CHAPTER

7

Vectors

Animation 7.1: Cross Product of Vectors
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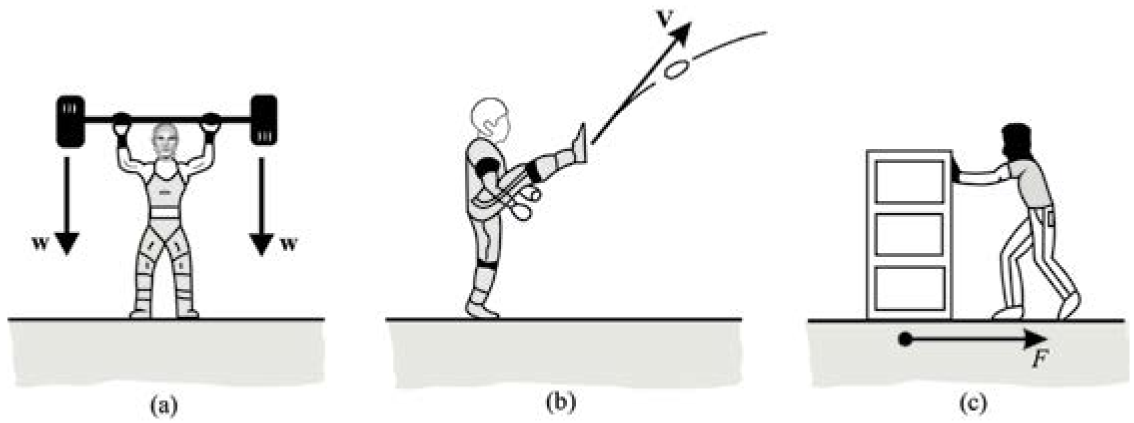
7.1 INTRODUCTION

In physics, mathematics and engineering, we encounter with two important quantities, known as **"Scalars and Vectors"**.

A **scalar quantity**, or simply a **scalar**, is one that possesses only magnitude. It can be specified by a number alongwith unit. In Physics, the quantities like mass, time, density, temperature, length, volume, speed and work are examples of scalars.

A **vector quantity**, or simply a **vector**, is one that possesses both magnitude and direction. In Physics, the quantities like displacement, velocity, acceleration, weight, force, momentum, electric and magnetic fields are examples of vectors.

In this section, we introduce vectors and their fundamental operations we begin with a geometric interpretation of vector in the plane and in space.



7.1.1 Geometric Interpretation of vector

Geometrically, a vector is represented by a directed line segment $\overrightarrow{AB}$ with *A* its initial point and *B* its terminal point. It is often found convenient to denote a vector by an arrow and is written either as $\overrightarrow{AB}$ or as a boldface symbol like $\boldsymbol{v}$ or in underlined form $\underline{v}$.

(i) The magnitude or length or norm of a vector $\overrightarrow{AB}$ or $\underline{v}$, is its absolute value and is written as $|\overrightarrow{AB}|$ or simply *AB* or $|\underline{v}|$.

(ii) A unit vector is defined as a vector whose magnitude is unity. Unit vector of vector $\underline{v}$ is written as $\hat{\underline{v}}$ (read as $\underline{v}$ hat) and is defined by $\hat{\underline{v}} = \dfrac{\underline{v}}{|\underline{v}|}$

(iii) If terminal point *B* of a vector $|\overrightarrow{AB}|$ coincides with its initial point *A*, then magnitude $AB = 0$ and $|\overrightarrow{AB}| = 0$, which is called zero or null vector.

(iv) Two vectors are said to be negative of each other if they have same magnitude but opposite direction.

$$\text{If} \quad \overrightarrow{AB} = \underline{v}, \quad \text{then} \quad \overrightarrow{BA} = -\overrightarrow{AB} = -\underline{v}$$

$$\text{and} \quad |\overrightarrow{BA}| = |-\overrightarrow{AB}|$$

7.1.2 Multiplication of Vector by a Scalar

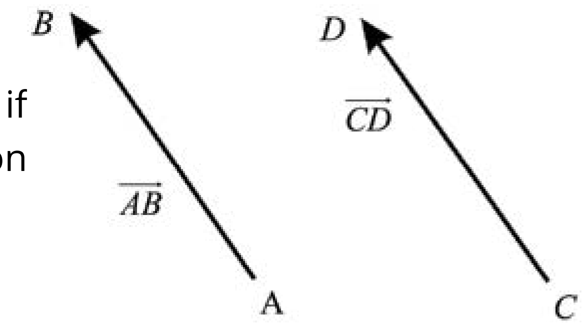
We use the word scalar to mean a real number. Multiplication of a vector $\underline{v}$ by a scalar '*k*' is a vector whose magnitude is *k* times that of $\underline{v}$. It is denoted by $k\underline{v}$.

(i) If *k* is +ve, then $\underline{v}$ and $k\underline{v}$ are in the same direction.

(ii) If *k* is –ve, then $\underline{v}$ and $k\underline{v}$ are in the opposite direction

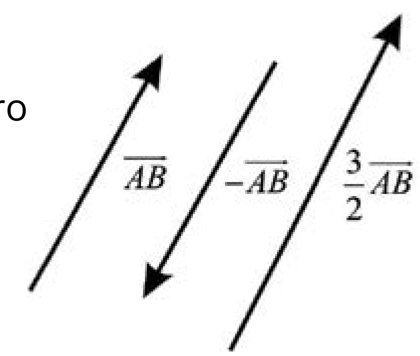
(a) Equal vectors

Two vectors $\overrightarrow{AB}$ and are said to be equal, if they have the same magnitude and same direction i.e., $|\overrightarrow{AB}| = |\overrightarrow{CD}|$



(b) Parallel vectors

Two vectors are parallel if and only if they are non-zero scalar multiple of each other, (see figure).



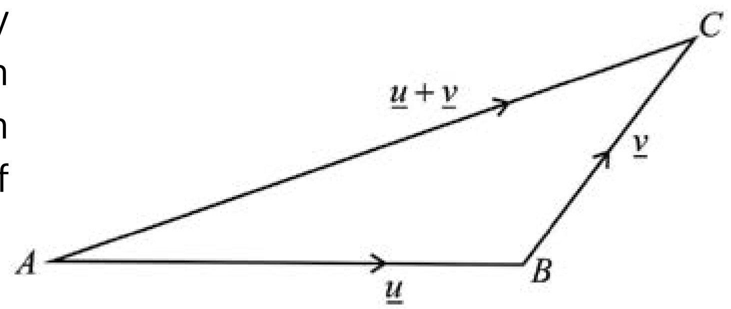
7.1.3 Addition and Subtraction of Two Vectors

Addition of two vectors is explained by the following two laws:

(i) Triangle Law of Addition

If two vectors $\underline{u}$ and $\underline{v}$ are represented by the two sides *AB* and *BC* of a triangle such that the terminal point of $\underline{u}$ coincide with the initial point of $\underline{v}$, then the third side *AC* of the triangle gives vector sum $\underline{u}+\underline{v}$, that is

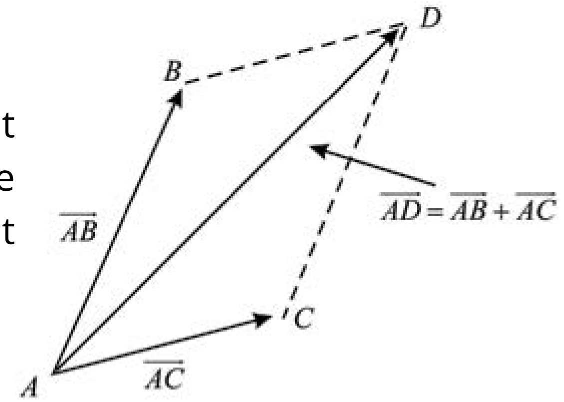
$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC} \quad \Rightarrow \quad \underline{u}+\underline{v}=\overrightarrow{AC}$$



(ii) Parallelogram Law of Addition

If two vectors $\underline{u}$ and $\underline{v}$ are represented by two adjacent sides AB and *AC* of a parallelogram as shown in the figure, then diagonal *AD* give the sum or resultant of $\overrightarrow{AB}$ and $\overrightarrow{AC}$, that is

$$\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{AC}=\underline{u}+\underline{v}$$



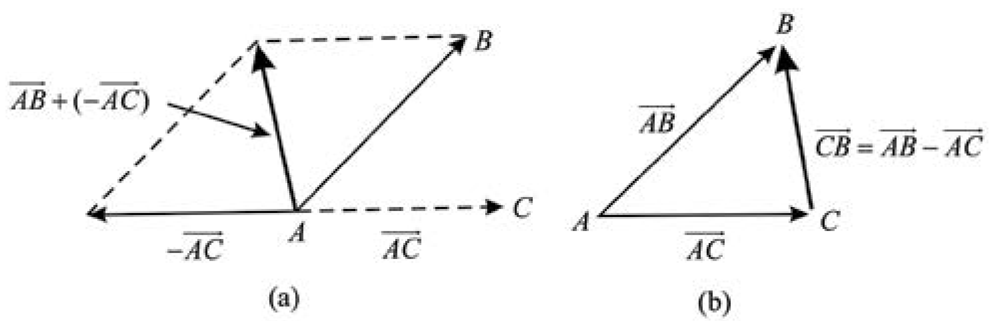
Note:	This law was used by Aristotle to describe the combined action of two forces.

