$

$$f''(x)=2a_2+6a_3(x-a)+12a_4(x-a)^2+\ldots+n(n-1)a_n(x-a)^{n-2}+\ldots$$

$$f'''(x)=6a_3+24a_4(x-a)+\ldots\ldots$$

Putting $x=a$, we get $f'(a)=a_1; f''(a)=2a_2 \Rightarrow a_2=\frac{f''(a)}{\underline{|2}};\ f'''(a)=6a_3$

$$\Rightarrow \quad a_3=\frac{f'''(a)}{\underline{|3}}$$

Following the above pattern, we have $\frac{f^{(\ )}(a)}{\underline{|\ }}$

Substituting the values of $a_0,a_1,a_2,a_3,\ldots$, w e g e t

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{\underline{|2}}(x-a)^2+\frac{f'''(a)}{\underline{|3}}(x-a)^3+\ldots$$

$$+\frac{f^{(n)}(a)}{\underline{|n}}(x-a)^n+\ldots$$

This expansion is the Taylor series for $f$ at $x=a$. The expansionisonly valid if it is convergent .

If $a$ = 0, then the above expansion becomes

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{\underline{|2}}x^2+\frac{f''(0)}{\underline{|3}}x^3+\ldots+\frac{f^{(n)}(0)}{\underline{|n}}x^n+\ldots$$

which is the Maclaurin series for f at $x=a$.

Replacing $x$ by $x+h$ and $a$ by $x$, the expansion in (A) can be written as

$$f(x+h)=f(x)+f'(x)h+\frac{f''(x)}{\underline{|2}}h^2+\frac{f'''(x)}{\underline{|3}}h^3+\ldots+\frac{f^{(n)}(x)}{\underline{|n}}h^n+\ldots \quad \text{(B)}$$

The expansions in (B) is termed as **Taylor's Theorem** and can be stated as: If $x$ and $h$ are two independent quantities and $f(x+h)$ can be expanded in ascending power of $h$ as an infinite series, then

$$f(x+h)=f(x)+f'(x)h+\frac{f''(x)}{\underline{|2}}h^2+\frac{f'''(x)}{\underline{|3}}h^3+\ldots+\frac{f^{(n)}(x)}{\underline{|n}}h^n+\ldots$$

**Example 1: Find the Taylor series expansion of ln (1 + $x$) at $x$ = 2.**

**Solution:** Let $f(x)=\ln(1+x)$, then $f(2)=\ln(1+2)=\ln 3$

Finding he successive derivatives of $\ln(1+x)$ and evaluating them at $x$ = 2

$f'(x)=\frac{1}{1+x}$ and $f'(2)=\frac{1}{1+2}=\frac{1}{3}$

$f''(x)=(-1)(1+x)$ and $f''(2)=-(1+2)^{-2}=-\frac{1}{9}$

$f'''(x)=(-1)(-2)(1+x)^{-3}$ and $f'''(2)=\underline{|2}\,.\,(1+2)^{-3}=\frac{\underline{|2}}{27}$

$f^{(4)}(x)=(-1)(-2)(-3)(1+x)^{-4}=(-1)^3\underline{|3}(1+x)^{-4}$ and $f^{(4)}(2)=-\frac{\underline{|3}}{81}$

The Taylor series expansions of f at $x=a$ is

$$f(x)=f(a)+f'(a).(x-a)+\frac{f''(a)}{\underline{|2}}(x-a)^2+\frac{f'''(a)}{\underline{|3}}(x-a)^3+\ldots\ldots$$

Now substituting the relative values, we have

$$\ln(1+x)=\ln 3+\frac{1}{3}(x-2)+\frac{-\frac{1}{9}}{\underline{|2}}(x-2)^2+\frac{\frac{\underline{|2}}{27}}{\underline{|3}}(x-2)^3+\frac{-\frac{\underline{|3}}{81}}{\underline{|4}}(x-2)^4+\ldots.$$

$$=\ln 3+\frac{x-2}{1.3}-\frac{(x-2)^2}{2.3^2}+\frac{(x-2)^3}{3.3^3}-\frac{(x-2)^4}{4.3^4}+\ldots.$$

**Example 2: Use the Taylor series expansion to find the value of sin 31⁰.**

**Solution:** We take $a$ = 30° = $\frac{\pi}{6}$

Let $f(x)=\sin x$, then $f\left(\frac{\pi}{6}\right)=\sin\frac{\pi}{6}=\frac{1}{2}$





Now taking the successive derivative of sin $x$ and evaluating them at $\frac{\pi}{6}$ , we have

$f'(x)=\cos x$ and $f'\left(\frac{\pi}{6}\right)=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$

$f''(x)=-\sin x$ and $f''\left(\frac{\pi}{6}\right)=-\sin\frac{\pi}{6}=-\frac{-1}{2}$

$f'''(x)=-\cos x$ and $f'''\left(\frac{\pi}{6}\right)=-\cos\frac{\pi}{6}=-\frac{\sqrt{3}}{2}$

$f^{(4)}(x)=-(-\sin x)=\sin x$ and $f^{(4)}\left(\frac{\pi}{6}\right)=\sin\frac{\pi}{6}=\frac{1}{2}$

Thus the Taylor series expansion at $a=\frac{\pi}{6}$ is

$$\sin x=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)+\frac{-\frac{1}{2}}{\underline{|2}}\left(x-\frac{\pi}{6}\right)^2+\frac{-\frac{\sqrt{3}}{2}}{\underline{|3}}\left(x-\frac{\pi}{6}\right)^3+\ldots$$

$$=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)-\frac{1}{2\underline{|2}}\left(x-\frac{\pi}{6}\right)^2-\frac{\sqrt{3}}{2\underline{|3}}\left(x-\frac{\pi}{6}\right)^3+\ldots.$$

For $x=31^0$, $x-\frac{\pi}{6}=\left(31^0-30^0\right)=1^0\approx .017455$

$$\sin 31^0\approx\frac{1}{2}+\frac{\sqrt{3}}{2}(.017455)-\frac{1}{4}(.017455)^2-\frac{\sqrt{3}}{12}(.017455)^3$$

$$\approx .5+.015116-0.000076\approx .5150$$

**Example 3: Prove that $e^{x+h}=e^x\left\{1+h+\frac{h^2}{\underline{|2}}+\frac{h^3}{\underline{|3}}+\ldots.\right\}$**

**Solution:** Let $f(x+h)=e^{x+h}$, then $f(x)=e^x$ ...(i)

By successive derivatives of (i) w.r.t '$x$' we have

$f'(x)=e^x$, $f''(x)=e^x$, $f'''(x)=e^x$ etc.

By Taylor's Theorem we have

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{\lfloor 2}f''(x)+\frac{h^3}{\lfloor 3}+f'''(x)+\dots$$

Putting the relative values, we get

$$e^{x+h}=e^x+h\ e^x+\frac{h^2}{\lfloor 2}e^x+\frac{h^3}{\lfloor 3}e^x+\dots$$

$$=e^x\left[1+h+\frac{h^2}{\lfloor 2}+\frac{h^3}{\lfloor 3}+\dots\right]$$

## EXERCISE 2.8

**1.** Apply the Maclaurin series expansion to prove that:

(i) $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{2}-\frac{x^4}{2}+\dots\dots$

(ii) $\cos x=1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}-\frac{x^6}{\lfloor 6}+\dots\dots$

(iii) $\sqrt{1+x}=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}+\dots\dots$

(iv) $e^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\dots\dots$

(v) $e^{2x}=1+2x+\frac{4x^2}{\lfloor 2}+\frac{8x^3}{\lfloor 3}+\dots\dots$

**2.** Show that:

$$\cos(x+h)=\cos x-h\sin x-\frac{h^2}{\lfloor 2}\cos x+\frac{h^3}{\lfloor 3}\sin x+\dots\dots$$

and evaluate cos 61°.

