CHAPTER

5

Linear Inequalities and Linear Programming

Animation 5.1: Feasible Solution Set
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5.1 INTRODUCTION

Many real life problems involve linear inequalities. Here we shall consider those problems (relating to trade, industry and agriculture etc.) which involve systems of linear inequalities in two variables. Linear inequalities in such problems are used to prescribe limitations or restrictions on allocation of available resources (material, capital, machine capacities, labour hours, land etc.). In this chapter, our main goal will be to optimize (maximize or minimize) a quantity under consideration subject to certain restrictions.

The method under our discussion is called the linear programming method and it involves solutions of certain linear inequalities.

5.2 LINEAR INEQUALITIES

Inequalities are expressed by the following four symbols;

> (greater than); < (less than); ≥ (greater than or equal to); ≤ (less than or equal to)

For example (i) $ax < b$ (ii) $ax + b \geq c$ (iii) $ax + by > c$ (iv) $ax + by \leq c$ are inequalities. Inequalities (i) and (ii) are in one variable while inequalities (iii) and (iv) are in two variables.

The following operations will not affect the order (or sense) of inequality while changing it to simpler equivalent form:

(i) Adding or subtracting a constant to each side of it.

(ii) Multiplying or dividing each side of it by a positive constant.

Note that the order (or sense) of an inequality is changed by multiplying or dividing its each side by a negative constant.

Now for revision we consider inequality, $x < \frac{3}{2}$ (A)

All real numbers $< \frac{3}{2}$ are in the solution set of (A).

Thus the interval $\left(-\infty, \frac{3}{2}\right)$ or $-\infty < x < \frac{3}{2}$ is the solution set of the inequality (A) which is shown in the figure 5.21

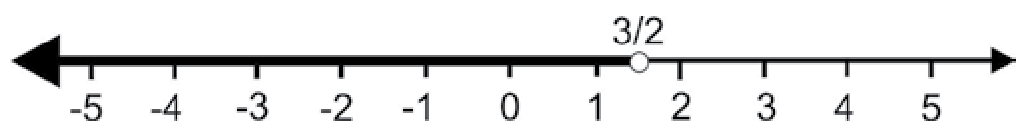

Fig. 5.21

We conclude that the solution set of an inequality consists of all solutions of the inequality.

5.2.1 Graphing of A Linear Inequality in Two Variables

Generally a linear inequality in two variables x and y can be one of the following forms:

$ax + by < c$; $ax + by > c$; $ax + by \leq c$; $ax + by \geq c$

where a, b, c are constants and a, b are not both zero.

We know that the graph of linear equation of the form $ax + by = c$ is a line which divides the plane into two disjoint regions as stated below:

(1) The set of ordered pairs (x, y) such that $ax + by < c$

(2) The set of ordered pairs (x, y) such that $ax + by > c$

The regions (1) and (2) are called **half planes** and the line $ax + by = c$ is called the **boundary of each half plane**.

Note that a **vertical line** divides the plane into **left and right half planes** while a **non-vertical line** divides the plane into **upper and lower half planes**.

A solution of a linear inequality in x and y is an ordered pair of numbers which satisfies the inequality.

For example, the ordered pair (1, 1) is a solution of the inequality $x + 2y < 6$ because $1 + 2(1) = 3 < 6$ which is true.

There are infinitely many ordered pairs that satisfy the inequality $x + 2y < 6$, so its graph will be a half plane.

Note that the linear equation $ax + by = c$ is called **"associated or corresponding equation"** of each of the above mentioned inequalities.

Procedure for Graphing a linear Inequality in two Variables

(i) The corresponding equation of the inequality is first graphed by using 'dashes' if the inequality involves the symbols > or < and a solid line is drawn if the inequality involves the symbols ≥ or ≤.

(ii) A test point (not on the graph of the corresponding equation) is chosen which determines that the half plane is on which side of the boundary line.

Example 1. Graph the inequality $x + 2y < 6$.

Solution. The associated equation of the inequality

$$x + 2y < 6 \qquad \text{(i)}$$

is

$$x + 2y = 6 \qquad \text{(ii)}$$

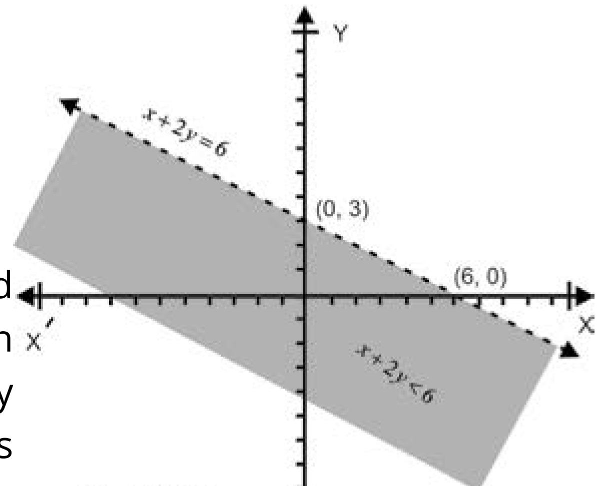

Fig. 5.22(a)

The line (ii) intersects the x-axis and y-axis at (6, 0) and (0. 3) respectively. As no point of the line (ii) is a solution of the inequality (i), so the graph of the line (ii) is shown by using dashes. We take $O(0, 0)$ as a test point because it is not on the line (ii).

Substituting $x = 0$, $y = 0$ in the expression $x + 2y$ gives $0 – 2(0) = 0 < 6$, so the point (0, 0) satisfies the inequality (i).

Any other point below the line (ii) satisfies the inequality (i), that is all points in the half plane containing the point (0, 0) satisfy the inequality (i).

Thus the graph of the solution set of inequality (i) is the a region on the origin-side of the line (ii), that is, the region below the line (ii). A portion of the open halfplane below the line (ii) is shown as shaded region in figure 5.22(a)

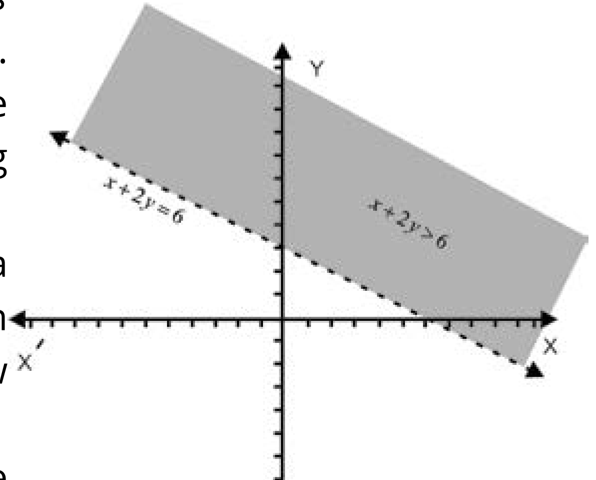

Fig. 5.22(b)

All points above the dashed line satisfy the inequality $x + 2y > 6$ (iii)

A portion of the open half plane above the line (ii) is shown by shading in figure 5.22(b)

Note: 1. The graph of the inequality $x + 2y \le 6$ ..(iv) includes the graph of the line (ii),' so the open half-plane below the line (ii) including the graph of the line (ii) is the graph of the inequality (iv). A portion of the graph of the inequality (iv) is shown by shading in figure 5.22(c)

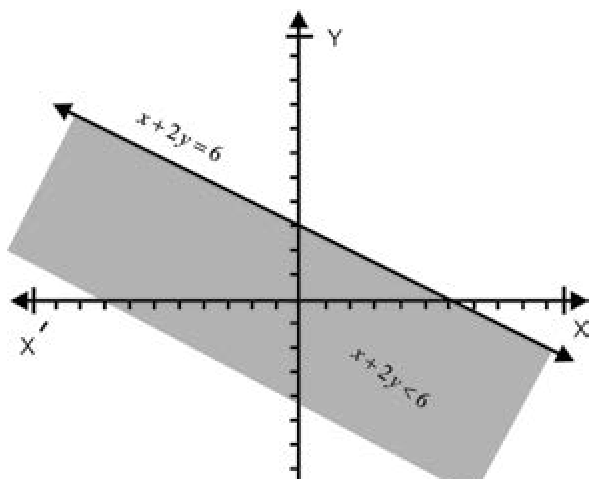

Fig. 5.22(c)

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Note: 2 All points on the line (ii) and above the line (ii) satisfy the inequality $x + 2y \ge 6$ .... (v). This means that the solution set of the inequality (v) consists of all points above the line (ii) and all points on the lines (ii). The graph of the inequality (v) is partially shown as shaded region in figure 5.22(d)

Note: 3 that the graphs of $x + 2y \le 6$ and $x + 2y \ge 6$ are closed half planes.