(b) Subtraction of two vectors

The difference of two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is defined by

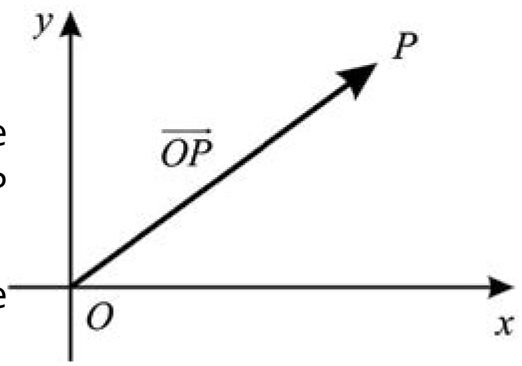
$$\overrightarrow{AB}-\overrightarrow{AC}=\overrightarrow{AB}+(-\overrightarrow{AC})$$

$$\underline{u}-\underline{v}=\underline{u}+(-\underline{v})$$



In figure, this difference is interpreted as the main diagonal of the parallelogram with sides $\overrightarrow{AB}$ and $-\overrightarrow{AC}$. We can also interpret the same vector difference as the third side of a triangle with sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$. In this second interpretation, the vector difference $\overrightarrow{AB}-\overrightarrow{AC}=\overrightarrow{CB}$ points the terminal point of the vector from which we are subtracting the second vector.

7.1.4 Position Vector



The vector, whose initial point is the origin *O* and whose terminal point is *P*, is called the position vector of the point P and is written as $\overrightarrow{OP}$.

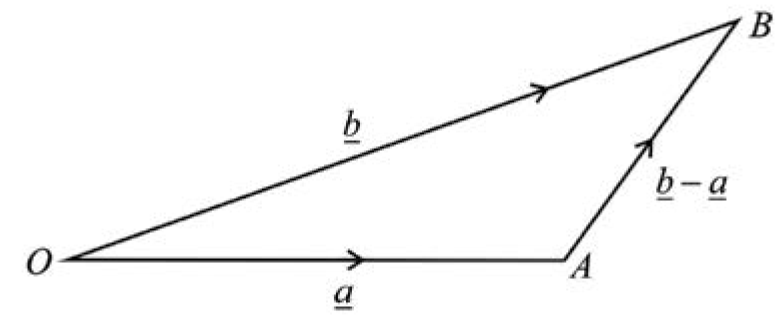
The position vectors of the points *A* and *B* relative to the origin *O* are defined by $\overrightarrow{OA}=\underline{a}$ and $\overrightarrow{OB}=\underline{b}$ respectively.

In the figure, by triangle law of addition,

$$\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$$

$$\underline{a}+\overrightarrow{AB}=\underline{b}$$

$$\Rightarrow \quad \overrightarrow{AB}=\underline{b}-\underline{a}$$

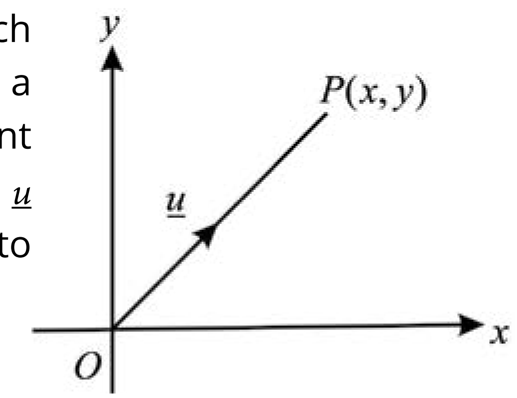


7.1.5 Vectors in a Plane

Let *R* be the set of real numbers. The Cartesian plane is defined to be the $R^2=\{(x, y) : x, y \in R\}$.

An element $(x, y) \in R^2$ represents a point $P(x, y)$ which is uniquely determined by its coordinate *x* and *y*. Given a vector $\underline{u}$ in the plane, there exists a unique point $P(x, y)$ in the plane such that the vector $\overrightarrow{OP}$ is equal to $\underline{u}$ (see figure). So we can use rectangular coordinates (x, y) for *P* to associate a unique ordered pair $[x, y]$ to vector $\underline{u}$.

We define addition and scalar multiplication in R^2 by:



(i) **Addition:** For any two vectors $\underline{u}=[x,y]$ and $\underline{v}=[x',y']$, we have

$$\underline{u}+\underline{v}=[x,y]+[x',y']=[x+x',y+y']$$

(ii) **Scalar Multiplication:** For $\underline{u}=[x, y]$ and $\alpha \in R$, we have

$$\alpha\underline{u}=\alpha[x, y]=[\alpha x, \alpha y]$$

Definition: The set of all ordered pairs $[x, y]$ of real numbers, together with the rules of addition and scalar multiplication, is called the set of **vectors** in R^2.

For the vector $\underline{u} = [x, y]$, x and y are called the components of $\underline{u}$.

Note: The vector $[x, y]$ is an ordered pair of numbers, not a point (x, y) in the plane.

(a) Negative of a Vector

In scalar multiplication (ii), if $\alpha = -1$ and $\underline{u} = [x, y]$ then

$$\alpha\underline{u} = (-1)\,[x, y] = [-x, -y]$$

which is denoted by $-\underline{u}$ and is called the **additive inverse** of $\underline{u}$ or **negative vector** of $\underline{u}$.

(b) Difference of two Vectors

We define $\underline{u} - \underline{v}$ as $\underline{u} + (-\underline{v})$

If $\underline{u} = [x, y]$ *and* $\underline{v} = [x', y']$, then

$\underline{u} - \underline{v} = \underline{u} + (-\underline{v})$

$$= [x, y] + [-x' - y'] = [x - x', y - y']$$

(c) Zero Vector

Clearly $\underline{u} + (-u) = [x, y] + [-x, -y] = [x - x, y - y] = [0,0] = \underline{0}$.

$\underline{0} = [0,0]$ is called the **Zero (Null) vector**.

(d) Equal Vectors

Two vectors $\underline{u} = [x, y]$ and $\underline{v} = [x', y']$ of R^2 are said to be equal if and only if they have the same components. That is,

$[x, y] = [x', y']$ if and only if $x = x'$ and $y = y'$

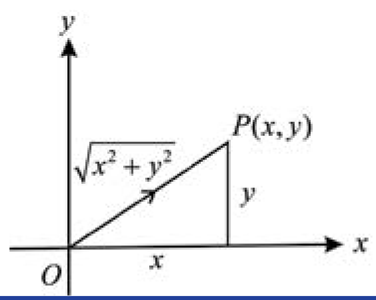
and we write $\underline{u} = \underline{v}$

(e) Position Vector

For any point $P(x, y)$ in R^2, a vector $\underline{u} = [x, y]$ is represented by a directed line segment $\overrightarrow{OP}$, whose initial point is at origin. Such vectors are called position vectors because they provide a unique correspondence between the points (positions) and vectors.