**3.** Show that $2^{x+h}=2^x\{1+(\ln 2)h+\frac{(\ln 2)^2h^2}{\lfloor 2}+\frac{(\ln 2)^3h^3}{\lfloor 3}+\dots\}$





# 2.18 GEOMETRICAL INTERPRETATION OF A DERIVATIVE

Let *AB* be the arc of the graph of *f* defined by the equation $y=f(x)$.

Let $P(x,f(x))$ and $Q(x+\delta x.f(x+\delta x))$ be two neighbouring points on the arc *AB* where $x$, $x+\delta x\in D_f$.

The line $PQ$ is secant of the curve and it makes $\angle XSQ$ with the positive direction of the $x$-axis. (See the figure 2.21.1)

Drawing the ordinates $PM, QN$ and perpendicular $PR$ to $NQ$, we have

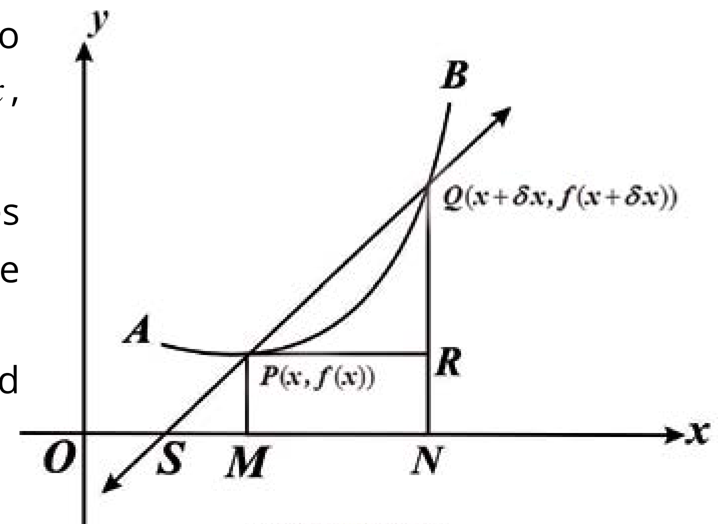


FIGURE 2.21.1

$$RQ=NQ-NR=NQ-MP=f(x+\delta x)-f(x)$$

and $PR=MN=ON-OM=x+\delta x-x=\delta x$

Thus $\tan m\angle XSQ=\tan m\angle RPQ$

$$=\frac{RQ}{PR}=\frac{f(x+\delta x)-f(x)}{\delta x}$$

Revolving the secant line $PQ$ about *P* towards *P*, some of its successive positions $PQ_1, PQ_2, PQ_3,\dots$ are shown in the figure 2.21.2. Points $Q_i\ (i=1,2,3,\dots)$ are getting closer and closer to the point *P* and $PR_i$ i.e; $\delta x_i$ (i = 1, 2, 3, ...) are approaching to zero.

In other words we can say that the revolving secant line approaches the tangent line *PT* as its limiting position at *P* while $\delta x$ approaches zero, that is,

$\tan m\angle XSQ\to\tan m\angle XTP$ when $\delta x\to 0$

or $\frac{f(x+\delta x)-f(x)}{\delta x}\to\tan m\angle XTP$ as $\delta x\to 0$

so $\lim\limits_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}=\tan m\angle XTP$

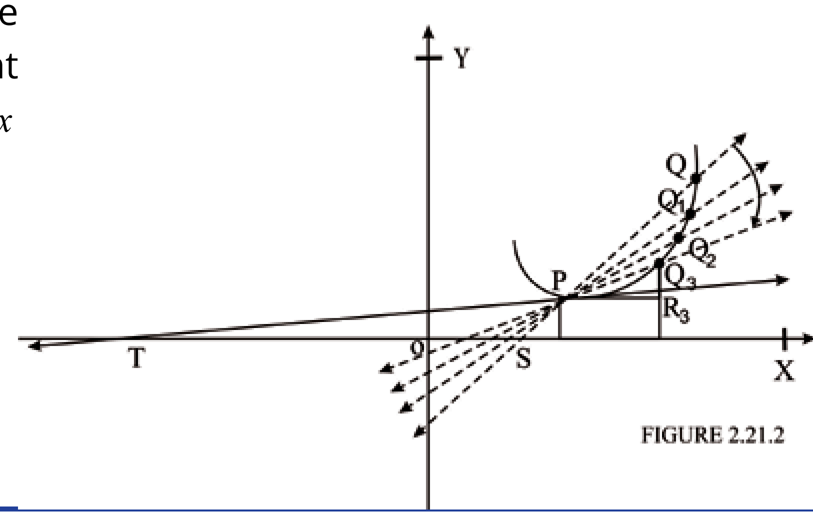


FIGURE 2.21.2

or $\quad f'(x) = \tan m\angle XTP$

Thus the slope of the tangent line to the graph of $f$ at $(x, f(x))$ is $f'(x)$.

**Example 1:** **Discuss the tangent line to the graph of the function $|x|$ at $x = 0$.**

**Solution:** Let $\quad f(x) = |x|$

$f(0) = |0| = 0 \quad and,$

$f(0+\delta x) = |0+\delta x| = |\delta x|,$

so $\quad f(0+\delta x) - f(0) = |\delta x| - 0$

and $\quad \dfrac{f(0+\delta x) - f(0)}{\delta x} = \dfrac{|\delta x|}{\delta x}$

Thus $\quad f'(0) = \lim\limits_{\delta x \to 0} \dfrac{|\delta x|}{\delta x}$

Because $\quad |\delta x| = \delta x \quad$ when $\delta x > 0$

and $\quad |\delta x| = -\delta x \quad$ when $\delta x < 0$

so we consider one-sided limits

$$\lim_{\delta x \to 0^+} \frac{|\delta x|}{\delta x} = \lim_{\delta x \to 0^+} \frac{\delta x}{\delta x} = 1$$

and $$\lim_{\delta x \to 0^-} \frac{|\delta x|}{\delta x} = \lim_{\delta x \to 0^-} \frac{-\delta x}{\delta x} = -1$$

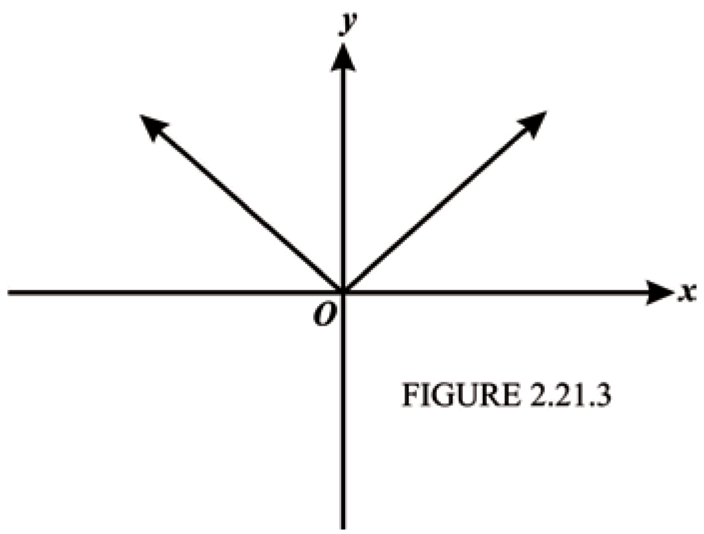



FIGURE 2.21.3

The right hand and left hand limits are not equal, therefore, the $\lim\limits_{\delta x \to 0} \dfrac{|\delta x|}{\delta x}$ does not exist.

This means that $f'(0)$,the derivative of f at $x = 0$ does not exist and there is no tangent line to the graph of $f$ and $x = 0$ (see the figure 2.21.3).