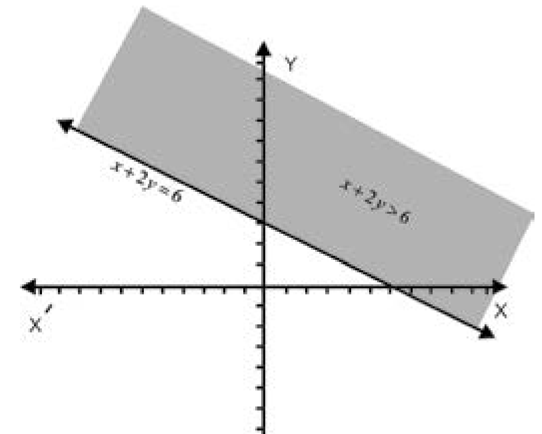


Fig. 5.22(d)

Example 2. Graph the following linear inequalities in xy-plane;

(i) $2x \ge -3$ (ii) $y \le 2$

Solution. The inequality (i) in xy-plane is considered as $2x + 0y \ge -3$ and its solution set consists of all point (x, y) such that $x, y \in R$ and $x \ge -\dfrac{3}{2}$

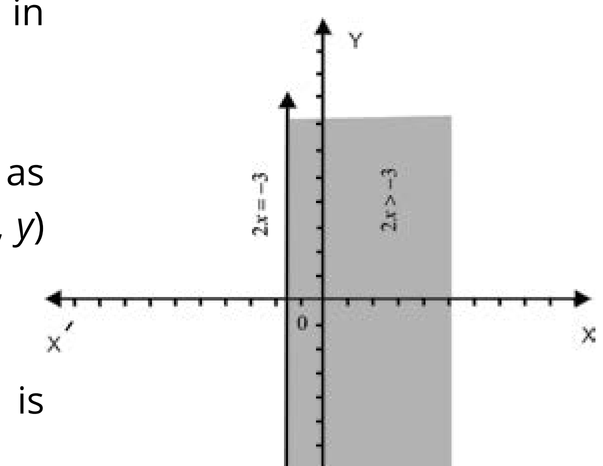


Fig. 5.23(a)

The corresponding equation of the inequality (i) is

$$2x = -3 \qquad (1)$$

which a vertical line (parallel to the y-axis) and its graph is drawn in figure 5.23(a).

The graph of the inequality $2x > -3$ is the open half plane to the right of the line (1).

Thus the graph of $2x \ge -3$ is the closed half-plane to the right of the line (1).

(ii) The associated equation of the inequality (ii) is

$$y = 2 \qquad (2)$$

which is a horizontal line (parallel to the x-axis) and its graph is shown in figure 5.23(b) Here the solution set of the inequality $y < 2$ is the open half plane below the boundary line $y = 2$. Thus the graph of $y \le 2$ consists of the boundary line and the open half plane below it.

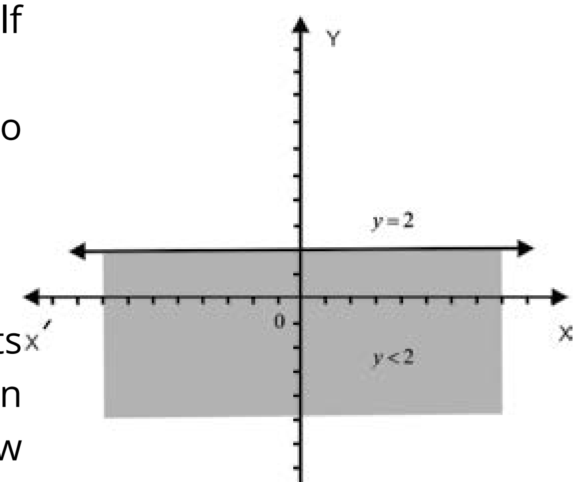


Fig. 5.23(b)

Note that the intersection of graphs of $2x \geq -3$ and $y \leq 2$ is partially shown in the adjoining figure 5.23(c).

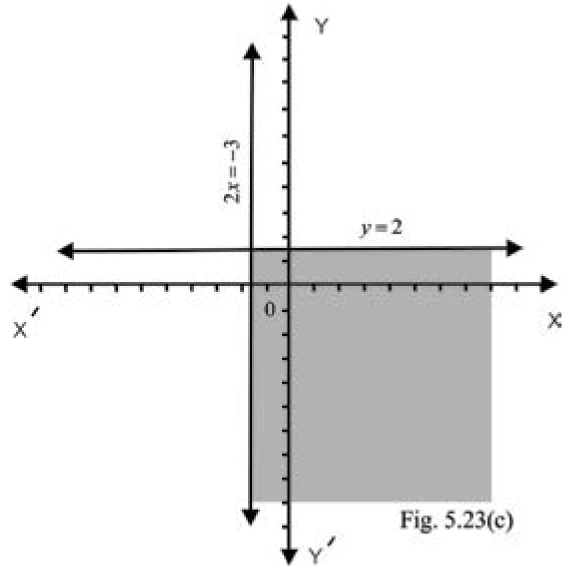


Fig. 5.23(c)

5.3 REGION BOUNDED BY 2 OR 3 SIMULTANEOUS INEQUALITIES

The graph of a system of inequalities consists of the set of all ordered pairs (x, y) in the xy-plane which simultaneously satisfy all the inequalities in the system. Find the graph of such a system, we draw the graph of each inequality in the system on the same coordinate axes and then take intersection of all the graphs. The common region so obtained is called the solution region for the system of inequalities.

Example 1: **Graph the system of inequalities**

$$x - 2y \leq 6$$
$$2x + y \geq 2$$

Solution.

The graph of the line $x - 2y = 6$ is drawn by joining the point (6, 0) and (0, –3). The point (0,0) satisfy the inequality $x - 2y < 6$ because $0 - 2(0) = 0 < 6$. Thus the graph of $x - 2y \leq 6$ is the upper half-plane including the graph of the line $x - 2y = 6$. The closed half-plane is partially shown by shading in figure 5.31(a).

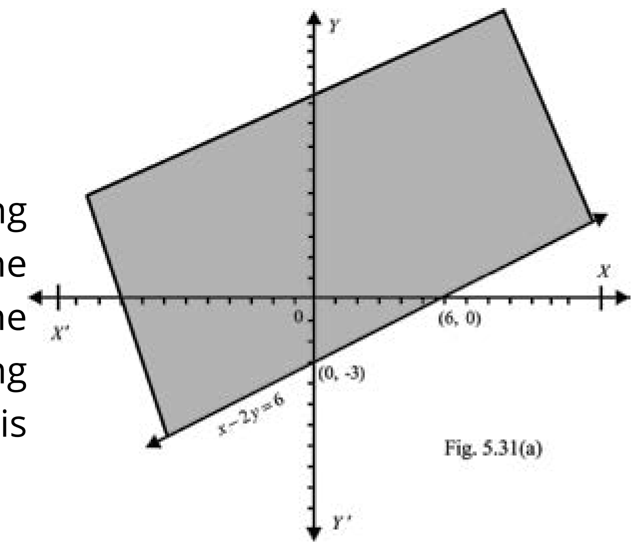


Fig. 5.31(a)

We draw the graph of the line $2x + y = 2$ joining the points (1, 0) and (0, 2). The point (0, 0) does not satisfy the inequality $2x + y > 2$ because $2(0) + 0 = 0 \not> 2$. Thus the graph of the inequality $2x + y \geq 2$ is the closed half-plane not on the origin-side of the line $2x + y = 2$.

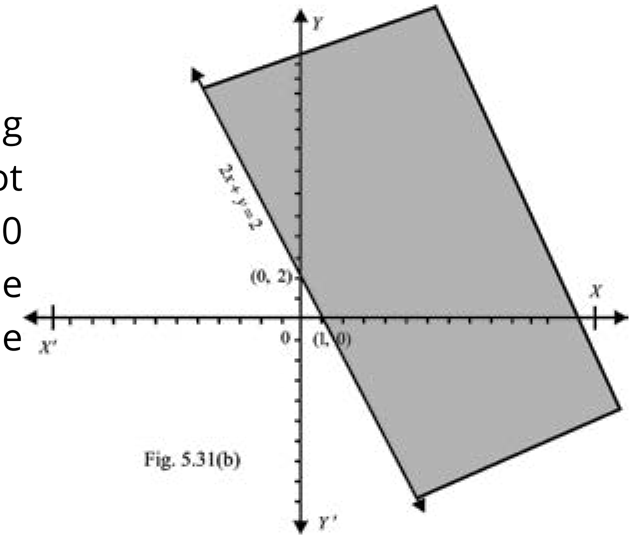


Fig. 5.31(b)

Thus the closed half-plane is shown partially as a shaded region in figure 5.31(b). The solution region of the given system of inequalities is the intersection of the graphs indicated in figures 5.31(a) and 5.31(b) and is shown as shaded region in figure 5.31(c).

The intersection point (2, – 2) can be found by solving the equations $x - 2y = 6$ and $2x + y = 2$.

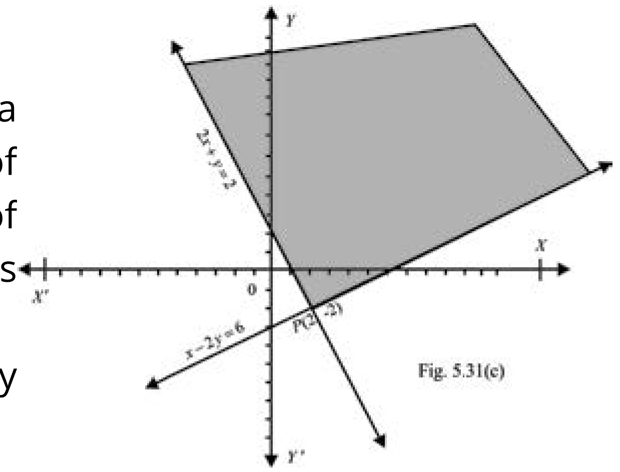
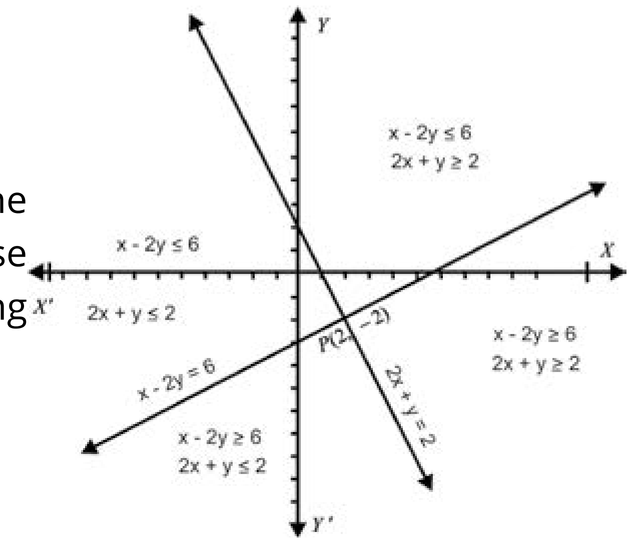


Fig. 5.31(c)

Note that the line $x - 2y = 6$ and $2x + y = 2$ divide the xy-plane into four region bounded by these lines. These four (bounded) regions are displayed in the adjoining figure.