(f) Magnitude of a Vector

For any vector $\underline{u} = [x, y]$ in R^2, we define the **magnitude** or **norm** or **length** of the vector as of the point $P(x, y)$ from the origin O



$$\therefore \quad \text{Magnitude of } \overrightarrow{OP} = \left|\overrightarrow{OP}\right| = |\underline{u}| = \sqrt{x^2 + y^2}$$

7.1.6 Properties of Magnitude of a Vector

Let $\underline{v}$ be a vector in the plane or in space and let c be a real number, then

(i) $|\underline{v}| \geq 0$, and $|\underline{v}| = 0$ if and only if $\underline{v} = \underline{0}$

(ii) $|c\underline{v}| = |c|\ |\underline{v}|$

Proof: (i) We write vector $\underline{v}$ in component form as $\underline{v} = [x, y]$, then

$|\underline{v}| = \sqrt{x^2 + y^2} \geq 0$ for all x and y.

Further $|\underline{v}| = \sqrt{x^2 + y^2} = 0$ if and only if $x = 0$, $y = 0$

In this case $\underline{v} = [0,0] = \underline{0}$

(ii) $|c\underline{v}| = |cx, cy| = \sqrt{(cx)^2 + (cy)^2} = \sqrt{c^2}\sqrt{x^2 + y^2} = |c||\underline{v}|$

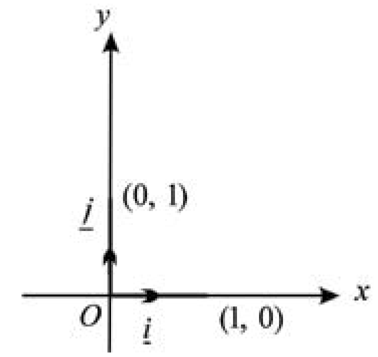
7.1.7 Another notation for representing vectors in plane

We introduce two special vectors,

$$\underline{i} = [1,0],\quad \underline{j} = [0,1] \text{ in } R^2$$

As magnitude of $\underline{i} = \sqrt{1^2 + 0^2} = 1$

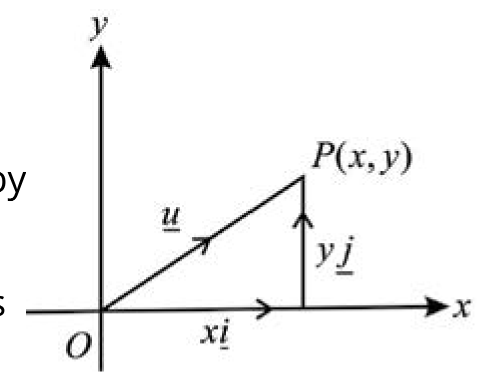
magnitude of $\underline{j} = \sqrt{0^2 + 1^2} = 1$



So $\underline{i}$ and $\underline{j}$ are called unit vectors along x-axis, and along y-axis respectively. Using the definition of addition and scalar multiplication, the vector $[x, y]$ can be written as

$$\begin{aligned} \underline{u} = [x, y] &= [x,0] + [0, y] \\ &= x[1,0] + y[0,1] \\ &= x\underline{i} + y\underline{j} \end{aligned}$$

Thus each vector $[x, y]$ in R^2 can be uniquely represented by $x\underline{i} + y\underline{j}$.



In terms of unit vector $\underline{i}$ and $\underline{j}$, the sum $\underline{u} + \underline{v}$ of two vectors $\underline{u} = [x, y]$ and $\underline{v} = [x', y']$ is written as

$$\underline{u} + \underline{v} = [x + x', y + y']$$

$$= (x + x')\underline{i} + (y + y')\underline{j}$$

7.1.8 A unit vector in the direction of another given vector.

A vector $\underline{u}$ is called a **unit vector**, if $|\underline{u}| = 1$
Now we find a unit vector u in the direction of any other given vector $\underline{v}$.
We can do by the use of property (ii) of magnitude of vector, as follows:

$$\because \quad \left|\frac{1}{|\underline{v}|}\right| = \frac{1}{|\underline{v}|}|\underline{v}| = 1$$

$$\therefore \quad \text{the vector } \underline{v} = \frac{1}{|\underline{v}|}\underline{v} \text{ is the required unit vector}$$

It points in the same direction as v, because it is a positive scalar multiple of $\underline{v}$.

Example 1:

For $\underline{v}$ = [1, –3] and $\underline{w}$ = [2,5]
(i) $\underline{v}$ + $\underline{w}$ = [1, –3] + [2,5] = [1 + 2, –3 + 5] = [3,2]
(ii) 4$\underline{v}$ + 2$\underline{w}$ = [4, –12] + [4,10] = [8,–2]
(iii) $\underline{v}$ – $\underline{w}$ = [1, –3]– [2,5] = [l – 2, –3 –5] = [–1,–8]
(iv) $\underline{v}$ – $\underline{v}$ = [l –1, –3 + 3] = [0,0] = 0
(v) $|\underline{v}| = \sqrt{(1)^2 + (-3)^2} = \sqrt{1+9} = \sqrt{10}$

Example 2: Find the unit vector in the same direction as the vector $\underline{v}$ = [3, –4].

Solution: $\underline{v} = [3,-4] = 3\underline{i} - 4\underline{j}$

$$|\underline{v}| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$$

Now $\underline{u} = \frac{1}{|\underline{v}|}\underline{v} = \frac{1}{5}[3,-4]$ ($\underline{u}$ is unit vector in the direction of v)

$$= \left[\frac{3}{5}, \frac{-4}{5}\right]$$

Verification: $|\underline{u}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{-4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$

Example 3: Find a unit vector in the direction of the vector
(i) $\underline{v} = 2\underline{i} + 6\underline{j}$ (ii) $\underline{v}$ =[–2,4]

Solution: (i) $\underline{v} = 2\underline{i} + 6\underline{j}$

$$|\underline{v}| = \sqrt{(2)^2 + (6)^2} = \sqrt{4+36} = \sqrt{40}$$

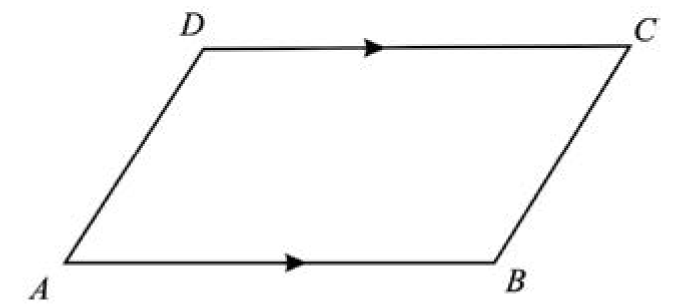
$$\therefore \text{ A unit vector in the direction of } \underline{v} = \frac{\underline{v}}{|\underline{v}|} = \frac{2}{\sqrt{40}}\underline{i} + \frac{6}{\sqrt{40}}\underline{j} = \frac{1}{\sqrt{10}}\underline{i} + \frac{3}{\sqrt{10}}\underline{j}$$

(ii) $\underline{v} = [-2,4] = -2\underline{i} + 4\underline{j}$

$$|\underline{v}| = \sqrt{(-2)^2 + (4)^2} = \sqrt{4+16} = \sqrt{20}$$

$$\therefore \text{ A unit vector in the direction of } \underline{v} = \frac{\underline{v}}{|\underline{v}|} = \frac{-2}{\sqrt{20}}\underline{i} + \frac{4}{\sqrt{20}}\underline{j} = \frac{-1}{\sqrt{5}}\underline{i} + \frac{2}{\sqrt{5}}\underline{j}$$

Example 4: If *ABCD* is a parallelogram such that the points *A*, *B* and *C* are respectively (–2, –3), (1,4) and (0, –5). Find the coordinates of *D*.



Solution: Suppose the coordinates of *D* are (*x*, *y*)
As *ABCD* is a parallelogram

$\therefore \quad \overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AB} \parallel \overrightarrow{DC}$

$\Rightarrow \quad \overrightarrow{AB} = \overrightarrow{DC}$

$\therefore \quad (1+2)\underline{i} + (4+3)\underline{j} = (0-x)\underline{i} + (-5-y)\underline{j}$

$\Rightarrow \quad 3\underline{i} + 7\underline{j} \quad = -x\underline{i} + (-5-y)\underline{j}$

Equating horizontal and vertical components, we have
$-x = 3 \Rightarrow x = -3$
and $-5 - y = 7 \Rightarrow y = -12$
Hence coordinates of *D* are (–3, 12).