**Example 2:** Find the equations of the tangents to the curve $x^2 - y^2 - 6y = 0$ at the point whose abscissa is 4.

**Solution.** Given that $x^2 - y^2 - 6y = 0 \qquad$ (i)

We first find the y-coordinates of the points at which the equations of the tangents are to be found. Putting $x = 4$ is (i) gives

$$16 - y^2 - 6y = 0 \qquad \Rightarrow y^2 + 6y - 16 = 0$$

or $\quad y = \dfrac{-6 \pm \sqrt{36+64}}{2} = \dfrac{-6 \pm \sqrt{100}}{2} = \dfrac{-6 \pm 10}{2}$, that is ,

$$y = \frac{-6+10}{2} = \frac{4}{2} = 2 \qquad \text{or} \qquad y = \frac{-6-10}{2} = \frac{-16}{2} = -8$$

Thus the points are (4, 2) and (4, - 8).

Differentiating (i) w.r.t. '$x$' we have

$$2x - 2y\frac{dy}{dx} - 6\frac{dy}{dx} = 0 \qquad \Rightarrow 2\frac{dy}{dx}(y+3) = 2x \qquad \Rightarrow \frac{dy}{dx} = \frac{x}{y+3}$$

The slope of the tangent to (i) at (4, 2) = $= \dfrac{4}{2+3} = \dfrac{4}{5}$ .

Therefore, the equation of the tangent to (i) at (4, 2) is

$$y - 2 = \frac{4}{5}(x-4) \qquad \Rightarrow 5y - 10 = 4x - 16$$

or $\quad 5y = 4x - 6$

The slope of the tangent to (i) at (4, - 8) = $\dfrac{4}{-8+3} = -\dfrac{4}{5}$

Therefore the equation of the tangent to (i) at (4, - 8) is

$$y - (-8) = -\frac{4}{5}(x-4)$$

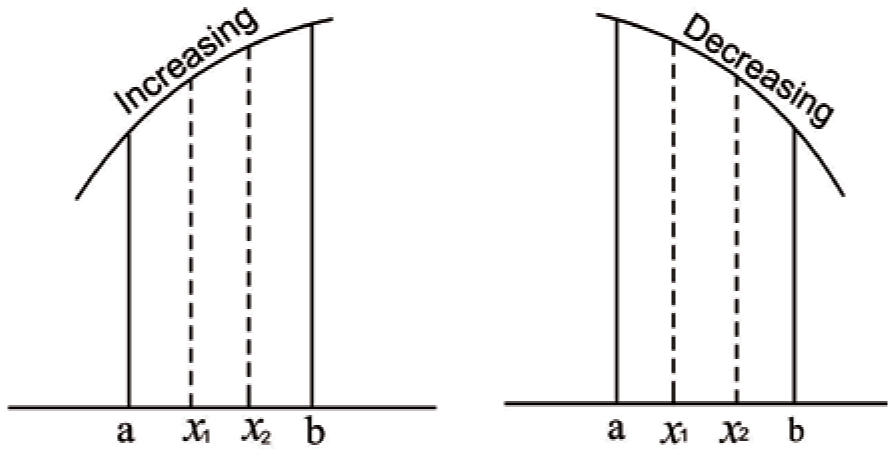
$$5y + 40 = -4x + 16 \qquad \Rightarrow 4x + 5y + 24 = 0$$

## 2.19 INCREASING AND DECREASING FUNCTIONS

Let $f$ be defined on an interval (a, b) and let $x_1, x_2 \in (a,b)$. Then

(i) $f$ is increasing on the interval $(a, b)$ if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$

(ii) $f$ is decreasing on the interval $(a, b)$ if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$




$f(x_2)>f(x_1)$ if $x_2> x_1$ $\qquad$ $f(x_2)<f(x_1)$ if $x_2> x_1$

We see that a differentiable function $f$ is increasing on (a,b) if tangent lines to its graph at all points $(x, f(x))$ where $x \in (a, b)$ have positive slopes, that is,

$f'(x) > 0$ for all $x$ such that $a < x < b$

and $f$ is decreasing on $(a, b)$ if tangent lines to its graph at all points $(x, f(x))$ where $x \in (a,b)$, have negative slopes, that is, $f'(x) < 0$ for all $x$ such that $a < x < b$

Now we state the above observation in the following theorem.

**Theorem:**

Let $f$ be a differentiable function on the open interval (a,b). Then

(i) $f$ is increasing on (a,b) if $f'(x) > 0$ for each $x \in (a,b)$

(ii) $f$ is decreasing on (a,b) if $f'(x) < 0$ for each $x \in (a,b)$

Let $f(x) = x^2$, then

$$f(x_2) - f(x_1) = x_2^2 - x_1^2 = (x_2 - x_1)(x_2 + x_1)$$

If $x_1, x_2 \in (-\infty, 0)$ and $x_2 > x_1$ ,, then





$f(x_2) - f(x_1) < 0 \qquad (\because x_2 - x_1 > 0 \text{ and } x_2 + x_1 < 0)$

$\Rightarrow f(x_2) < f(x_1)$

$\Rightarrow f$ is decreasing on the interval $(-\infty, 0)$

If $x_1, x_2 \in (0, \infty)$ and $x_2 > x_1$ , then

$f(x_2) - f(x_1) > 0 \qquad (\because x_2 - x_1 > 0 \text{ and } x_2 + x_1 > 0)$

$\Rightarrow f(x_2) > f(x_1)$

$\Rightarrow f$ is increasing on the interval $(0, \infty)$

Here $f'(x) = 2x$ and $f'(x) < 0$ for all $x \in (-\infty, 0)$, therefore,

$f$ is decreasing on the interval $(-\infty, 0)$

Also $f'(x) > 0$ for all $x \in (0, \infty)$, so $f$ is increasing on the interval $(0, \infty)$.

From the above theorem we can conclude that

1. $f'(x_1) < 0 \Rightarrow f$ is decreasing at $x_1$
2. $f'(x_1) = 0 \Rightarrow f$ is neither increasing nor decreasing at $x_1$
3. $f'(x_1) > 0 \Rightarrow f$ is increasing at $x_1$

Now we illustrate the ideas discussed so far considering the function f defined as

$f(x) = 4x - x^2 \qquad$ (I)

To draw the graph of $f$, we form a table of some ordered pairs which belongs to $f$

| $x$ | –1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| $y = f(x)$ | –5 | 0 | 3 | 4 | 3 | 0 | –5 |

The graph of *f* is shown in the figure 2.22.1.