Example 2. **Graph the solution region for the following system of inequalities:**

$x - 2y \le 6, \quad 2x + y > 2, \quad x + 2y \ge 10$

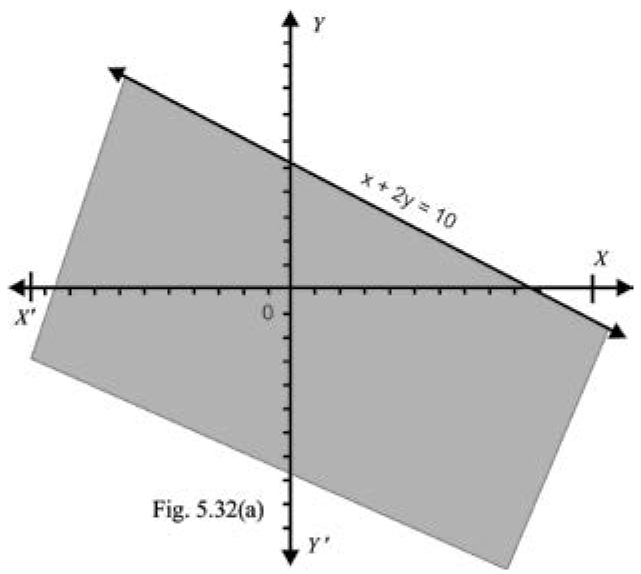


Fig. 5.32(a)

Solution: The graph of the inequalities $x - 2y \le 6$ and $2x + y \ge 2$ have already drawn in figure 5.31(a) and 5.31(b) and their intersection is partially shown as a shaded region in figure 5.31(c) of the example 1 Art (5.3). Following the procedure of the example 1 of Art (5.3) the graph of the inequality $x + 2y \le 10$ is shown partially in the figure 5.32(a).

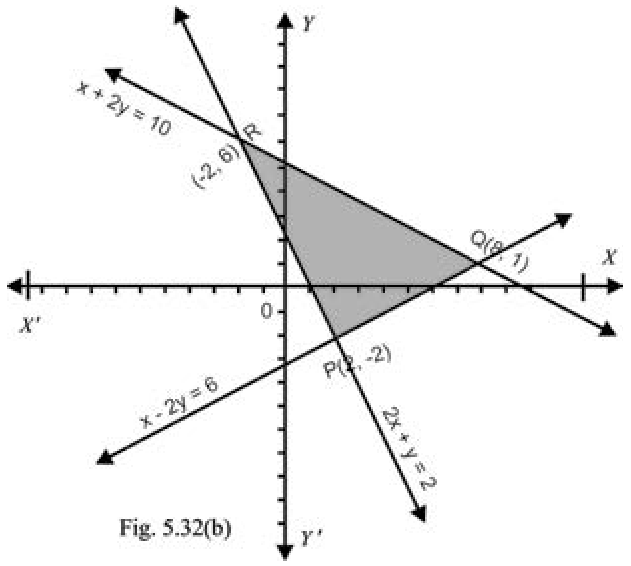


Fig. 5.32(b)

The intersection of three graphs is the required solution region which is the shaded triangular region *PQR* (including its sides) shown partially in the figure 5.32(b).

Now we define a corner point of a solution region.

DEFINITION:

A point of a solution region where two of its boundary lines intersect, is called a **corner point** or **vertex** of the solution region.

Such points play a useful role while solving linear programming problems. In example 2, the following three corner points are obtained by corresponding equations (of linear inequalities given in the example 2) in pairs.

Corresponding lines of inequalities:		Corner Points
$x - 2y = 6,$	$2x + y = 2$	$P(2, -2)$
$x - 2y = 6,$	$x + 2y = 10$	$Q(8, 1)$
$2x + y = 2,$	$x + 2y = 10$	$R(-2, 6)$

Example 3. **Graph the following systems of inequalities.**

(i) $2x + y \ge 2$, $x + 2y \le 10$, $y \ge 0$

(ii) $2x + y \ge 2$, $x + 2y \le 10$, $x \ge 0$

(iii) $2x + y \ge 2$, $x + 2y \le 10$, $x \ge 0, \ y \ge 0$

Solution:

(i) The corresponding equations of the inequalities

$2x + y \ge 2$ and $x + 2y \le 10$ are

$2x + y = 2 \quad (I)$ and $x + 2y = 10 \quad (II)$

For the partial graph of $2x + y \ge 2$ see figure 5.31(b) of the example 1 and the graph of the inequality $x + 2y \le 10$ is partially shown in figure 5.32(a) of the example 2.

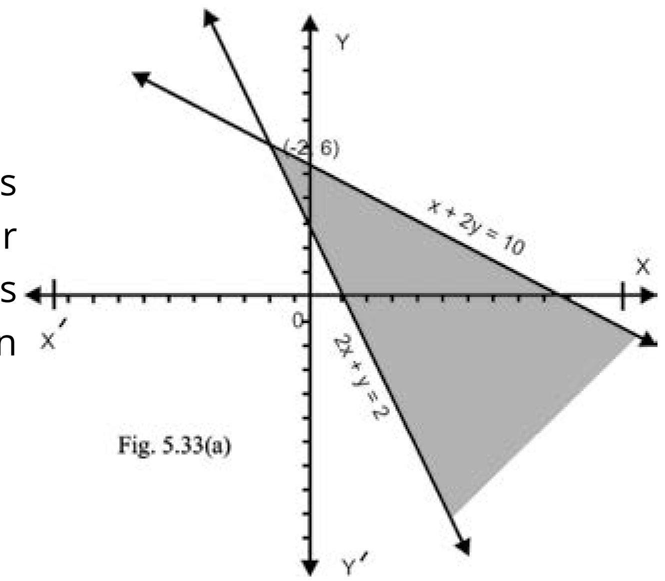


Fig. 5.33(a)

The solution region of the inequalities $2x + y \ge 2$ and $x + 2y \le 10$ is the intersection of their individual graphs. The common region of the graphs of inequalities is partially shown as a shaded region in figure 5.33(a).

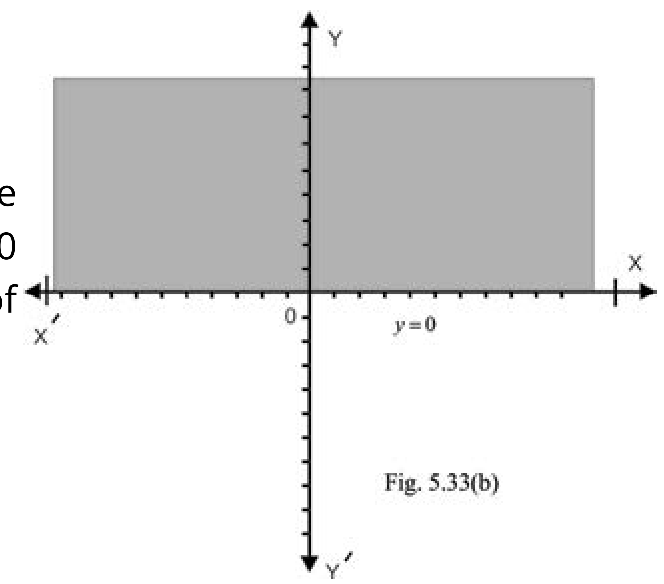


Fig. 5.33(b)

The graph of $y \ge 0$ is the upper half plane including the graph of the corresponding line $y = 0$ (the x-axis) of the linear inequality $y \ge 0$. The graph of $y \ge 0$ is partially displayed in figure 5.33(b).

The solution region of the system of inequalities in (i) is the intersection of the graphs shown in figure 5.33(a) and 5.33(b). This solution region is displayed in figure 5.33(c).

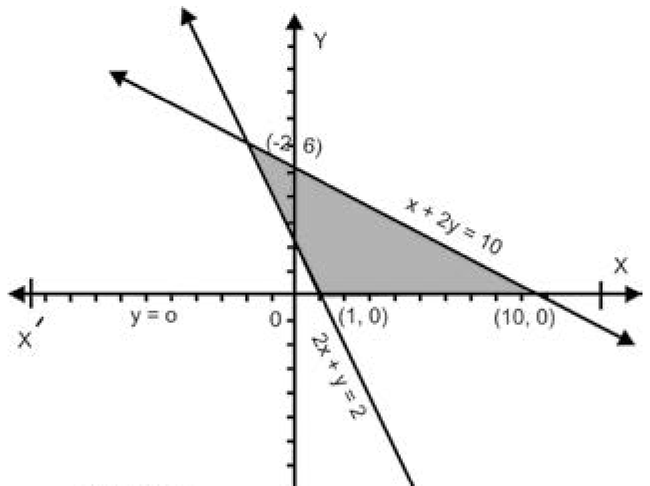


Fig. 5.33(c)

(ii) See figure 5.33(a) for the graphs of the inequalities $2x + y \geq 2$ and $x + 2y \leq 10$.

The graph of $x \geq 0$ consists of the open half-plane to the right of the corresponding line $x = 0$ (y-axis) of the inequality $x \geq 0$ and its graph. See figure 5.34(a).