7.1.9 The Ratio Formula

Let *A* and *B* be two points whose position vectors (p.v.) are $\underline{a}$ and $\underline{b}$ respectively. If a point *P* divides *AB* in the ratio *p* : *q*, then the position vector of *P* is given by

$$\underline{r} = \frac{q\underline{a} + p\underline{b}}{p+q}$$

Proof: Given $\underline{a}$ and $\underline{b}$ are position vectors of the points *A* and *B* respectively. Let *r* be the position vector of the point *P* which divides the line segment *AB* in the ratio *p* : *q*. That is

$$m\overrightarrow{AP} : m\overrightarrow{PB} = p : q$$

So $\dfrac{m\overrightarrow{AP}}{m\overrightarrow{PB}} = \dfrac{p}{q}$

$\Rightarrow \quad q\left(m\overrightarrow{AP}\right) = p\left(m\overrightarrow{PB}\right)$

Thus $\quad q\left(\overrightarrow{AP}\right) = p\left(\overrightarrow{PB}\right)$

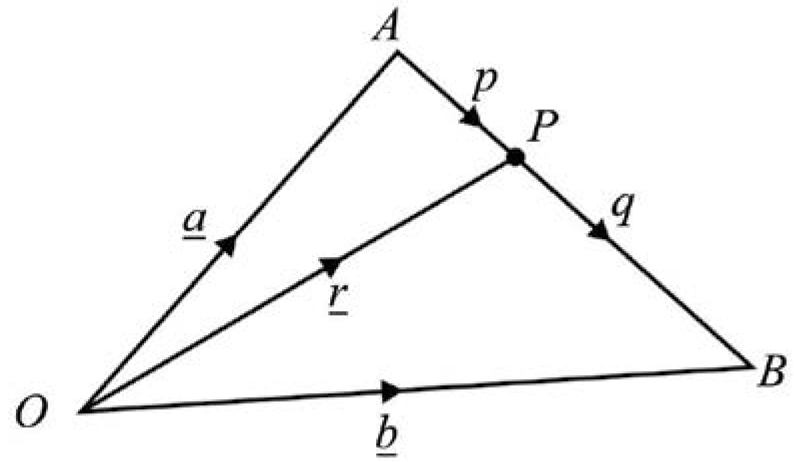
$\Rightarrow \quad q(\underline{r} - \underline{a}) = p(\underline{b} - \underline{r})$

$\Rightarrow \quad q\underline{r} - q\underline{a} = p\underline{b} - p\underline{r}$

$\Rightarrow \quad p\underline{r} + q\underline{r} = q\underline{a} + p\underline{b}$

$\Rightarrow \quad \underline{r}(p+q) = q\underline{a} + p\underline{b}$

$\Rightarrow \quad \underline{r} = \dfrac{q\underline{a} + p\underline{b}}{q+p}$



Corollary: If *P* is the mid point of *AB*, then *p* : *q* = 1 : 1

$$\therefore \text{ positive vector of } \quad P = \underline{r} = \frac{\underline{a}+\underline{b}}{2}$$

7.1.10 Vector Geometry

Let us now use the concepts of vectors discussed so far in proving Geometrical Theorems. A few examples are being solved here to illustrate the method.

Example 5: If $\underline{a}$ and $\underline{b}$ be the p.vs of *A* and *B* respectively w.r.t. origin *O*, and *C* be a point on $\overline{AB}$ such that $\overrightarrow{OC} = \dfrac{a+b}{2}$, then show that *C* is the mid-point of *AB*.

Solution: $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$ and $\overrightarrow{OC} = \dfrac{1}{2}(\underline{a} + \underline{b})$

Now $\quad 2\overrightarrow{OC} = \underline{a} + \underline{b}$

$\Rightarrow \quad \overrightarrow{OC} + \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$

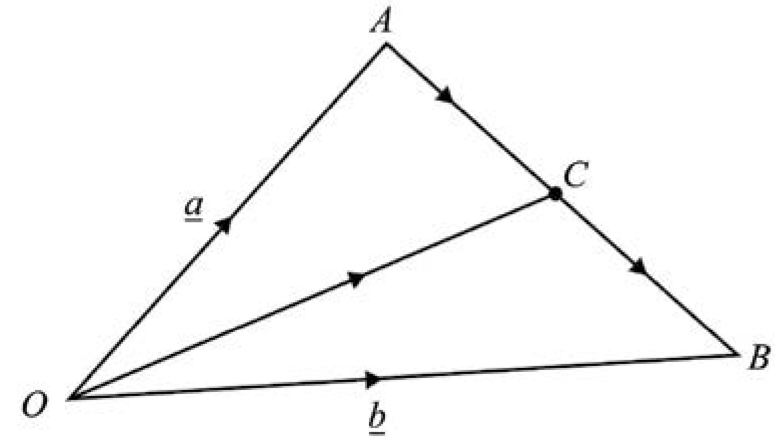
$\Rightarrow \quad \overrightarrow{OC} - \overrightarrow{OA} = \overrightarrow{OB} - \overrightarrow{OC}$

$\Rightarrow \quad \overrightarrow{OC} + \overrightarrow{AO} = \overrightarrow{OB} + \overrightarrow{CO}$

$\Rightarrow \quad \overrightarrow{AO} + \overrightarrow{OC} = \overrightarrow{CO} + \overrightarrow{OB}$

$\therefore \quad \overrightarrow{AC} = \overrightarrow{CB}$

Thus $\quad m\overrightarrow{AC} = m\overrightarrow{CB}$



$\Rightarrow$ *C* is equidistant from *A* and *B*, but *A*, *B*, *C* are collinear.
Hence *C* is the mid point of *AB*.

Example 6: Use vectors, to prove that the diagonals of a parallelogram bisect each other.

Solution: Let the vertices of the parallelogram be *A*, *B*, *C* and *D* (see figure)
Since $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$, the vector from *A* to the mid point of diagonal $\overrightarrow{AC}$ is

$$\underline{v} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AD}\right)$$

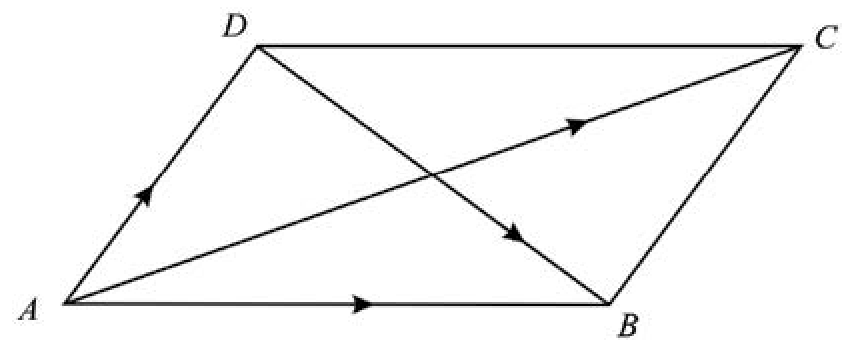
Since $\overrightarrow{DB} = \overrightarrow{AB} - \overrightarrow{AD}$, the vector from *A* to the mid point of diagonal $\overrightarrow{DB}$ is

$\underline{w} = \overrightarrow{AD} + \dfrac{1}{2}\left(\overrightarrow{AB} - \overrightarrow{AD}\right)$

$= \overrightarrow{AD} + \dfrac{1}{2}\overrightarrow{AB} - \dfrac{1}{2}\overrightarrow{AD}$

$= \dfrac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AD}\right)$

$= \underline{v}$



Since $\underline{v} = \underline{w}$, these mid points of the diagonals $\overrightarrow{AC}$ and $\overrightarrow{DB}$ are the same.
Thus the diagonals of a parallelogram bisect each other.