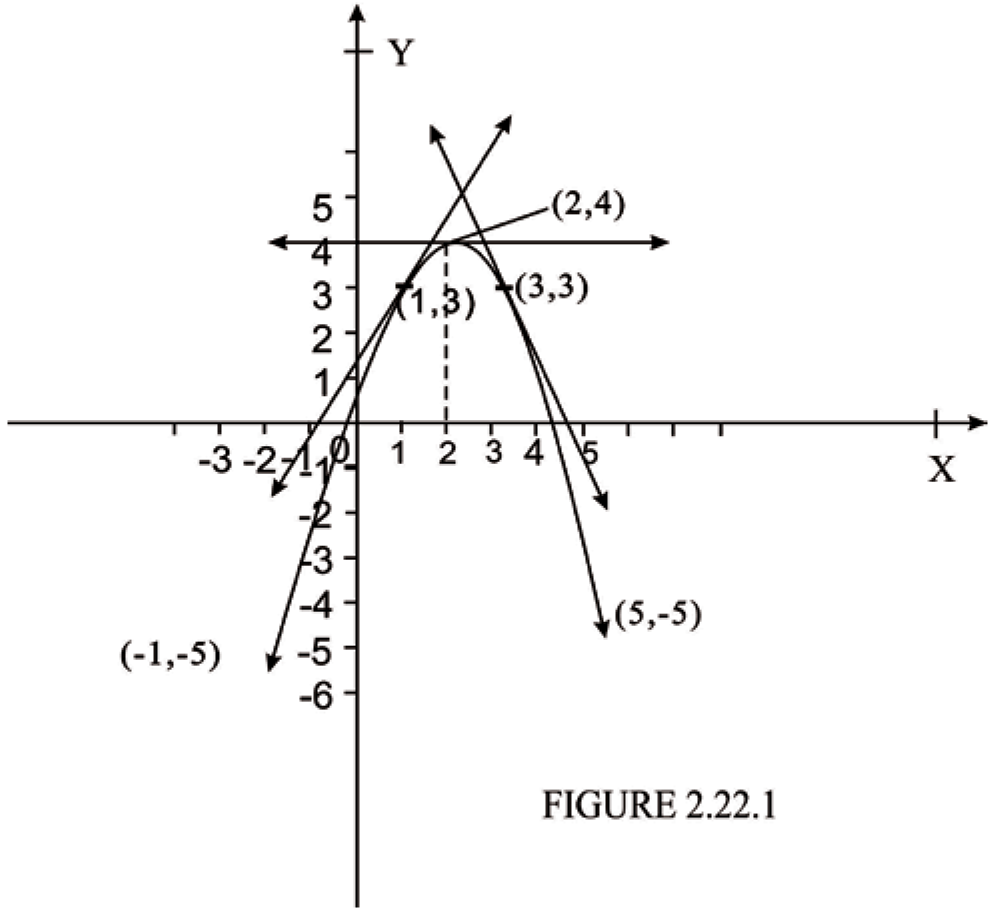



FIGURE 2.22.1

From the graph of *f*, it is obvious that *y* rises from 0 to 4 as $x$ increases from 0 to 2 and *y* falls from 4 to 0 as $x$ increases from 2 to 4.

In other words, we can say that the function *f* defined as in (I) is increasing in the interval $0<x<2$ and is decreasing in the interval $2 < x < 4$.

The slope of the tangent to the graph of *f* at any point in the interval $0<x<2$, in which the function *f* is increasing is positive because it makes an acute angle with the positive direction of x-axis. (See the tangent line to the graph of *f* at (1, 3)).

But the slope of the tangent line to the graph of *f* at any pointin the interval $2<x<4$ in which the function *f* is decreasing is negative as it makes an obtuse angle with the positive direction of *x*-axis. (See the tangent line to the graph of *f* at (3, 3)).

As we know that the slope of the tangent line to the graph of *f* at $(x, f(x))$ is $f'(x)$, so the derivative of the function *f* i.e., $f'(x)$, is positive in the interval in which *f* is increasing and $f'(x)$, is negative in the interval in which *f* is decreasing.

The function *f* under consideration is actually increasing at each *x* for which $f'(x)>0$.





i.e. $4-2x>0 \quad \Rightarrow -2x>-4 \quad \Rightarrow x<2$

Thus it is increasing in the interval $(-\infty, 2)$. Similarly we can show that it is decreasing, in the interval $(2, \infty)$.

Now we give an analytical approach to the above discussion.

Let *f* be an increasing function in some interval in which it is differentiable. Let *x* and $x+\delta x$ be two, points in that interval such that $x+\delta x > x$.

As the function *f* is increasing in the interval, it conveys the fact that $f(x + \delta x) > f(x)$.

Consequently we have, $f(x+\delta x)-f(x)>0$ and $(x+\delta x)-x>0$, that is,

$f(x + \delta x) - f(x) > 0$ and $\delta x > 0$

or $\dfrac{f(x+\delta x)-f(x)}{\delta x} > 0$

The above difference quotient becomes one-sided limit

$$\lim_{\delta x \to 0^+} \frac{f(x+\delta x)-f(x)}{\delta x}$$

As *f* is differentiable, so $f'(x)$ exists and one sided limit must equal to $f'(x)$.

Thus $f'(x) > 0$

**Example 1: Determine the values of *x* for which *f* defined as $f(x)=x^2+2x-3$ is**
**(i) increasing (ii) decreasing.**
**(iii) find the point where the function is neither increasing nor decreasing.**

**Solution:** The table of some ordered pairs satisfying $f(x)=x^2+2x-3$ is given below:

| *x* | –4 | –3 | –2 | –1 | 0 | 1 | 2 |
|---|---|---|---|---|---|---|---|
| *y* = *f*(*x*) | 5 | 0 | –3 | –4 | –3 | 0 | 5 |

The graph of $f$ is shown in the figure2.22.2.

$f'(x) = 2x + 2$

(i) The condition $f'(x) > 0$ $\Rightarrow 2x + 2 > 0$

$\Rightarrow 2x > -2$

which gives $x > -1$, so the function $f$ defined as $f(x) = x^2 + 2x - 3$ is increasing in the interval $(-1, \infty)$.

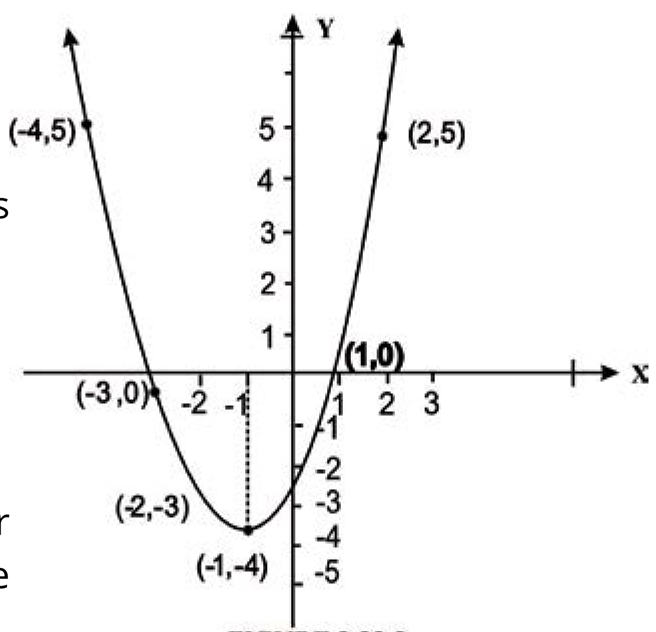


FIGURE 2.22.2

(ii) And the condition $f'(x) < 0$ $\Rightarrow 2x + 2 < 0$

$\Rightarrow 2x < -2$

which gives $x < -1$, so the function $f$ under consideration in the example I is decreasing in the interval $(-\infty, -1)$.

(iii) The function is neither increasing nor decreasing where $f'(x) = 0$, that is,

$2x + 2 = 0$ $\Rightarrow x = -1$.

If $x = -1$ then $f(-1) = (-1)^2 + 2(-1) - 3 = -4$. Thus $f$ is neither increasing nor deceasing at the point (-1, -4).

**Note:** Any point where $f$ is neither increasing nor decreasing is called a **stationary point**, provided that $f'(x) = 0$ at that point.