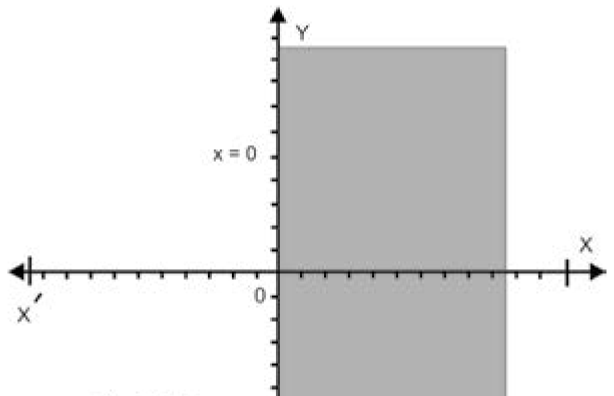


Fig. 5.34(a)

Thus the solution region of the inequalities in (ii) is partially shown in figure 5.34(b). This region is the intersection of graphs in figure 5.33(a) and 5.34(a).

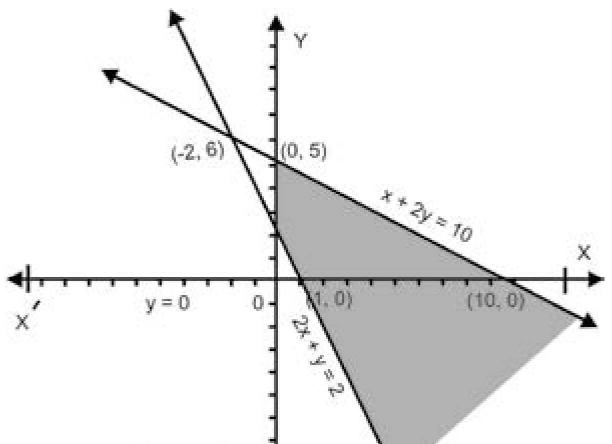


Fig. 5.34(b)

(iii) The graphs of the system of inequalities in (iii) are drawn in the solution of (i) and (ii). The solution region in this case, is shown as shaded region *ABCD* in figure 5.34. (c).

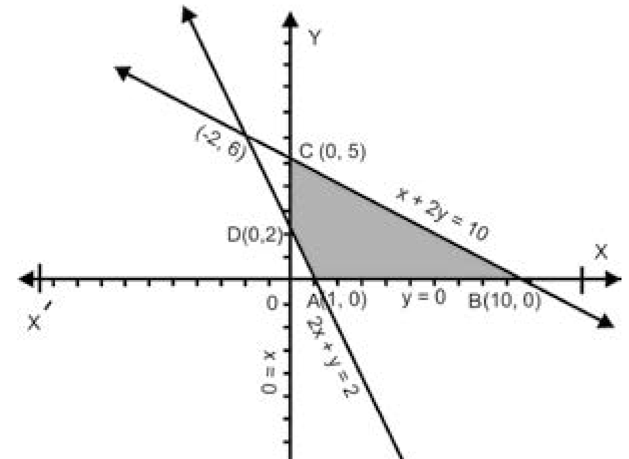


Fig. 5.34(c)

EXERCISE 5.1

1. Graph the solution set of each of the following linear inequality in xy-plane:

(i) $2x + y \leq 6$ (ii) $3x + 7y \geq 21$ (iii) $3x - 2y \geq 6$
(iv) $5x - 4y \leq 20$ (v) $2x + 1 \geq 0$ (vi) $3y - 4 \leq 0$

2. Indicate the solution set of the following systems of linear inequalities by shading:

(i) $2x - 3y \leq 6$, $2x + 3y \leq 12$
(ii) $x + y \geq 5$, $-y + x \leq 1$
(iii) $3x + 7y \geq 21$, $x - y \leq 2$
(iv) $4x - 3y \leq 12$, $x \geq -\dfrac{3}{2}$
(v) $3x + 7y \geq 21$, $y \leq 4$

3. Indicate the solution region of the following systems of linear inequalities by shading:

(i) $2x - 3y \leq 6$, $2x + 3y \leq 12$, $y \geq 0$
(ii) $x + y \leq 5$, $y - 2x \leq 2$, $x \geq 0$
(iii) $x + y \geq 5$, $x - y \geq 1$, $y \geq 0$
(iv) $3x + 7y \leq 21$, $x - y \leq 2$, $x \geq 0$
(v) $3x + 7y \leq 21$, $x - y \leq 2$, $y \geq 0$
(vi) $3x + 7y \leq 21$, $2x - y \geq -3$, $x \geq 0$

4. Graph the solution region of the following system of linear inequalities and find the corner points in each case.

(i)	$2x - 3y \le 6$	(ii)	$x + y \le 5$	(iii)	$3x + 7y \le 21$
	$2x + 3y \le 12$		$-2x + y \le 2$		$2x - y \le -3$
	$x \ge 0$		$y \ge 0$		$y \ge 0$
(iv)	$3x + 2y \ge 6$	(v)	$5x + 7y \le 35$	(vi)	$5x + 7y \le 35$
	$x + 3y \le 6$		$-x + 3y \le 3$		$x - 2y \le 2$
	$y \ge 0$		$x \ge 0$		$x \ge 0$

5. Graph the solution region of the following system of linear inequalities by shading.

(i)	$3x - 4y \le 12$	(ii)	$3x - 4y \le 12$	(iii)	$2x + y \le 4$
	$3x + 2y \ge 3$		$x + 2y \le 6$		$2x - 3y \ge 12$
	$x + 2y \le 9$		$x + y \ge 1$		$x + 2y \le 6$
(iv)	$2x + y \le 10$	(v)	$2x + 3y \le 18$	(vi)	$3x - 2y \ge 3$
	$x + y \le 7$		$2x + y \le 10$		$x + 4y \le 12$
	$-2x + y \le 4$		$-2x + y \le 2$		$3x + y \le 12$

5.4 PROBLEM CONSTRAINTS

In the beginning we described that linear inequalities prescribe limitations and restrictions on allocation of available sources. While tackling a certain problem from every day life each linear inequality concerning the problem is named as **problem constraint**. The system of linear inequalities involved in the problem concerned are called **problem constraints**. The variables used in the system of linear inequalities relating to the problems of every day life are non-negative and are called **non-negative constraints**. These non-negative constraints play an important role for taking decision. So these variables are called **decision variables**.

5.5 Feasible solution set

We see that solution region of the inequalities in example 2 of Art 5.3 is not within the first quadrant. If the nonnegative constraints $x \ge 0$ and $y \ge 0$ are included with the system of inequalities given in the example 2, then the solution region is restricted to the first quadrant.

It is the polygonal region *ABCDE* (including its sides) as shown in the figure 5.51.

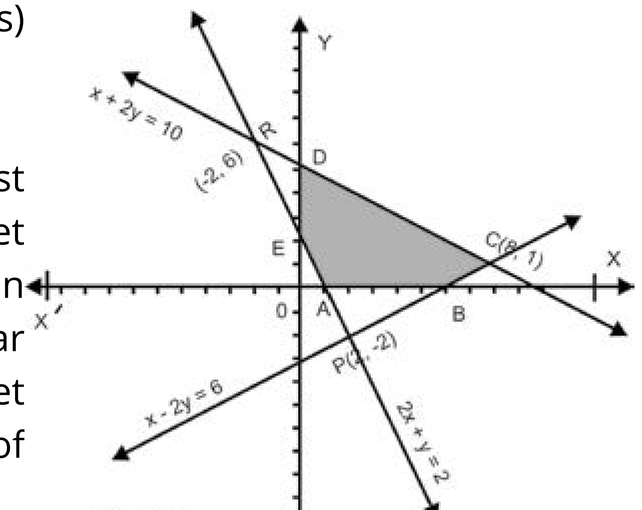


Fig. 5.51

Such a region (which is restricted to the first quadrant) is referred to as a **feasible region** for the set of given constraints. Each point of the feasible region is called a **feasible solution** of the system of linear inequalities (or for the set of a given constraints). A set consisting of all the feasible solutions of the system of linear inequalities is called a **feasible solution set**.

Example 1. Graph the feasible region and find the corner points for the following system of inequalities (or subject to the following constraints).

$$x - y \le 3$$
$$x + 2y \le 6, \quad x \ge 0, \quad y \ge 0$$

Solution: The associated equations for the inequalities

$x - y \le 3$ (i) and $x + 2y \le 6$ (ii)

are $x - y = 3$ (1) and $x + 2y = 6$ (2)

As the point (3, 0) and (0, –3) are on the line (1), so the graph of $x - y = 3$ is drawn by joining the points (3, 0) and (0, –3) by solid line.

Similarly line (2) is graphed by joining the points (6, 0) and (0, 3) by solid line. For $x = 0$ and $y = 0$, we have;

$$0 - 0 = 0 \le 3 \text{ and } 0 + 2(0) = 0 \le 6,$$

so both the closed half-planes are on the origin sides of the lines (1) and (2). The intersection of these closed half-planes is partially displayed as shaded region in figure 5.52(a).

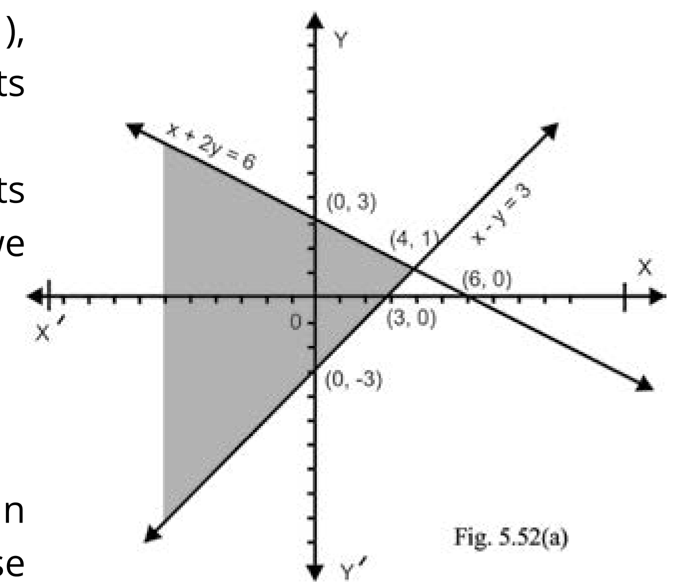


Fig. 5.52(a)

For the graph of $y \geq 0$, see figure 5.33(b) of the example 3 of art 5.3.