EXERCISE 7.1

1. Write the vector $\overrightarrow{PQ}$ in the form $x\underline{i}+y\underline{j}$.
(i) $P(2,3)$, $Q(6,-2)$ (ii) $P(0,5)$, $Q(-1,-6)$

2. Find the magnitude of the vector $\underline{u}$:
(i) $\underline{u}=2\underline{i}-7\underline{j}$ (ii) $\underline{u}=\underline{i}+\underline{j}$ (iii) $\underline{u}=[3,-4]$

3. If $\underline{u}=2\underline{i}-7\underline{j}$, $\underline{v}=\underline{i}-6\underline{j}$ and $\underline{w}=-\underline{i}+\underline{j}$. Find the following vectors:
(i) $\underline{u}+\underline{v}-\underline{w}$ (ii) $2\underline{u}-3\underline{v}+4\underline{w}$ (iii) $\frac{1}{2}\underline{u}+\frac{1}{2}\underline{v}+\frac{1}{2}\underline{w}$

4. Find the sum of the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$, given the four points $A(1,-1)$, $B(2,0)$, $C(-1,3)$ and $D(-2,2)$.

5. Find the vector from the point A to the origin where $\overrightarrow{AB}=4\underline{i}-2\underline{j}$ and B is the point $(-2,5)$.

6. Find a unit vector in the direction of the vector given below:
(i) $\underline{v}=2\underline{i}-\underline{j}$ (ii) $\underline{v}=\frac{1}{2}\underline{i}+\frac{\sqrt{3}}{2}\underline{j}$ (iii) $\underline{v}=-\frac{\sqrt{3}}{2}\underline{i}-\frac{1}{2}\underline{j}$

7. If A, B and C are respectively the points $(2,-4)$, $(4,0)$ and $(1,6)$. Use vector method to find the coordinates of the point D if:
(i) $ABCD$ is a parallelogram (ii) $ADBC$ is a parallelogram

8. If B, C and D are respectively $(4,1)$, $(-2,3)$ and $(-8,0)$. Use vector method to find the coordinates of the point:
(i) A if $ABCD$ is a parallelogram. (ii) E if $AEBD$ is a parallelogram.

9. If O is the origin and $\overrightarrow{OP}=\overrightarrow{AB}$, find the point P when A and B are $(-3,7)$ and $(1,0)$ respectively.

10. Use vectors, to show that $ABCD$ is a parallelogram, when the points A, B, C and D are respectively $(0,0)$, $(a,0)$, (b,c) and $(b-a,c)$.

11. If $\overrightarrow{AB}=\overrightarrow{CD}$, find the coordinates of the point A when points B, C, D are $(1,2)$, $(-2,5)$, $(4,11)$ respectively.

12. Find the position vectors of the point of division of the line segments joining the following pair of points, in the given ratio:

(i) Point C with position vector $2\underline{i}-3\underline{j}$ and point D with position vector $3\underline{i}+2\underline{j}$ in the ratio 4 : 3

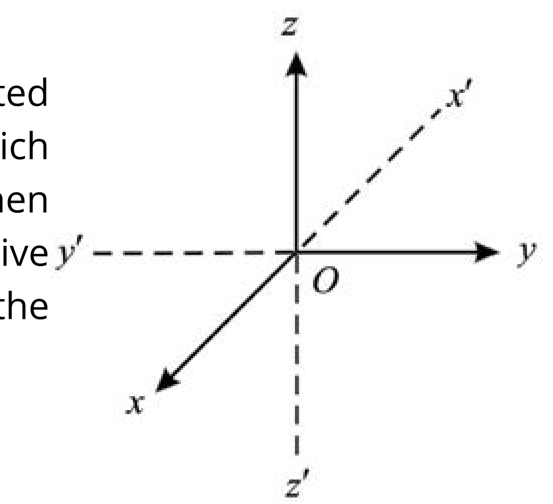
(ii) Point E with position vector $5\underline{j}$ and point F with position vector $4\underline{i}+\underline{j}$ in ratio 2 : 5

13. Prove that the line segment joining the mid points of two sides of a triangle is parallel to the third side and half as long.

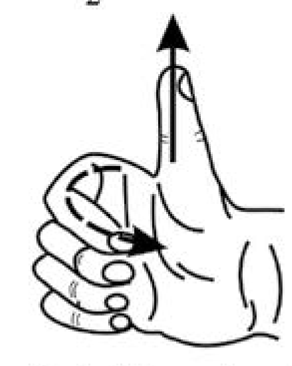
14. Prove that the line segments joining the mid points of the sides of a quadrilateral taken in order form a parallelogram.

7.2 INTRODUCTION OF VECTOR IN SPACE

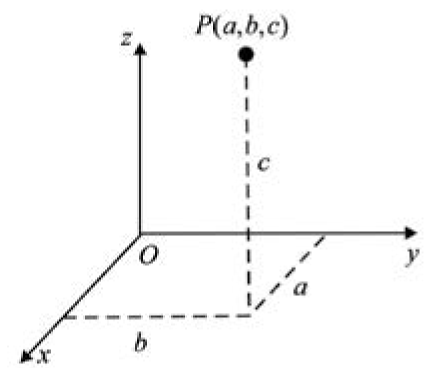
In space, a rectangular coordinate system is constructed using three mutually orthogonal (perpendicular) axes, which have orgin as their common point of intersection. When sketching figures, we follow the convention that the positive x-axis points towards the reader, the positive y-axis to the right and the positive z-axis points upwards.



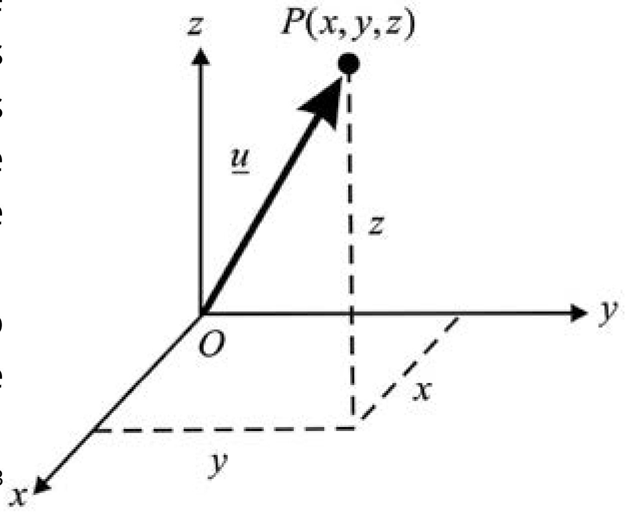
These axis are also labeled in accordance with the right hand rule. If fingers of the right hand, pointing in the direction of positive x-axis, are curled toward the positive y-axis, then the thumb will point in the direction of positive z-axis, perpendicular to the xy-plane. The broken lines in the figure represent the negative axes.


right hand rule

A point P in space has three coordinates, one along x-axis, the second along y-axis and the third along z-axis. If the distances along x-axis, y-axis and z-axis respectively are a, b, and c, then the point P is written with a unique triple of real numbers as $P=(a,b,c)$ (see figure).



7.2.1 Concept of a vector in space



The set R^3 = {(x, y, z) : x, y, z $\in$ R} is called the 3-dimensional space. An element (x, y, z) of R^3 represents a point P(x, y, z), which is uniquely determined by its coordinates x, y and z. Given a vector $\underline{u}$ in space, there exists a unique point P(x, y, z) in space such that the vector $\overrightarrow{OP}$ is equal to $\underline{u}$ (see figure).

Now each element (x, y, z) $\in$ P^3 is associated to a unique ordered triple [x, y, z], which represents the vector $\underline{u}$ = $\overrightarrow{OP}$ = [x, y, z].

We define addition and scalar multiplication in R^3 by:

(i) **Addition:** For any two vectors $\underline{u}$ = [x, y, z] and $\underline{v}=[x',y',z']$, we have

$$\underline{u}+\underline{v}=[x,y,z]+[x',y',z']=[x+x',y+y',z+z']$$

(ii) **Scalar Multiplication:** For $\underline{u}$ = [x, y, z] and α $\in$ R, we have

$\alpha\underline{u}$ $=\alpha[x, y, z] = [\alpha x, \alpha y, \alpha z]$

Definition: The set of all ordered triples [x, y, z] of real numbers, together with the rules of addition and scalar multiplication, is called the set of **vectors** in R^3.

For the vector $\underline{u}$ = [x, y, z], x, y and z are called the components of $\underline{u}$.