**Example 2:** Determine the intervals in which $f$ is increasing or it is decreasing if

$f(x) = x^3 - 6x^2 + 9x$

**Solution.** $f'(x) = 3x^2 - 12x + 9$

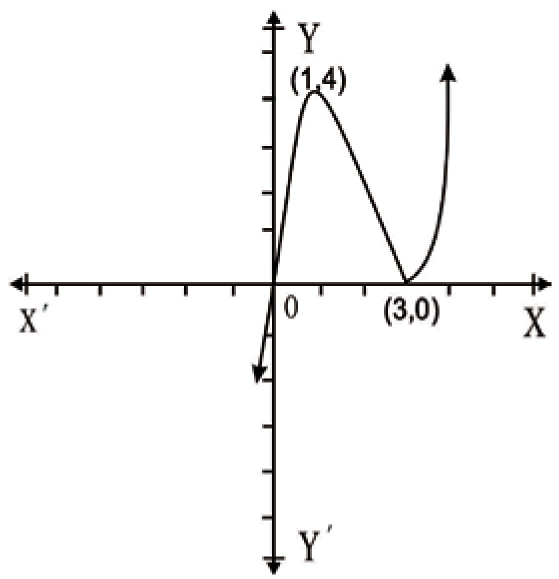
$= 3(x^2 - 4x + 3)$

$= 3(x-1)(x-3)$

$f'(x) > 0$

$\Rightarrow$ $x^2 - 4x + 3 > 0$

$\Rightarrow$ $(x-1)(x-3) > 0$








$(x-1)(x-3) > 0$' in the intervals $(-\infty, 1)$ and $(3, \infty)$

$f'(x) < 0$ $\Rightarrow (x-1)(x-3) < 0$

$(x-1)(x-3) < 0$ if $x > 1$ and $x < 3$ that is $1 < x < 3$

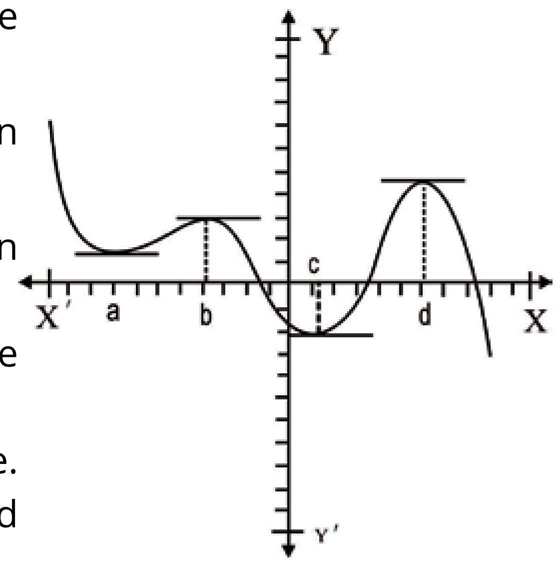
## 2.20 RELATIVE EXTREMA

Let $(c - \delta x, c + \delta x) \subseteq D_f$, , (domain of a function $f$), where $\delta x$ is small positive number.

If $f(c) \geq f(x)$ for all $x \in (c - \delta x, c + \delta x)$ then the function $f$ is said to have a relative maxima at $x = c$.

Similarly if $f(c) \leq f(x)$ for all $x \in (c - \delta x, c + \delta x)$, then the function f has relative minima at $x = c$.

Both relative maximum and relative minimum are called in general **relative extrema**.




The graph of a function is shown in the adjoining figure. It has relative maxima at $x = b$ and $x = d$. But at $x = a$ and $x = c$, it has relative minima.

Note that the relative maxima at $x = d$ is not the highest point of the graph.

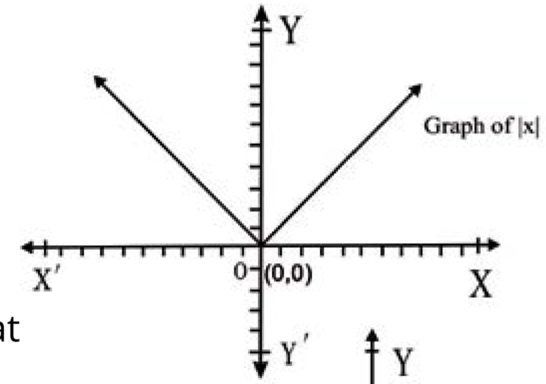
## 2.21 CRITICAL VALUES AND CRITICAL POINTS

If $c \in Df$ and $f'(c) = 0$ or $f'(c)$ does not exist, then the number $c$ is called a critical value for $f$ while the point ($c$. $f(c)$) on the graph of $f$ is named as a critical point.

**Note:** There are functions which have extrema (maxima or minima) at the points where their derivatives do not exist. For example, the derivatives of the function $f$ and $\phi$ defined as.

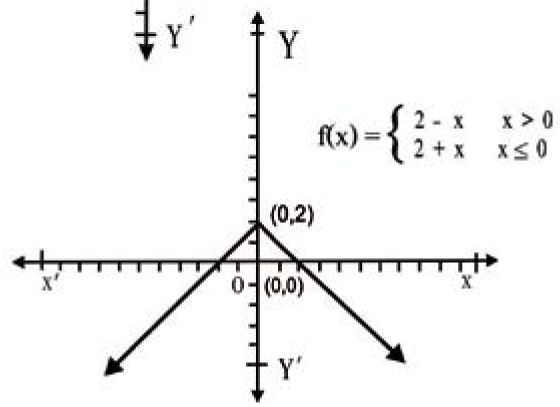
$$f(x) = |x|$$

and $\phi(x) = \begin{cases} 2-x & x>0 \\ 2+x & x \le 0 \end{cases}$




do not exist at (0, 0) and (0, 2) respectively.

But $f$ has minima at (0, 0) and $\phi$ has maxima at (0, 2). See the adjoining figures.




Those critical points on the graph of $f$ at which $f'(x)=0$ are called stationary points of $f$.

Now we discuss relative maxima and relative minima of the differentiable function $f$ defined as:

$$y = f(x) = x^3 - 3x^2 + 4 \ldots\text{(1)}$$

Graph of $f$ is drawn with the help of some ordered pairs tabulated as below:

| X | –3/2 | –1 | –1/2 | 0 | 1/2 | 1 | 3/2 | 2 | 5/2 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|
| Y | –49/8 | 0 | 25/8 | 4 | 27/8 | 2 | 5/8 | 0 | 7/8 | 4 |

Now differentiating (i) w.r.t. $'x'$ we get

$f'(x) = 3x^2 - 6x = 3x(x-2)$

$f'(x) = 0 \qquad \Rightarrow 3x(x-2)=0 \qquad \Rightarrow x = 0 \text{ or } x = 2$

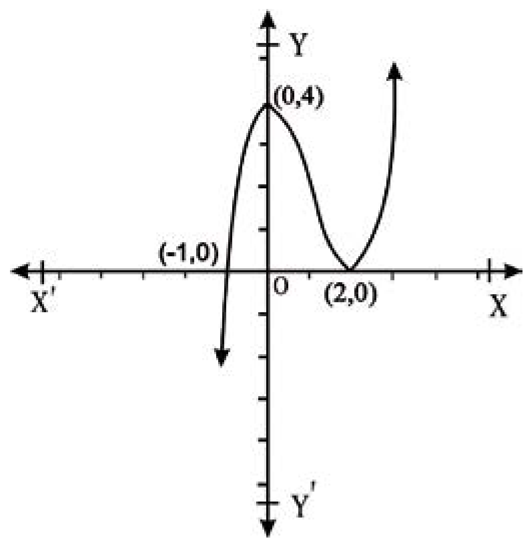
Now we consider an interval $(-\delta x, \delta x)$ in the neighbourhood of $x=0$. Let $0-\varepsilon$ is a point in the interval $(-\delta x, 0)$ We see that

$$f'(0-\varepsilon) = 3(-\varepsilon)(-\varepsilon-2) \qquad (\because f'(x)=3x(x-2))$$
$$= 3\varepsilon(\varepsilon+2) > 0 \qquad (\because \varepsilon>0, \varepsilon+2>0)$$

That is $f'(x)$ is positive for all $x \in (-\delta x, 0)$.

Let $0+\varepsilon_1$ is a point in the interval $(0, \delta x)$, then we have

$$f'(0+\varepsilon_1) = 3(\varepsilon_1)(\varepsilon_1-2)$$
$$= -3\varepsilon_1(2-\varepsilon_1) < 0 \ (\because 2-\varepsilon_1>0, \varepsilon_1>0)$$, that is,




$f'(x)$ is negative for all $x \in (0, \delta x)$





We note that $f'(x)>0$ before $x=0, f'(x) = 0$ at $x=0$ and $f'(x)<0$ after $x=0$.