The intersection of graphs shown in figures 5.52(a) and 5.33(b) is partially graphed as shaded region in figure 5.52(b).

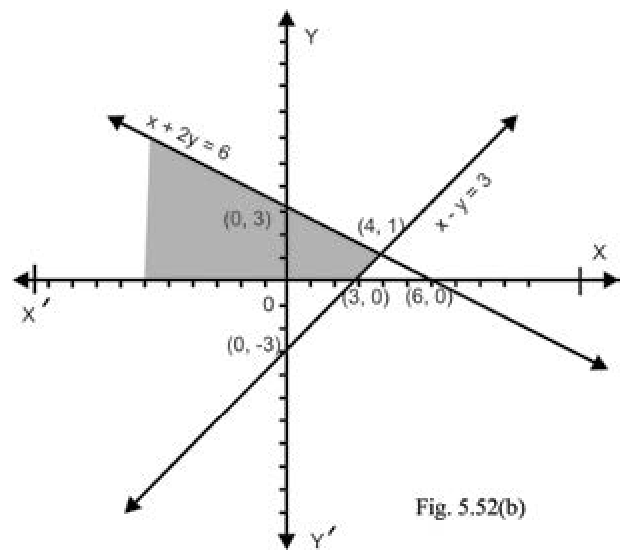


Fig. 5.52(b)

The graph of $x \geq 0$ is drawn in figure 5.34(a). The intersection of the graphs shown in figures 5.52(a) and 5.34(a) is graphed in figure 5.52(c).

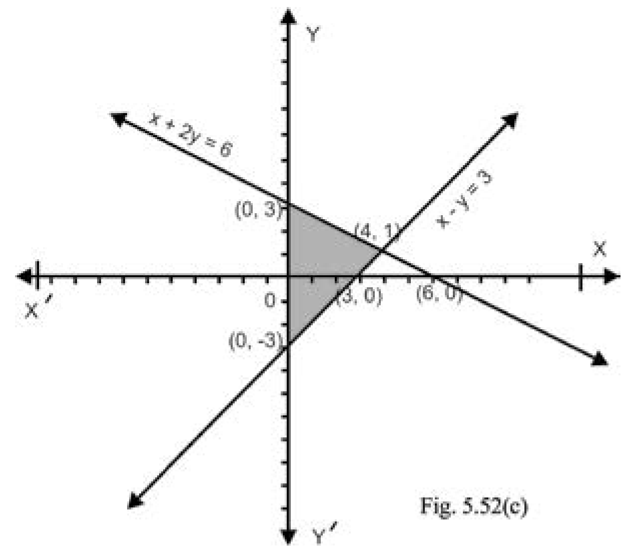


Fig. 5.52(c)

Finally the graph of the given system of linear inequalities is displayed in figure 5.52(d) which is the feasible region for the given system of linear inequalities. The points (0, 0), (3, 0), (4, 1) and (0, 3) are corner points of the feasible region.

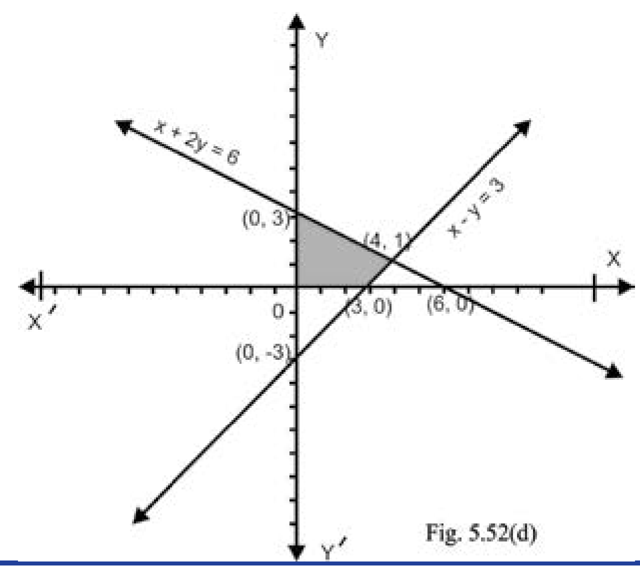


Fig. 5.52(d)

Example 2. A manufacturer wants to make two types of concrete. Each bag of Agrade concrete contains 8 kilograms of gravel (small pebbles with coarse sand) and 4 kilograms of cement while each bag of B-grade concrete contains 12 kilograms of gravel and two kilograms of cement. If there are 1920 kilograms of gravel and 480 kilograms of cement, then graph the feasible region under the given restrictions and find corner points of the feasible region.

Solution: Let x be the number of bags of A-grade concrete produced and y denote the number of bags of B-grade concrete produced, then $8x$ kilograms of gravel will be used for A-grade concrete and $12y$ kilograms of gravel will be required for B-grade concretes so $8x + 12y$ should not exceed 1920, that is,

$$8x + 12y \leq 1920$$

Similarly, the linear constraint for cement will be

$$4x + 2y \leq 480$$

Now we have to graph the feasible region for the linear constraints

$$8x + 12y \leq 1920$$

$$4x + 2y \leq 480, \quad x \geq 0, \quad y \geq 0$$

Taking the one unit along x-axis and y-axis equal to 40 we draw the graph of the feasible region required.

The shaded region of figure 5.53(a) shows the graph of $8x + 12y \leq 1920$ including the nonnegative constraints $x \geq 0$ and $y \geq 0$

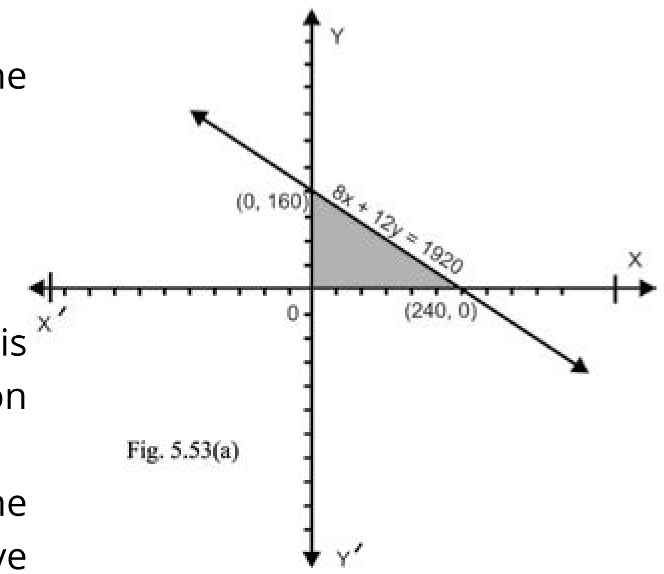


Fig. 5.53(a)

In the figure 5.53(b), the graph of $4x + 2y \leq 480$ including the non-negative constraints $x \geq 0$ and $y \geq 0$ is displayed as shaded region.

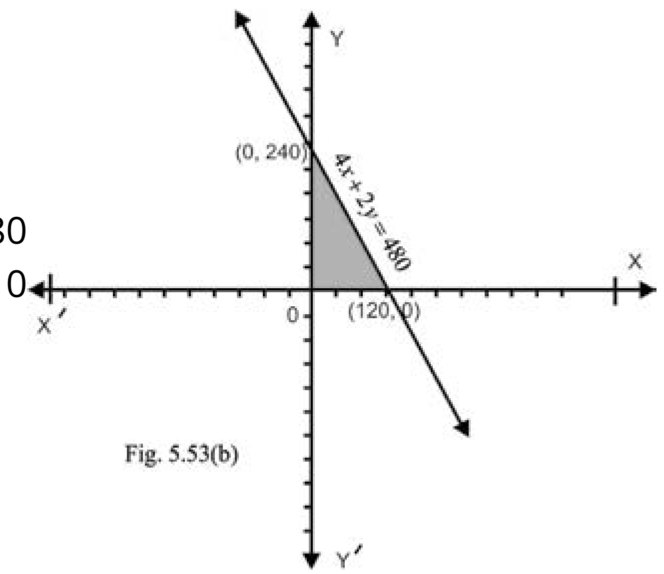


Fig. 5.53(b)

The intersection of these two graphs is shown as shaded region in figure 5.53(c), which is the feasible region for the given linear constraints.

The point (0, 0), (120, 0), (60, 120) and (0, 160) are the corner points of the feasible region.

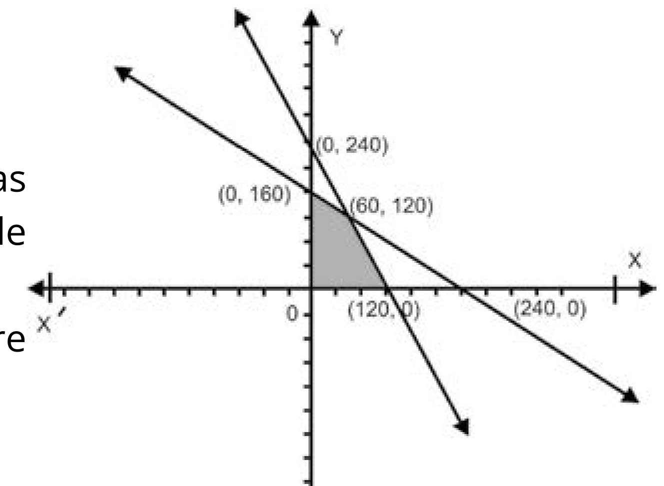


Fig. 5.53(c)

Example 3. Graph the feasible regions subject to the following constraints.