The definition of vectors in R^3 states that vector addition and scalar multiplication are to be carried out for vectors in space just as for vectors in the plane. So we define in R^3:

a) The **negative** of the vector $\underline{u}=[x;y,z]$ as $-\underline{u}=(-1)\underline{u}=[-x,-y,-z]$

b) The **difference** of two vectors $\underline{v}=[x',y',z']$ and $\underline{w}=[x'',y'',z'']$ as $\underline{v}-\underline{w}=\underline{v}+(-\underline{w})=[x'-x'',y'-y'',z'-z'']$

c) The **zero vector** as 0 = [0,0,0]

d) **Equality** of two vectors $\underline{v}=[x',y',z']$ and $\underline{w}=[x'',y'',z'']$ by $\underline{v}=\underline{w}$ if and only $x'=x'',y'=y''$ and $z'=z''$.

e) **Position Vector**

For any point P(x, y, z) in R^3, a vector $\underline{u}$ = [x, y, z] is represented by a directed line segment $\overrightarrow{OP}$, whose initial point is at origin. Such vectors are called position vectors in R^3.

f) **Magnitude** of a vector: We define the **magnitude** or **norm** or **length** of a vector $\underline{u}$ in space by the distance of the point P(x, y, z) from the origin O.

$$\therefore \quad \left|\overrightarrow{OP}\right|=\left|\underline{u}\right|=\sqrt{x^2+y^2+z^2}$$

Example 1: For the vectors, $\underline{v}$ = [2,1,3] and $\underline{w}$ = [–1,4,0], we have the following

(i) $\underline{v}+\underline{w}$ = [2 – 1, 1 + 4, 3 + 0]= [l,5,3]

(ii) $\underline{v}-\underline{w}$ = [2 + 1,1 – 4, 3 – 0]= [3, –3, 3]

(iii) $2\underline{w}$ =2[–1, 4, 0] = [–2, 8, 0]

(iv) $\left|\underline{v}-2\underline{w}\right|=\left\|[2+2,1-8,3-0]\right\|=\left\|[4,-7,3]\right\|=\sqrt{(4)^2+(-7)^2+(3)^2}=\sqrt{16+49+9}=\sqrt{74}$

7.2.2 Properties of Vectors

Vectors, both in the plane and in space, have the following properties:

Let $\underline{u}$, $\underline{v}$ and $\underline{w}$ be vectors in the plane or in space and let a, b $\in$ R, then they have the following properties

(i)	$\underline{u}+\underline{v}=\underline{v}+\underline{u}$	(Commutative Property)
(ii)	$(\underline{u}+\underline{v})+\underline{w}=\underline{u}+(\underline{v}+\underline{w})$	(Associative Property)
(iii)	$\underline{u}+(-1)\underline{u}=\underline{u}-\underline{u}=0$	(Inverse for vector addition)
(iv)	$a(\underline{v}+\underline{w})=a\underline{v}+a\underline{w}$	(Distributive Property)
(v)	$a(b\underline{u})=(ab)\underline{u}$	(Scalar Multiplication)

Proof: Each statement is proved by writing the vector/vectors in component form in R^2 / R^3 and using the properties of real numbers. We give the proofs of properties (i) and (ii) as follows.

(i) Since for any two real numbers a and b

$a + b = b + a$, it follows, that

for any two vectors $\underline{u}$ = [x, y] and $\underline{v}=[x',y']$ in R^2, we have

$$\begin{aligned}\underline{u}+\underline{v}&=[x,y]+[x'+y']\\&=[x+x',y+y']\\&=[x'+x,y'+y]\\&=[x',y']+[x,y]\\&=\underline{v}+\underline{u}\end{aligned}$$

So addition of vectors in R^2 is commutative

(ii) Since for any three real numbers a, b, c,

$(a + b) + c = a + (b + c)$, it follows that

for any three vectors, $\underline{u} = [x, y]$, $\underline{v} = [x', y']$ and $w = [x'', y'']$ in R^2, we have

$$\begin{aligned}(\underline{u}+\underline{v})+\underline{w} &= [x+x', y+y'] + [x'', y''] \\ &= [(x+x')+x'', (y+y')+y''] \\ &= [x+(x'+x''), y+(y'+y'')] \\ &= [x, y] + [x'+x'', y'+y''] \\ &= \underline{u} + (\underline{v}+\underline{w})\end{aligned}$$

So addition of vectors in R^2 is associative

The proofs of the other parts are left as an exercise for the students.

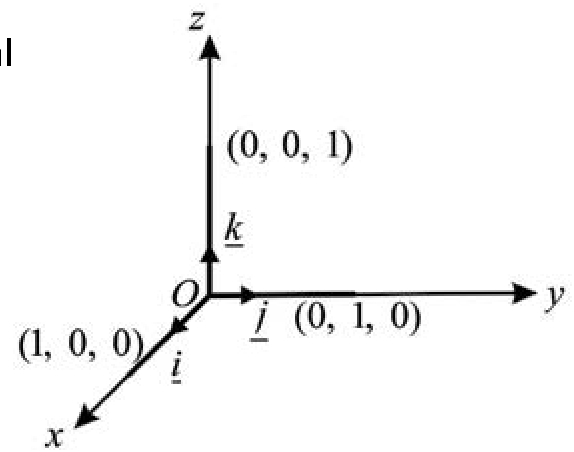
7.2.3 Another notation for representing vectors in space

As in plane, similarly we introduce three special vectors

$\underline{i} = [1,0,0]$, $\underline{j} = [0,1,0]$ and $\underline{k} = [0,0,1]$ in R^3.

As magnitude of $\underline{i}$ $= \sqrt{1^2+0^2+0^2} = 1$

magnitude of $\underline{j}$ $= \sqrt{0^2+1^2+0^2} = 1$



and magnitude of $\underline{k}$ $= \sqrt{0^2+0^2+1^2} = 1$ So $\underline{i}, \underline{j}$ and $\underline{k}$ are called unit vectors along x-axis, along y-axis and along z-axis respectively. Using the definition of addition and scalar multiplication, the vector $[x, y, z]$ can be written as

$$\begin{aligned}\underline{u} = [x, y, z] &= [x,0,0] + [0,y,0] + [0,0,z] \\ &= x[1,0,0] + y[0,1,0] + z[0,0,,1] \\ &= x\underline{i} + y\underline{j} + z\underline{k}\end{aligned}$$

Thus each vector $[x, y, z]$ in R^3 can be uniquely represented by $x\underline{i} + y\underline{j} + z\underline{k}$.

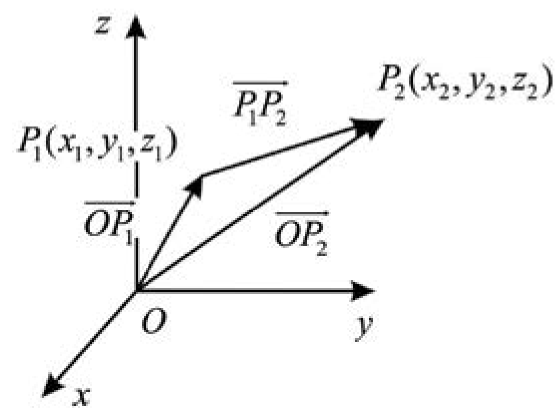
In terms of unit vector $\underline{i}$, $\underline{j}$ and $\underline{k}$, , the sum $\underline{u} + \underline{v}$ of two vectors

$\underline{u} = [x, y, z]$ and $\underline{v} = [x', y', z']$ is written as

$$\begin{aligned}\underline{u}+\underline{v} &= [x+x', y+y', z+z'] \\ &= (x+x')\underline{i} + (y+y')\underline{j} + (z+z')\underline{k}\end{aligned}$$

7.2.4 Distance Between two Points in Space

If $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ are the position vectors of the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$



The vector $\overrightarrow{P_1P_2}$, is given by

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = [x_2 - x_1, y_2 - y_1, z_2 - z_1]$$

$\therefore$ Distance between P_1 and P_2 $= \left|\overrightarrow{P_1P_2}\right|$

$$= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

This is called **distance formula** between two points P_1 and P_2 in R^3,

Example 2: If $\underline{u} = 2\underline{i} + 3\underline{j} + \underline{k}$, $\underline{v} = 4\underline{i} + 6\underline{j} + 2\underline{k}$ and $\underline{w} = -6\underline{i} - 9\underline{j} - 3\underline{k}$, then

(a) Find

(i) $\underline{u} + 2\underline{v}$ (ii) $|\underline{u} - \underline{v} - \underline{w}|$

(b) Show that $\underline{u}$, $\underline{v}$, and $\underline{w}$ are parallel to each other.