The graph of $f$ shows that it has relative maxima at x = 0.

Thus we conclude that **a function has relative maxima at $x=c$ if $f'(x)>0$, before $x=c$ $f'(c)=0$ and $f'(x)<0$ after $x=c$.**

Considering an interval (2 – δ$x$, 2 + δ$x$) in the neighbourhood of $x$ = 2 we find the values of $f'$ (2–ε) and $f'$ (2 + ε) when 2 – ε ∈ (2 – δ$x$, 2) and 2 + ε ∈ (2, 2 + δ$x$)

$$f'(2-\varepsilon) = 3(2-\varepsilon)(2-\varepsilon-2) \qquad [\because f'(x)=3x(x-2)]$$
$$= 3(2-\varepsilon)(-\varepsilon)$$
$$= -3\varepsilon(2-\varepsilon) < 0 \qquad (\because \varepsilon>0, 2-\varepsilon>0)$$

and
$$f'(2+\varepsilon) = 3(2+\varepsilon)(2+\varepsilon-2)$$
$$= 3\varepsilon(2+\varepsilon) > 0 \qquad (\because \varepsilon>0, 2+\varepsilon>0)$$

We see that $f'(x)<0$ before $x$ = 2, $f'(x)=0$ at $x=2$ and $f'(x)>0$ after $x=2$.

It is obvious from the graph that it has relative minima at $x=2$.

Thus we conclude that **a function has relative minima at $x=c$ if $f'(x)<0$ before $x=c, f'(x)=0$ at $x=c$ and $f'(x)>0$ after $x=c$.**

**First Derivative Rule:**

Let $f$ be differentiable in neighbourhood of $c$ where $f'(c)=0$.

1. If $f'(x)$ changes sign from positive to negative as $x$ increases through $c$, then $f(c)$ the relative maxima of $f$.

2. If $f'(x)$ changes sign from negative to positive as $x$ increases through $c$, then $f(c)$ is the relative minima of $f$.

**Note: 1.** A stationary point is called a turning point if it is either a maximum point or a minimum point.

**2.** If $f'(x) > 0$ before the point $x = a$, $f'(x) = 0$ at $x = 0$ and $f'(x) > 0$ after $x = 0$, then $f$ does not has a relative maxima.

See the graph of $f(x) = x^3$. In this case, we have

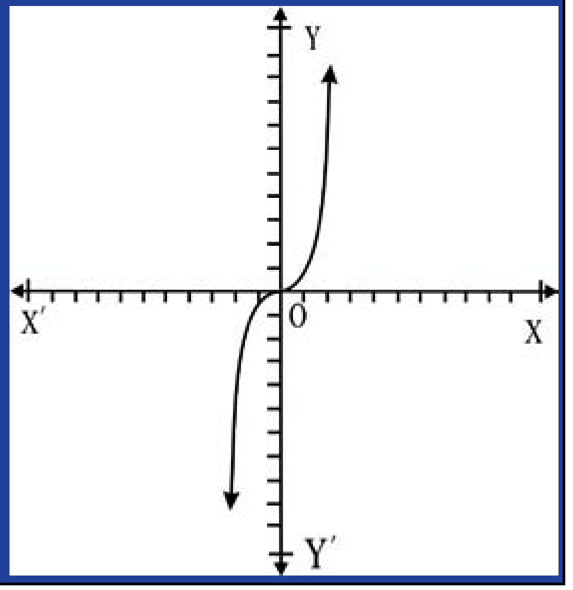
$f'(x) = 3x^2$, that is,

$f'(0-\varepsilon) = 3(-\varepsilon)^2 = 3\varepsilon^2 > 0$

and $f'(0+\varepsilon) = 3(\varepsilon)^2 = 3\varepsilon^2 > 0$

The function $f$ is increasing before $x = 0$ and also it is increasing after $x = 0$.

Such a point of the function is called the **point of inflexion**.




**Second Derivative Test:**

We have noticed that the first derivative $f'(x)$ of a function changes its sign from positive to negative at the point where f has relative maxima, that is, $f'$ is a decreasing function in the neighbouring interval containing the point where $f$ has relative maxima.

Thus $f''(x)$ is negative at the point where $f$ has a relative maxima.

But $f'(x)$ of a function f changes its sign from negative to positive at the point where f has relative minima, that is, $f'$ is an increasing function in the neighbouring interval containing the point where $f$ has relative minima.

Thus $f''(x)$ is positive at the point where f has relative minima.

**Second Derivative Rule**

Let $f$ be differential function in a neighbourhood of $c$ where $f'(c) = 0$. Then

1. $f$ has relative maxima at c if $f''(c) < 0$.
2. $f$ has relative minima at c if $f''(c) > 0$.





**Example 1:** Examine the function defined as

$\boldsymbol{f(x) = x^3 - 6x^2 + 9x}$ **for extreme values.**

**Solution:** $f'(x) = 3x^2 - 12x + 9$

$= 3(x^2 - 4x + 3) = 3(x-1)(x-3)$

**First Method**

If $x = 1-\varepsilon$ where $\varepsilon$ is very very small positive number, then

$(x-1)(x-3) = (1-\varepsilon-1)(1-\varepsilon-3) = (-\varepsilon)(-\varepsilon-2) = \varepsilon(2+\varepsilon) > 0$ that is ,

$f'(x) > 0$ before $x=1$. For $x = 1+\varepsilon$, we have

$(x-1)(x-3) = (1+\varepsilon-1)(1+\varepsilon-3) = (\varepsilon)(-2+\varepsilon) = -\varepsilon(2-\varepsilon) < 0$

That is, $f'(x) < 0$ after $x = 1$

As $f'(x) > 0$ before $x = 1$, $f'(x) = 0$ at $x = 1$ and $f'(x) < 0$ after $x = 1$

Thus f has relative maxima at $x = 1$ and $f(1) = -1 - 6 + 9 = 4$.

Let $x = 3 - \varepsilon$, then

$(x-1)(x-3) = (3-\varepsilon-1)(3-\varepsilon-3) = (2-\varepsilon)(-\varepsilon) = -\varepsilon(2-\varepsilon) < 0$

That is $f'(x) < 0$ before $x = 3$.

For $x = 3 + \varepsilon$

$(x-1)(x-3) = (3+\varepsilon-1)(3+\varepsilon-3) = (2+\varepsilon)(\varepsilon) > 0$

That is, $f'(x) > 0$ after $x = 3$.

As $f'(x) < 0$ before $x = 3$, $f'(x)$ at $x = 3$ and $f'(x) > 0$ after $x=3$,

Thus $f$ has relative minima at $x = 3$. and $f(3) = 3(3)^2 - 12(3) + 9 = 0$

**Second Method:** $f''(x) = 3(2x-4) = 6(x-2)$

$f''(1) = 6(1-2) = -6 < 0$ , therefore,

$f$ has relative maxima at $x = 1$ and $f(1) = (1)^3 - 6(1)^2 + 9(1)$

$= 1 - 6 + 9 = 4$

$f''(3) = 6(3-2) = 6 > 0$, therefore $f$ has relative minima at $x = 3$ and $f(3) = 27 - 54 + 27 = 0$

**Example 2:** Examine the function defined as $f(x)=1+x^3$ for extreme values

**Solution:** Given that $f(x)=1+x^3$

Differentiating w.r.t. $'x'$ we get $f'(x)=3x^2$

$$f'(x)=0 \qquad \Rightarrow 3x^2=0 \qquad \Rightarrow x=0$$

$$f''(x)=6x \qquad \text{and } f''(0)=6(0)=0$$

The second derivative does not help in determining the extreme values.

$$f'(0-\varepsilon)=3(0-\varepsilon)^2=3\varepsilon^2 > 0$$

$$f'(0+\varepsilon)=3(0+\varepsilon)^2=3\varepsilon^2 > 0$$

As the first derivative does not change sign at $x=0$, therefore (0, 0) is a point of inflexion.