(a) $2x - 3y \leq 6$, $2x + y \geq 2$, $x \geq 0, y \geq 0$

(b) $2x - 3y \leq 6$, $2x + y \geq 2$, $x + 2y < 8$, $x \geq 0$, $y \geq 0$

Solution: The graph of $2x - 3y \leq 6$ is the closed half-plane on the origin side of $2x - 3y = 6$. The portion of the graph of system $2x - 3y \leq 6$,

$$x \geq 0, \ y \geq 0$$

is shown as shaded region in figure 5.54(a).

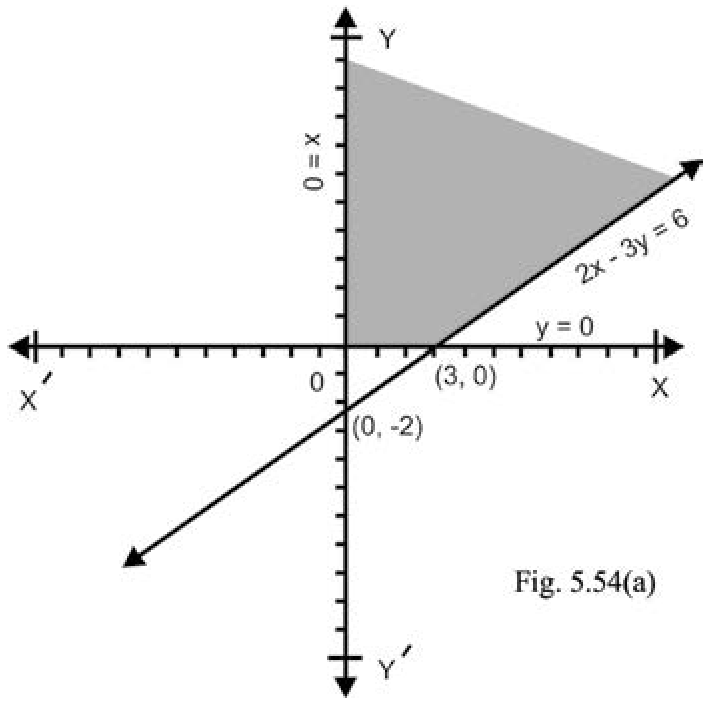


Fig. 5.54(a)

The graph of $2x + y \geq 2$ is the closed half-plane not on the origin side of $2x + y = 2$. The portion of the graph of the system $2x + y \geq 2$,

$$x \geq 0, y \geq 0$$

is displayed as shaded region in figure 5.54(b).

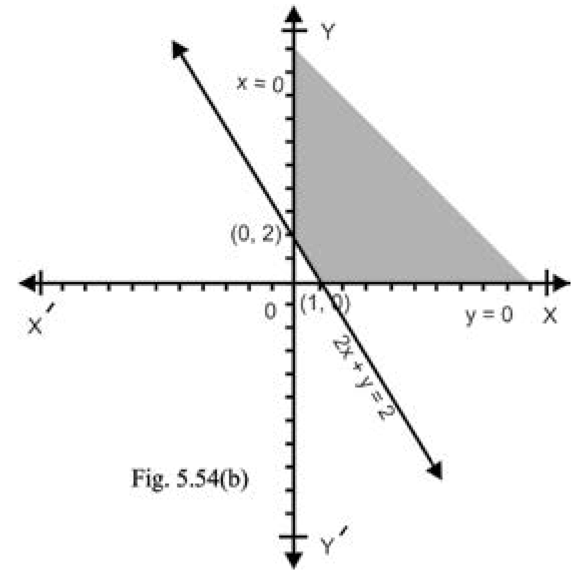


Fig. 5.54(b)

The graph of the system

$2x - 3y \leq 6$, $2x + y \leq 2$,

$x \geq 0$, $y \geq 0$

is the intersection of the graphs shown in figures 5.54(a) and 5.54(b) and it is partially displayed in figure 5.54(c) as shaded region.

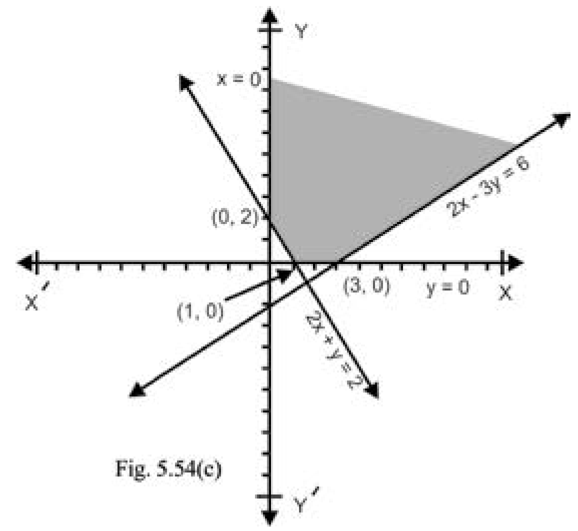


Fig. 5.54(c)

(b) The graph of system $x + 2y \leq 8$, $x \geq 0$, $y \geq 0$ is a triangular region indicated in figure 5.45(d).

Thus the graph of the system

$2x - 3y \leq 6$

$2x + y \geq 2$

$x + 2y \leq 8$ $\quad x \geq 0, y \geq 0$

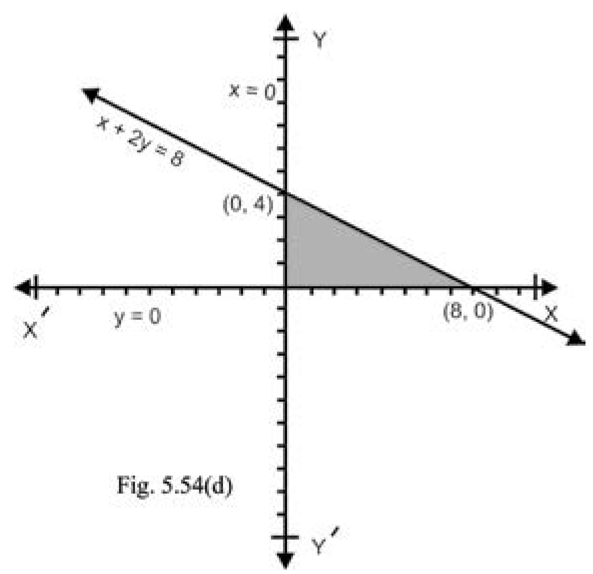


Fig. 5.54(d)

is the intersection of the graphs shown in figures 5.54(c) and 5.54(d). It is the shaded region indicated in the figure 5.54(e).

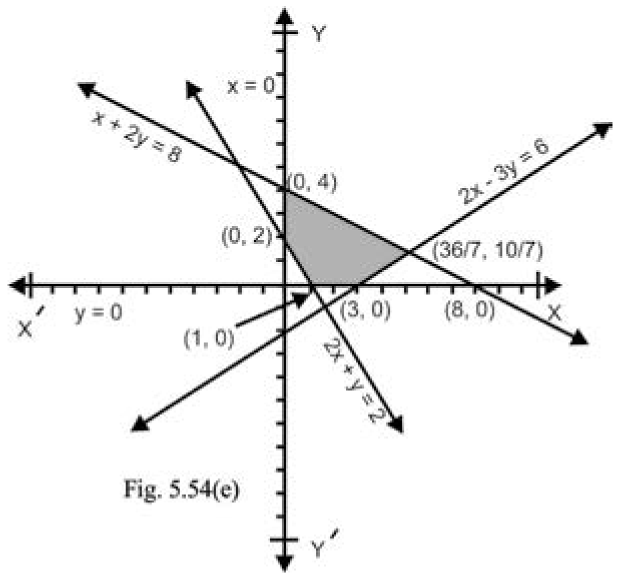
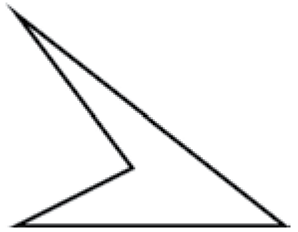


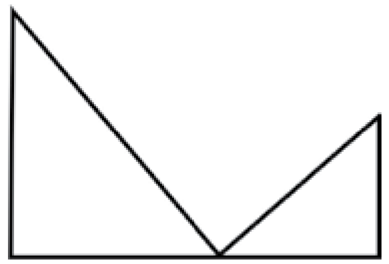
Fig. 5.54(e)

Note: The corner points of feasible region the set of constraints in (a) are (1, 0), (3, 0) and (0, 2) while the corner points of the feasible region for the set of constraints in (b) are (1, 0), (3, 0), $\left(\frac{36}{7}, \frac{10}{7}\right)$, (0, 4) and (0, 2)

We see that the feasible solution regions in example 3(a) and 3(b) are of different types. The feasible region in example 3(a) is unbounded as it cannot be enclosed in any circle how large it may be while the feasible region in example 3(b) can easily be enclosed within a circle, so it is bounded. If the line segment obtained by joining any two points of a region lies entirely within the region, then the region is called **convex**.

Both the feasible regions of example 3(a) and 3(b) are convex but the regions such as shown in the adjoining figures are not convex.