Solution: (a)

(i) $$\begin{aligned}\underline{u}+2\underline{v} &= 2\underline{i}+3\underline{j}+\underline{k}+2(4\underline{i}+6\underline{j}+2\underline{k}) \\ &= 2\underline{i}+3\underline{j}+\underline{k}+8\underline{i}+12\underline{j}+4\underline{k} \\ &= 10\underline{i}+15\underline{j}+5\underline{k}\end{aligned}$$

(ii) $$\begin{aligned}\underline{u}-\underline{v}-w &= (2\underline{i}+3\underline{j}+\underline{k})-(4\underline{i}+6\underline{j}+2\underline{k})-(-6\underline{i}-9\underline{j}-3\underline{k}) \\ &= (2-4+6)\underline{i}+(3-6+9)\underline{j}+(1-2+3)\underline{k} \\ &= 4\underline{i}+6\underline{j}+2\underline{k}\end{aligned}$$

(b) $\underline{v} = 4\underline{i}+6\underline{j}+2\underline{k} = 2(2\underline{i}+3\underline{j}+\underline{k})$

$\therefore$ $\underline{v} = 2\underline{u}$

$\Rightarrow$ $\underline{u}$ and $\underline{v}$ are parallel vectors, and have same direction

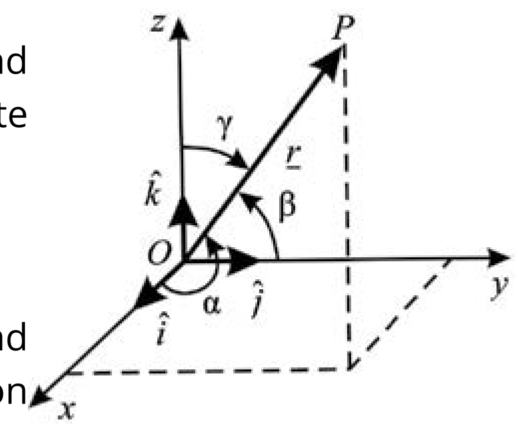
Again $$\begin{aligned}\underline{w} &= -6\underline{i}-9\underline{j}-3\underline{k} \\ &= -3(2\underline{i}+3\underline{j}+\underline{k})\end{aligned}$$

$\therefore$ $\underline{w} = -3\underline{u}$

$\Rightarrow$ $\underline{u}$ and $\underline{w}$ are parallel vectors and have opposite direction.

Hence $\underline{u}$, $\underline{v}$ and $\underline{w}$ are parallel to each other.

7.2.5 Direction Angles and Direction Cosines of a Vector

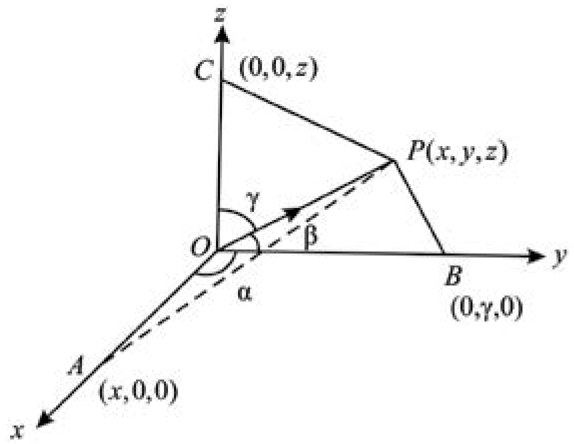


Let $\underline{r} = \overrightarrow{OP} = x\underline{i} + y\underline{j} + z\underline{k}$ be a non-zero vector, let α, β and γ denote the angles formed between $\underline{r}$ and the unit coordinate vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$ respectively.
such that

$0 \le \alpha \le \pi,\quad 0 \le \beta \le \pi,\quad \text{and}\quad 0 \le \gamma \le \pi,$

(i) the angles α, β, γ are called the direction angles and

(ii) the numbers cos α, cos β and cos γ are called direction cosines of the vector $\underline{r}$.



Important Result:

Prove that cos² α + cos² β + cos² γ = 1

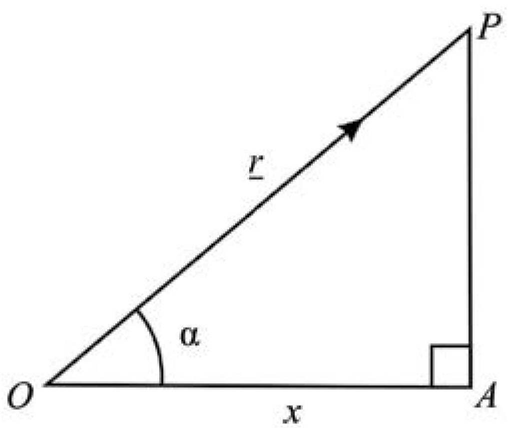
Solution:

Let $\underline{r} = [x, y, z] = x\underline{i} + y\underline{j} + z\underline{k}$

$\therefore \quad |\underline{r}| = \sqrt{x^2 + y^2 + z^2} = r$

then $\dfrac{\underline{r}}{|\underline{r}|} = \left[\dfrac{x}{r}, \dfrac{y}{r}, \dfrac{z}{r}\right]$ is the unit vector in the direction of the vector $\underline{r} = \overrightarrow{OP}$.

It can be visualized that the triangle *OAP* is a right triangle with $\angle A = 90^0$.
Therefore in right triangle *OAP*,



$$\cos\alpha = \frac{\overline{OA}}{\overline{OP}} = \frac{x}{r}, \text{ similarly}$$

$$\cos\beta = \frac{y}{r},\quad \cos\gamma = \frac{z}{r}$$

The numbers $\cos\alpha = \dfrac{x}{r},$, $\cos\beta = \dfrac{y}{r}$ and $\cos\gamma = \dfrac{z}{r}$ are called the direction cosines of $\overrightarrow{OP}$.

$$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = \frac{x^2 + y^2 + z^2}{r^2} = \frac{r^2}{r^2} = 1$$

EXERCISE 7.2

1. Let *A* = (2, 5), *B* = (–1,1) and *C* = (2, –6). Find
(i) $\overrightarrow{AB}$ (ii) $2\overrightarrow{AB} - \overrightarrow{CB}$ (iii) $2\overrightarrow{CB} - 2\overrightarrow{CA}$

2. Let $\underline{u} = \underline{i} + 2\underline{j} - \underline{k}$, $\underline{v} = 3\underline{i} - 2\underline{j} + 2\underline{k}$, $\underline{w} = 5\underline{i} - \underline{j} + 3\underline{k}$. Find the indicated vector or number.
(i) $\underline{u} + 2\underline{v} + \underline{w}$ (ii) $\underline{u} - 3\underline{w}$ (iii) $|3\underline{v} + \underline{w}|$

3. Find the magnitude of the vector $\underline{v}$ and write the direction cosines of $\underline{v}$.
(i) $\underline{v} = 2\underline{i} + 3\underline{j} + 4\underline{k}$ (ii) $\underline{v} = \underline{i} - \underline{j} - \underline{k}$ (iii) $\underline{v} = 4\underline{i} - 5\underline{j}$

4. Find α, so that $|\alpha\underline{i} + (\alpha + 1)\underline{j} + 2\underline{k}| = 3$.

5. Find a unit vector in the direction of $\underline{v} = \underline{i} + 2\underline{j} - \underline{k}$.

6. If $\underline{a} = 3\underline{i} - \underline{j} - 4\underline{k}$, $\underline{b} = -2\underline{i} - 4\underline{j} - 3\underline{k}$ and $\underline{c} = \underline{i} + 2\underline{j} - \underline{k}$.
Find a unit vector parallel to $3\underline{a} - 2\underline{b} + 4\underline{c}$.