**Example 3:** Discuss the function defined as $f(x)=\sin x+\frac{1}{2\sqrt{2}}\cos 2x$ for extreme values in the interval $(0, 2\pi)$.

**Solution:** Given that $f(x)=\sin x+\frac{1}{2\sqrt{2}}\cos 2x$

$$f'(x)=\cos x+\frac{1}{2\sqrt{2}}(-2\sin 2x)=\cos x-\frac{1}{\sqrt{2}}\sin 2x$$

$$=\cos x-\frac{1}{\sqrt{2}}(2\sin x\cos x)=\cos x-\sqrt{2}\sin x\cos x$$

$$=\cos x\left(1-\sqrt{2}\sin x\right)$$

Now $f'(x)=0 \qquad \Rightarrow \cos x\left(1-\sqrt{2}\sin x\right)=0$

$$\Rightarrow \cos x=0 \qquad \Rightarrow x=\frac{\pi}{2},\frac{3\pi}{2}$$

or $1-\sqrt{2}\sin x=0 \qquad \Rightarrow \sin x=\frac{1}{\sqrt{2}} \qquad \Rightarrow x=\frac{\pi}{4},\frac{3\pi}{4}$

Differentiating (i) w.r.t. $'x'$, we have





$$f''(x)=-\sin x-\frac{1}{\sqrt{2}}(\cos 2x)\times 2=-\sin x-\sqrt{2}\cos 2x$$

As $f''\left(\frac{\pi}{2}\right)=-\sin\frac{\pi}{2}-\sqrt{2}\cos\pi=-1-\sqrt{2}\times(-1)=\sqrt{2}-1>0$

and $f''\left(\frac{3\pi}{2}\right)=-\sin\frac{3\pi}{2}-\sqrt{2}\cos 3\pi=-(-1)-\sqrt{2}(-1)=1+\sqrt{2}>0$

Thus $f(x)$ has minimum values for $x=\frac{\pi}{2}$ and $x=\frac{3\pi}{2}$

As $f''\left(\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}-\sqrt{2}\cos\frac{\pi}{2}=-\frac{1}{\sqrt{2}}-\sqrt{2}\,.\,0=-\frac{1}{\sqrt{2}}<0$

and $f''\left(\frac{3\pi}{4}\right)=-\sin\frac{3\pi}{4}-\sqrt{2}\cos\frac{3\pi}{2}=-\frac{1}{\sqrt{2}}-\sqrt{2}\,.\,0=-\frac{1}{\sqrt{2}}<0$

Thus $f(x)$ has minimum values for $x=\frac{\pi}{4}$ and $x=\frac{3\pi}{4}$

## EXERCISE 2.9

**1.** Determine the intervals in which $f$ is increasing or decreasing for the domain mentioned in each case.

(i) $f(x) = \sin x$ ; $x\in(-\pi,\pi)$

(ii) $f(x) = \cos x$ ; $x\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

(iii) $f(x) = 4-x^2$ ; $x\in(-2,2)$

(iv) $f(x) = x^2+3x+2$ ; $x\in(-4,1)$

**2.** Find the extreme values for the following functions defined as:

(i) $f(x) = 1-x^3$ (ii) $f(x) = x^2-x-2$

(iii) $f(x) = 5x^2-6x+2$ (iv) $f(x) = 3x^2$

(v) $f(x) = 3x^2-4x+5$ (vi) $f(x) = 2x^3-2x^2-36x+3$

(vii) $f(x) = x^4 - 4x^2$ (viii) $f(x) = (x-2)^2(x-1)$

(ix) $f(x) = 5 + 3x - x^3$

**3.** Find the maximum and minimum values of the function defined by the following equation occurring in the interval $[0, 2\pi]$

$f(x) = \sin x + \cos x.$

**4.** Show that $y = \dfrac{\ln x}{x}$ has maximum value at $x = e$.

**5.** Show that $y = x^x$ has a minimum value at $x = \dfrac{1}{e}$.

**Application of Maxima and Minima**

Now we apply the concept of maxima and minima to the practical problems. We first form the functional relation of the form *y* = *f*(*x*) from the given information and find the maximum or minimum value of *f* as required. Here we solve some examples relating to maxima and minima problems.

**Example 1: Find two positive integers whose sum is 9 and the product of one with the square of the other will be maximum.**

**Solution:** Let $x$ and $9-x$ be the two required positive integers such that

$x(9-x)^2$ will be maximum.

Let $f(x) = x(9-x)^2$. Then

$$f'(x) = 1.(9-x)^2 + x.2(9-x)\times(-1)$$
$$= (9-x)[9-x-2x] = (9-x)(9-3x) = 3(9-x)(3-x)$$
$$f'(x) = 0 \Rightarrow 3(9-x)(3-x) = 0 \Rightarrow x = 9 \text{ or } x = 3$$

In this case $x = 9$ is not possible because

$9 - x = 9 - 9 = 0$ which is not positive integer.

$$f''(x) = 3[(-1)(3-x) + (9-x)\times(-1)] = 3[-3+x-9+x]$$





$$= 3[2x-12] = 6(x-6)$$

As $f''(3) = 6(3-6) = 6(-3) = -18$ which is negative.

Thus $f(x)$ gives the maximum value if $x = 3$, so the other positive integer is 6 because 9 – 3 = 6.

**Example 2: What are the dimensions of a box of a square base having largest volume if the sum of one side of the base and its height is 12 cm.**

**Solution:** Let the length of one side of the base (in cm) be $x$ and the height of the box (in cm) be h, then $V = x^2 h$

It is given that $x + h = 12 \quad \Rightarrow h = 12 - x$

Thus $V = x^2(12-x)$ and

$$\frac{dV}{dx} = 2x(12-x) + x^2(-1) = 24x - 3x^2 = 3x(8-x)$$

$\dfrac{dV}{dx} = 0 \Rightarrow 3x(8-x) = 0$. In this case x cannot be zero,

so $8 - x = 0 \Rightarrow x = 8.$

$\dfrac{d^2V}{dx^2} = 24 - 6x$ which is negative for $x = 8$

Thus *V* is maximum if *x* = 8*(cm)* and *h* = 12 – 8 = 4*(cm)*

**Example 3: The perimeter of a triangle is 20 centimetres. If one side is of length 8 centimetres, what are lengths of the other two sides for maximum area of the triangle?**

**Solution:** Let the length of one unknown side (in cm) be $x$, then the length of the other unknown side (in cm) will be $20 - x - 8 = 12 - x$.

Let *y* denote the square of the area of the triangle, then we have

$y = 10(10-8)(10-x)(10-12+x)$ $\left(\because s = \dfrac{20}{2} = 10 \text{ and area of the triangle } \sqrt{s(s-a)(s-b)(s-c)}\right)$

$= 10.2(10-x)(x-2) = 20(-x^2 + 12x - 20)$

$$\frac{dy}{dx}=20(-2x+12)=-40(x-6)$$

$$\frac{dy}{dx}=0 \qquad \Rightarrow x=6$$

As $\frac{d^2y}{dx^2}$ is -ve, so $x=6$ gives the maximum area of the triangle.