EXERCISE 5.2

1. **Graph the feasible region of the following system of linear inequalities and find the corner points in each case.**

(i)	$2x - 3y \le 6$ $2x + 3y \le 12$ $x \ge 0, y \ge 0$	(ii)	$x + y \le 5$ $-2y + y \le 2$ $x \ge 0, y \ge 0$	(iii)	$x + y \le 5$ $-2x + y \ge 2$ $x \ge 0$
(iv)	$3x + 7y \le 21$ $x - y \le 3$ $x \ge 0,\ y \ge 0$	(v)	$3x + 2y > 6$ $x + y \le 4$ $x \ge 0,\ y \ge 0$	(vi)	$5x + 7y \le 35$ $x - 2y \le 4$ $x \ge 0,\ y \ge 0$

2. **Graph the feasible region of the following system of linear inequalities and find the corner points in each case.**

(i)	$2x + y \le 10$ $x + 4y \le 12$ $x + 2y \le 10$ $x \ge 0, y \ge 0$	(ii)	$2x + 3y \le 18$ $2x + y \le 10$ $x + 4y \le 12$ $x \ge 0 , y \ge 0$	(iii)	$2x + 3y \le 18$ $x + 4y \le 12$ $3x + y \le 12$ $x \ge 0, y \ge 0$
(iv)	$x + 2y \le 14$ $3x + 4y \le 36$ $2x + y \le 10$ $x \ge 0,\ y \ge 0$	(v)	$x + 3y \le 15$ $2x + y \le 12$ $4x + 3y \le 24$ $x \ge 0,\ y \ge\ 0$	(vi)	$2x + y \le 20$ $8x+15y \le 120$ $x + y \le 11$ $x \ge 0,\ y \ge 0$

5.6 LINEAR PROGRAMMING

A function which is to be maximized or minimized is called an **objective function**. Note that there are infinitely many feasible solutions in the feasible region. The feasible solution which maximizes or minimizes the objective function is called the **optimal solution**. The theorem of linear programming states that the maximum and minimum values of the objective function occur at corner points of the feasible region.

Procedure for determining optimal solution:

(i) Graph the solution set of linear inequality constraints to determine feasible region.

(ii) Find the corner points of the feasible region.

(iii) Evaluate the objective function at each corner point to find the optimal solution.

Example 1. Find the maximum and minimum values of the function defined as:

$f(x, y) = 2x + 3y$ subject to the constraints;

$x - y \le 2 \qquad x + y \le 4 \qquad 2x - y \le 6, \quad x \ge 0$

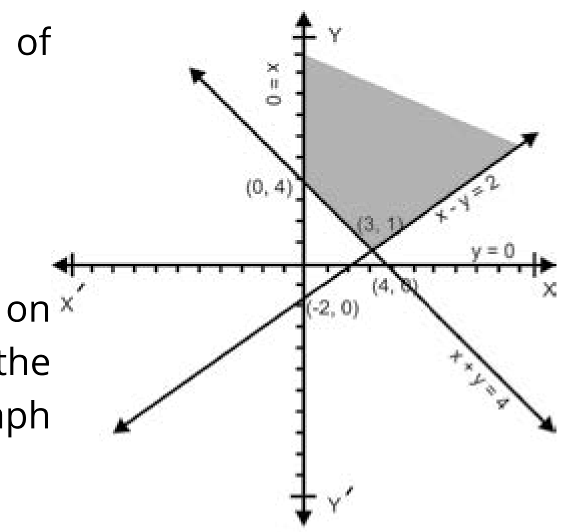


Fig. 5.61

Solution. The graphs of $x - y \le 2$ is the closed half plane on the origin side of $x - y = 2$ and the graph of $x + y \le 4$ is the closed half-plane not on the origin side of $x + y = 4$. The graph of the system

$x - y \le 2, x + y \ge 4$

including the non-negative constraints $x \geq 0$ is partially displayed as shaded region in the figure 5.61. The graph of $2x - y \leq 6$ consists of the graph of the line $2x - y = 6$ and the half plane on the origin side of the line $2x - y=6$. A portion of the solution region of the given system of inequalities is shaded in the figure 5.62.

We see that feasible region is unbounded upwards and its corner points are $A(0, 4)$, $B(3, 1)$ and $C(4, 2)$. Note that the point at which the lines $x + y = 4$ and $2x - y = 6$ intersect is not a corner point of the feasible region.

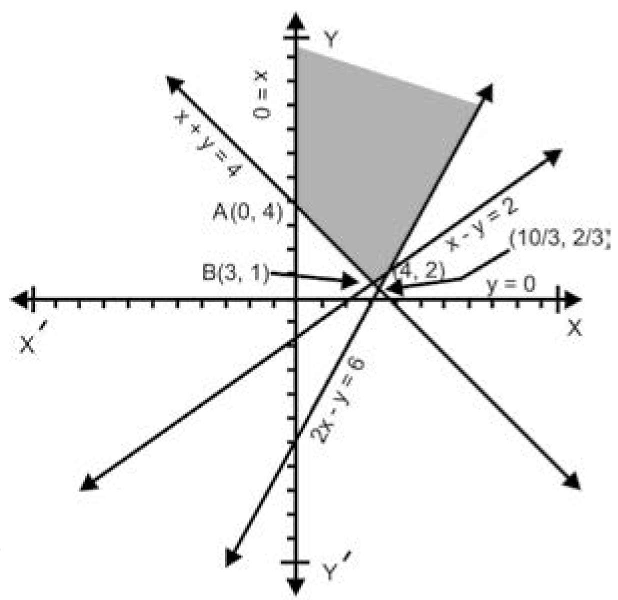


Fig. 5.62

It is obvious that the expression $2x + 3y$ does not posses a maximum value in the feasible region because its value can be made larger than any number by increasing x and y. We calculate the values of f at the corner points to find its minimum value:

$$f(0, 4) = 2(0) + 3 \times 4 = 12$$
$$f(3, 1) = 2 \times 3 + 3 \times 1 = 6 + 3 = 9$$
$$f(4, 2) = 2 \times 4 + 3 \times 2 = 8 + 6 = 14$$

Thus the minimum value of $2x + 3y$ is 9 at the corner point (3, 1).

Note: If $f(x, y) = 2x + 2y$, then $f(0, 4) = 2 \times 0 + 2 \times 4 = 8$, $f(3, 1) = 2 \times 3 + 2 \times 1 = 6 + 2 = 8$ and $f(4, 2) = 2 \times 4 + 2 \times 2 = 8 + 4 = 12$. The minimum value of $2x + 2y$ is the same at two corner points (0, 4) and (3, 1).

We observe that the minimum value of $2x + 2y$ at each point of the line segment AB is 8 as:

$$f(x, y) = 2x + 2(4 - x) \qquad (\because \; x + y = 4 \Rightarrow y = 4 - x)$$
$$= 2x + 8 - 2x = 8$$

Example 2. Find the minimum and maximum values of f and ϕ defined as:

$$f(x, y) = 4x + 5y, \qquad \phi(x, y) = 4x + 6y$$

under the constraints

$$2x - 3y \leq 6 \qquad 2x + y \geq 2 \qquad 2x + 3y \leq 12 \qquad x \geq 0, \quad y \geq 0$$

Solution. The graphs of $2x - 3y \leq 6$, $2x + y \geq 2$, are displayed in the example 3 of Art. 5.5. Joining the points (6. 0) and (0, 4), we obtain the graph of the line $2x + 3y = 12$. As $2(0) + 3(0) = 0 < 12$, so the graph of $2x + 3y < 12$ is the half plane below the line $2x + 3y = 12$. Thus the graph of $2x + 3y \leq 12$ consists of the graph of the line $2x + 3y = 12$ and the half plane below the line $2x + 3y = 12$. The solution region of $2x - 3y \leq 6$, $2x + y \geq 2$ and $2x + 3y \leq 12$ is the triangular region PQR shown in figure 5.63. The non-negative constraints $x \geq 0$, $y \geq 0$ indicated the first quadrant. Thus the feasible region satisfying all the constrains is shaded in the figure 5.63 and its corner points are (1, 0) (0, 2), (0, 4), $\left(\frac{9}{2}, 1\right)$ and (3, 0).

We find values of f at the corner points.

Corner point	$f(x, y) = 4x + 5y$
(1, 0)	$f(1, 0) = 4 \times 1 + 5.0 = 4$
(0, 2)	$f(0, 2) = 4 \times 0 + 5.2 = 10$
(0, 4)	$f(0, 4) = 4 \times 0 + 5.4 = 20$
(9/2, 1)	$f(9/2, 1) = 4 \times 9/2 + 5.1 = 23$
(3, 0)	$f(3, 0) = 4 \times 3 + 50 \times 0 = 12$

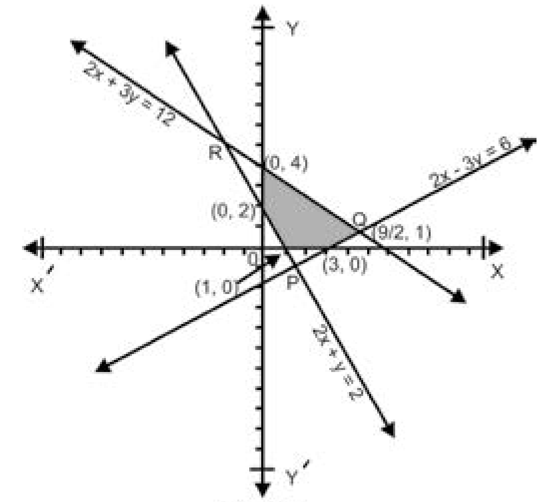


Fig. 5.63

From the above table, it follows that the minimum value of f is 4 at the corner point (1, 0) and maximum value of f is 23 at the corner point $\left(\frac{9}{2}, 1\right)$. The values of ϕ at the corner points are given below in tabular form.

Corner point	$\phi(x, y) = 4x + 5y$
(1, 0)	$\phi(1, 0) = 4.1 + 6.0 = 4$
(0, 2)	$\phi(0, 2) = 4.0 + 6.2 = 12$
(0, 4)	$\phi(0, 4) = 4.0 + 6.4 = 24$
(9/2, 1)	$\phi(9/2, 1) = 4 . 9/2 + 6.1 = 24$
(3, 0)	$\phi(3, 0) = 4 \times 3 + 6.0 = 12$

The minimum value of ϕ is 4 at the point (1, 0) and maximum value of ϕ is 24 at the corner points (0, 4) and $\left(\frac{9}{2}, 1\right)$. As observed in the above example, it follows that the function ϕ has maximum value at all the points of the line segment between the points

(0, 4) and $\left(\frac{9}{2}, 1\right)$.