The length of other unknown side $=12-6=6\,(\text{cm})$

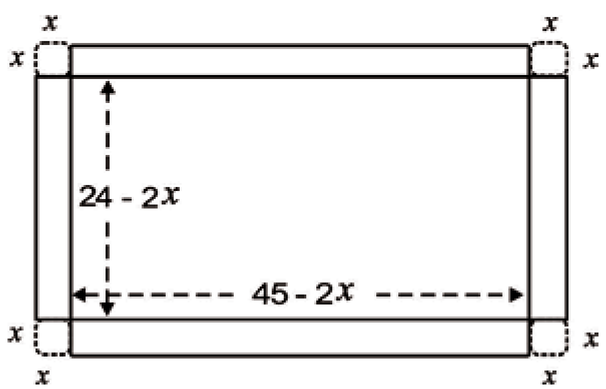
Thus the lengths of the other two sides are 6 cm and 6 cm.

**Example 4: An open box of rectangular base is to be made from 24 cm by 45cm cardboard by cutting out square sheets of equal size from each corner and bending the sides. Find the dimensions of corner squares to obtain a box having largest possible volume.**

**Solution:** Let $x$ (in cm) be the length of a side of each square sheet to be cut off from each comer of the cardboard. Then the length and breadth of the resulting box (in cm) will be $45-2x$ and $24-2x$ respectively. Obviously the height of the box (in cm) will be $x$. Thus the volume *V* of the box (in cubic cm) will be given by

$$V=x(24-2x)(45-2x)=2x(12-x)(45-2x)$$

$$=2x(540-69x+2x^2)$$




and $$\frac{dV}{dx}=2\left[1.(2x^2-69x+540)+x(4x-69)\right]$$

$$=2(6x^2-138x+540)$$

$$=12\left[x^2-23x+90\right]=12(x-5)(x-18)$$

$$\frac{dV}{dx}=0 \qquad \Rightarrow 12(x-5)(x-18)=0 \qquad \Rightarrow x=5 \text{ or } x=18$$

$\Rightarrow x=5\,[\because$ if $x=18$, then $12\text{-}x=12-18=-6$, that is ,

*V* is negative which is not possible]

$$\frac{d^2y}{dx^2}=12(2x-23)$$





$\frac{d^2V}{dx^2}$ is negative for $x=5$ because $12(2\times5-23)=12(-13)$

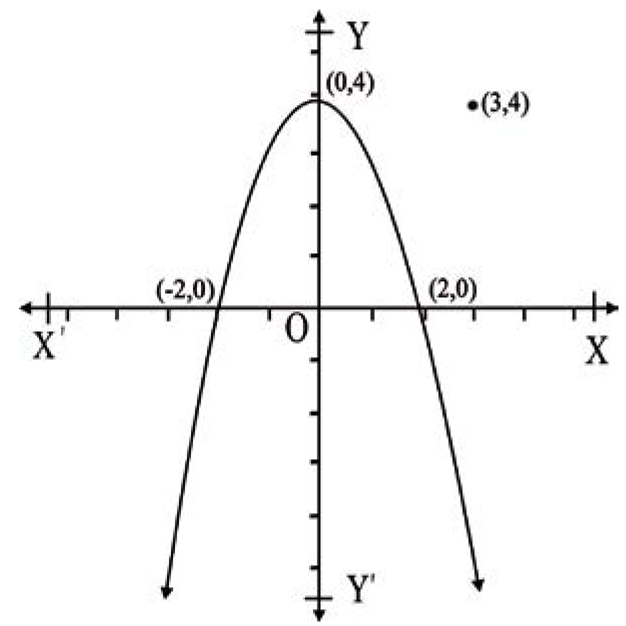
Thus V will be maximum if the length of a side of the corner square to be cut off is 5 cm.

**Example 5: Find the point on the graph of the curve $y = 4 - x^2$ which is closest to the point (3, 4).**

**Solution:** Let $l$ be distance between a point $(x,y)$ on the curve $y=4-x^2$ and the point (3 , 4). Then $l=\sqrt{(x-3)^2+(y-4)^2}$

$$=\sqrt{(x-3)^2+(4-x^2-4)^2} \qquad (\because (x,y) \text{ is on the curve } y=4-x^2)$$

$$=\sqrt{(x-3)^2+x^4}$$




Now we find $x$ for which $l$ is minimum.

$$\frac{dl}{dx}=\frac{1}{2.\sqrt{(x-3)^2+x^4}}.\left[\left(2(x-3)+4x^3\right)\right]$$

$$=\frac{1}{2l}.2(2x^3+x-3)$$

$$=\frac{1}{l}\left(2x^3+x-3\right)$$

$$=\frac{1}{l}(x-1)\left(2x^2+x-3\right)$$

$$\frac{dl}{dx}=0 \quad \Rightarrow \frac{1}{l}(x-1)\left(2x^2+2x+3\right)=0 \quad \Rightarrow x-1=0 \quad \text{or} \quad 2x^2+2x+3=0$$

$$\Rightarrow x=1 \quad \left(\because 2x^2+2x+3=0\right)$$

$l$ is positive for $1-\varepsilon$ and $1+\varepsilon$ where $\varepsilon$ is very very small positive real number.

Also $2x^2+2x+3=2\left(x^2+x+\frac{1}{4}\right)+\frac{5}{2}=2\left(x+\frac{1}{2}\right)^2+\frac{5}{2}$ is positive,for $x=1-\varepsilon$

and $x=1+\varepsilon$

The sign of $\frac{dl}{dx}$ depends on the factor $x-1$.

$x-1$ is negative for $x=1-\varepsilon$ because $x-1=1-\varepsilon-1=-\varepsilon$ ..... (i)

$x-1$ is positive for $x=1+\varepsilon$ because $x-1=1+\varepsilon-1=\varepsilon$ ..... (ii)

From (i) and (ii), we conclude that $\frac{dl}{dx}$ changes sign from –ve to +ve at $x$ = 1.

Thus $l$ has a minimum value at $x$ = 1.

Putting $x=1$ in $y=4-x^2$, we get the $y$-coordinate of the required point which is $4-(1)^2=3$

Hence the required point on the curve is (1, 3).

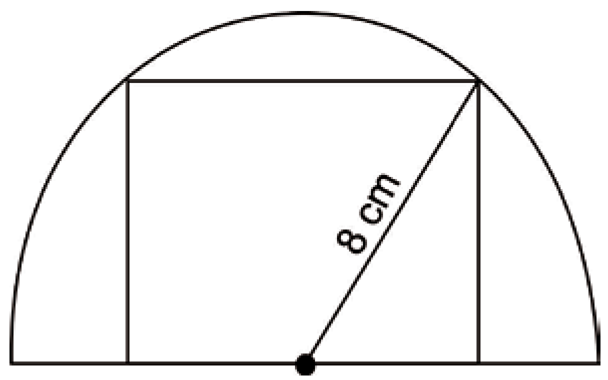
## EXERCISE 2.10

1. Find two positive integers whose sum is 30 and their product will be maximum.
2. Divide 20 into two parts so that the sum of their squares will be minimum.
3. Find two positive integers whose sum is 12 and the product of one with the square of the other will be maximum.
4. The perimeter of a triangle is 16 centimetres. If one side is of length 6 cm, what are length of the other sides for maximum area of the triangle?
5. Find the dimensions of a rectangle of largest area having perimeter 120 centimetres.





6. Find the lengths of the sides of a variable rectangle having area $36\,\text{cm}^2$ when its perimeter is minimum.
7. A box with a square base and open top is to have a volume of 4 cubic dm. Find the dimensions of the box which will require the least material.
8. Find the dimensions of a rectangular garden having perimeter 80 metres if its area is to be maximum.
9. An open tank of square base of side $x$ and vertical sides is to be constructed to contain a given quantity of water. Find the depth in terms of $x$ if the expense of lining the inside of the tank with lead will be least.
10. Find the dimensions of the rectangle of maximum area which fits inside the semi-circle of radius 8 cm as shown in the figure.




11. Find the point on the curve $y=x^2-1$ that is closest to the point (3, –1).
12. Find the point on the curve $y=x^2+1$ that is closest to the point (18, 1).