Note 1. Some times it may happen that each point of constraint line gives the optimal value of the objective function.

Note 2. For different value of k, the equation $4x + 5y = k$ represents lines parallel to the line $4x + 5y = 0$. For a certain admissible value of k, the intersection of $4x + 5y = k$ with the feasible region gives feasible solutions for which the profit is k.

5.7 LINEAR PROGRAMMING PROBLEMS

Convert a linear programming problem to a mathematical form by using variables, then follow the procedure given in Art 5.6.

Example 1: A farmer possesses 100 canals of land and wants to grow corn and wheat. Cultivation of corn requires 3 hours per canal while cultivation of wheat requires 2 hours per canal. Working hours cannot exceed 240. If he gets a profit of Rs. 20 per canal for corn and Rs. 15/- per canal for wheat, how many canals of each he should cultivate to maximize his profit?

Solution: Suppose that he cultivates x canals of corn and y canals of wheat. Then constraints can be written as:

$$x + y \leq 100$$
$$3x + 2y \leq 240$$

Non-negative constraints are $x \geq 0$, $y \geq 0$. Let $P(x, y)$ be the profit function, then

$$P(x, y) = 20x + 15y$$

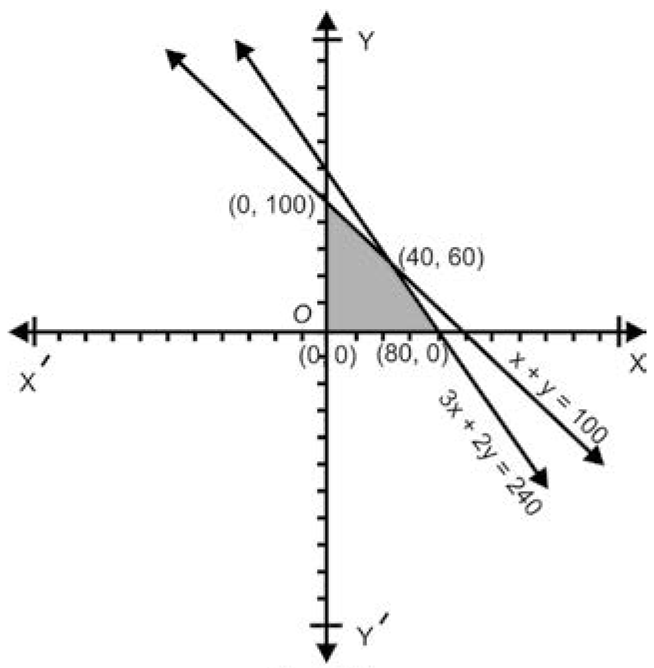


Fig. 5.71

Now the problem is to maximize the profit function P under the given constraints. Graphing the inequalities, we obtain the feasible region which is shaded in the figure 5.71. Solving the equations $x + y = 100$ and $3x + 2y = 240$ gives $x = 240 - 2(x + y) = 240 - 200 = 40$ and $y = 100 - 40 = 60$, that is; their point of intersection is (40, 60). The corner points of the feasible region are (0, 0),(0, 100), (40, 60) and (80, 0).

Now we find the values of P at the corner points.

Corner point	$P(x, y) = 20x + 15y$
(0, 0)	$P(0, 0)$ = 2 0 x 0 + 15 x 0 = 0
(0, 100)	$P(0, 100)$ = 20 x 0 + 15 x 100 = 1500
(40, 60)	$P(40, 60)$ = 20 x 40 + 15 x 60 = 1700
(80, 0)	$P(80, 0)$ = 20 x 80 + 15 x 0 = 1600

From the above table, it follows that the maximum profit is Rs. 1700 at the corner point (40, 60). Thus the farmer will get the maximum profit if he cultivates 40 canals of corn and 60 canals of wheat.

Exam ple 2. A factory produces bicycles and motorcycles by using two machines A and B. Machine A has at most 120 hours available and machine B has a maximum of 144 hours available. Manufacturing a bicycle requires 5 hours in machine A and 4 hours in machine B while manufacturing of a motorcycle requires 4 hours in machine A and 8 hours in machine B. If he gets profit of Rs. 40 per bicycle and profit of Rs. 50 per motorcycle, how many bicycles and motorcycles should be manufactured to get maximum profit?

Solution: Let the number of bicycles to be manufactured be x and the number of motor cycles to be manufactured be y.

Then the time required to use machine A for x bicycles and y motorcycles is $5x + 4y$ (hours) and the time required to use machine B for x bicycles and y motorcycles in $4x + 8y$ (hours). Thus the problem constraints are $5x + 4y \leq 120$

And $\quad 4x + 8y \leq 144$

$\Rightarrow \quad 2x + 4y \leq 72$.

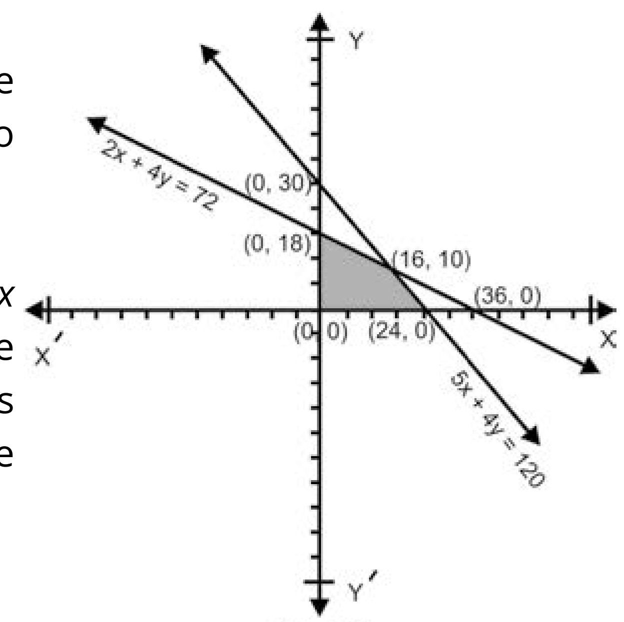


Fig. 5.72

Since the numbers of articles to be produced cannot be negative, so $x \geq 0, y \geq 0$.

Let $P(x, y)$ be the profit function, then $P(x, y) = 40x + 50y$.

Now the problem is to maximize P subject to the constraints:

$$5x + 4y \leq 120$$
$$2x + 4y \leq 72 \quad ; \quad x \geq 0, y \geq 0$$

Solving $5x + 4y = 120$ and $2x + 4y = 72$, gives $3x = 48 \Rightarrow x = 16$ and $4y = 72 - 2x = 72 - 32 = 40 \Rightarrow y = 10$.

Thus their point of intersection is (16, 10). Graphing the linear inequality constraints, the feasible region obtained is depicted in the figure 5.72 by shading. The corner points of the feasible region are (0, 0), (0, 18), (16, 10) and (24, 0).

Now we find the values of P at the comer points.

Corner point	$P(x, y) = 40x + 50y$
(0, 0)	$P(0, 0) = 40 \times 0 + 50 \times 0 = 0$
(0, 18)	$P(0, 18) = 40 \times 0 + 50 \times 18 = 900$
(16, 10)	$P(16, 10) = 40 \times 16 + 50 \times 10 = 1140$
(24, 0)	$P(24, 0) = 40 \times 24 + 50 \times 0 = 960$

From the above table, it follows, that the maximum profit is Rs. 1140 at the corner point (16, 10). Manufacturer gets the maximum profit if he manufactures 16 bicycles and 10 motorcycles.

EXERCISE 5.3

1. Maximize $f(x, y) = 2x + 5y$
subject to the constraints
$2y - x \leq 8$; $\quad x - y \leq 4$; $\quad x \quad 0 \geq 0$; $\quad y \geq 0$

2. Maximize $f(x , y) = x + 3y$
subject to the constraints
$2x + 5y \leq 30$; $\quad 5x + 4y \leq 20$; $\quad x \geq 0$; $\quad y \geq 0$

3. Maximize $z = 2x + 3y$; subject to the constraints:
$3x + 4y \leq 12$; $\quad 2x + y \leq 4$: $\quad 4x - y \leq 4$; $\quad x \geq 0$; $\quad y \geq 0$

4. Minimize $z = 2x + y$: subject to the constraints:
$x + y \geq 3$; $\quad 7x + 5y \leq 35$; $\quad x \geq 0$; $\quad y \geq 0$

5. Maximize the function defined as; $f(x, y) = 2x + 3y$ subject to the constraints:
$2x + y \leq 8$; $\quad x + 2y \leq 14$; $\quad x \geq 0$; $\quad y \geq 0$

6. Minimize $z = 3x + y$; subject to the constraints:
$3x + 5y \geq 15$; $\quad x + 6y \geq 9$; $\quad x \geq 0$; $\quad y \geq 0$

7. Each unit of food X costs Rs. 25 and contains 2 units of protein and 4 units of iron while each unit of food Y costs Rs. 30 and contains 3 units of protein and 2 unit of iron. Each animal must receive at least 12 units of protein and 16 units of iron each day. How many units of each food should be fed to each animal at the smallest possible cost?

8. A dealer wishes to purchase a number of fans and sewing machines. He has only Rs. 5760 to invest and has space atmost for 20 items. A fan costs him Rs. 360 and a sewing machine costs Rs. 240. His expectation is that the can sell a fan at a profit of Rs. 22 and a sewing machine at a profit of Rs. 18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit?

9. A machine can produce product A by using 2 units of chemical and 1 unit of a compound or can produce product B by using 1 unit of chemical and 2 units of the compound. Only 800 units of chemical and 1000 units of the compound are available. The profits per unit of A and B are Rs. 30 and Rs. 20 respectively, maximize the profit function.