7. Find a vector whose
(i) magnitude is 4 and is parallel to $2\underline{i} - 3\underline{j} + 6\underline{k}$
(ii) magnitude is 2 and is parallel to $-\underline{i} + \underline{j} + \underline{k}$

8. If $\underline{u} = 2\underline{i} + 3\underline{j} + 4\underline{k},$, $\underline{v} = -\underline{i} + 3\underline{j} - \underline{k}$ and $\underline{w} = \underline{i} + 6\underline{j} + z\underline{k}$ represent the sides of a triangle. Find the value of *z*.

9. The position vectors of the points *A*, *B*, *C* and *D* are $2\underline{i} - \underline{j} + \underline{k}$, $3\underline{i} + \underline{j}$, $2\underline{i} + 4\underline{j} - 2\underline{k}$ and $-\underline{i} - 2\underline{j} + \underline{k}$ respectively. Show that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

10. We say that two vectors $\underline{v}$ and $\underline{w}$ in space are parallel if there is a scalar *c* such that $\underline{v}$ = $c\underline{w}$. The vectors point in the same direction if *c* > 0, and the vectors point in the opposite direction if *c* < 0
(a) Find two vectors of length 2 parallel to the vector $\underline{v} = 2\underline{i} - 4\underline{j} + 4\underline{k}$.
(b) Find the constant *a* so that the vectors $\underline{v} = \underline{i} - 3\underline{j} + 4\underline{k}$ and $\underline{w} = a\underline{i} + 9\underline{j} - 12\underline{k}$ are parallel.
(c) Find a vector of length 5 in the direction opposite that of $\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$.
(d) Find *a* and *b* so that the vectors $3\underline{i} - \underline{j} + 4\underline{k}$ and $a\underline{i} + b\underline{j} - 2\underline{k}$ are parallel.

11. Find the direction cosines for the given vector:

(i) $\underline{v} = 3\underline{i} - \underline{j} + 2\underline{k}$ (ii) $6\underline{i} - 2\underline{j} + \underline{k}$

(iii) $\overrightarrow{PQ}$, where $P = (2,\ 1,\ 5)$ and $Q = (1,\ 3,\ 1)$.

12. Which of the following triples can be the direction angles of a single vector:

(i) $45^0, 45^0, 60^0$ (ii) $30^0, 45^0, 60^0$ (iii) $45^0, 60^0, 60^0$

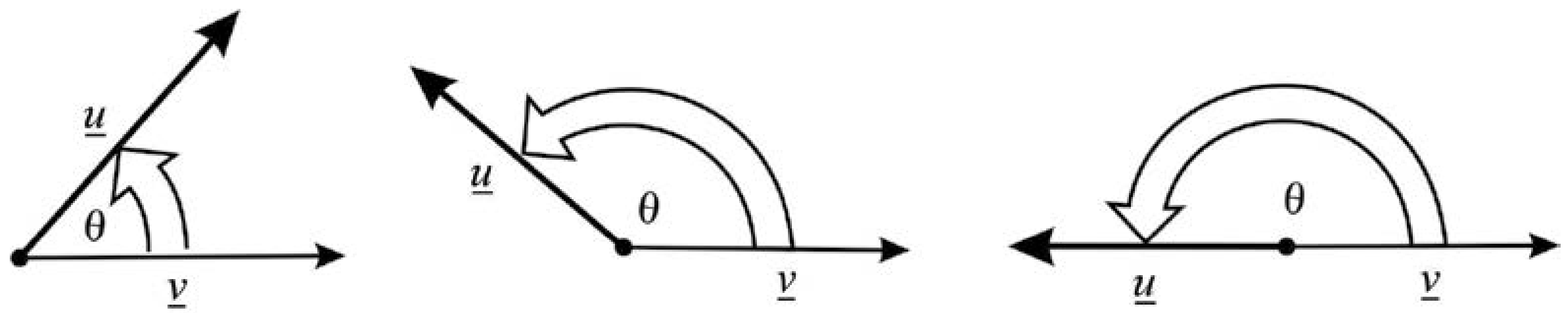
7.3 THE SCALAR PRODUCT OF TWO VECTORS

We shall now consider products of two vectors that originated in the study of Physics and Engineering. The concept of angle between two vectors is expressed in terms of a **scalar product of two vectors.**

Definition 1:

Let two non-zero **vectors** $\underline{u}$ and $\underline{v}$, in the plane or in space, have same initial point. The **dot** product of $\underline{u}$ and $\underline{v}$, written as $\underline{u}.\underline{v}$, is defined by

$$\underline{u}.\underline{v} = |\underline{u}|\ |\underline{v}|\ \cos\theta$$



where θ is the angle between $\underline{u}$ and $\underline{v}$ and $0 \le 6 \le \pi$

Definition 2:

(a) If $\underline{u} = a_1\underline{i} + b_1\underline{j}$ and $\underline{v} = a_2\underline{i} + b_2\underline{j}$.

are two non-zero vectors in the plane. The dot product $\underline{u}.\underline{v}$ is defined by

$\underline{u}.\underline{v} = a_1a_2 + b_1b_2$

(b) If $\underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$ and $\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$.

are two non-zero vectors in space. The dot product $\underline{u}.\underline{v}$ is defined by

$\underline{u}.\underline{v} = a_1a_2 + b_1b_2 + c_1c_2$

Note: The dot product is also referred to the **scalar** product or the **inner** product.

7.3.1 Deductions of the Important Results

By Applying the definition of dot product to unit vectors $\underline{i},\ \underline{j},\ \underline{k}$, we have,

(a) $\underline{i}.\underline{i} = |\underline{i}||\underline{i}|\ \cos 0^\circ = 1$ (b) $\underline{i}.\underline{j} = |\underline{i}||\underline{j}|\ \cos 90^\circ = 0$

$\underline{j}.\underline{j} = |\underline{j}||\underline{j}|\ \cos 0^\circ = 1$ $\underline{j}.\underline{k} = |\underline{j}||\underline{k}|\ \cos 90^\circ = 0$

$\underline{k}.\underline{k} = |\underline{k}||\underline{k}|\ \cos 0^\circ = 1$ $\underline{k}.\underline{i} = |\underline{k}||\underline{i}|\ \cos 90^\circ = 0$

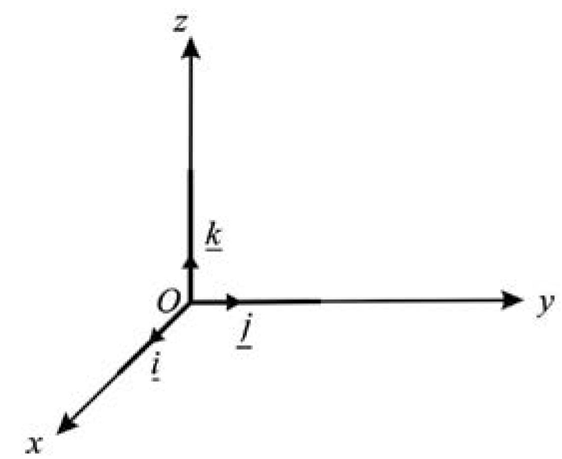
(c) $\underline{u}.\underline{v} = |\underline{u}||\underline{v}|\ \cos\theta$

$= |\underline{v}||\underline{u}|\ \cos(-\theta)$

$= |\underline{v}||\underline{u}|\ \cos\theta$

$\Rightarrow \underline{u}.\underline{v} = \underline{v}.\underline{u}$

$\therefore$ Dot product of two vectors is commutative.



7.3.2 Perpendicular (Orthogonal) Vectors

Definition: Two non-zero vectors $\underline{u}$ and $\underline{v}$ are perpendicular if and only if $\underline{u}.\underline{v} = 0$.

Since angle between $\underline{u}$ and $\underline{v}$ is $\frac{\pi}{2}$ and $\cos\frac{\pi}{2} = 0$

so $\underline{u}.\underline{v} = |\underline{u}|\ |\underline{v}|\ \cos\frac{\pi}{2}$

$\therefore \underline{u}.\underline{v} = 0$

Note: As $\underline{0}\ .\ \underline{b} = 0$, for every vector $\underline{b}$. So the zero vector is regarded to be perpendicular to every vector.

7.3.3 Properties of Dot Product

Let $\underline{u}$, $\underline{v}$ and $\underline{w}$ be vectors and let c be a real number, then

(i) $\underline{u}.\underline{v} = 0 \Rightarrow \underline{u} = 0$ or $\underline{v} = 0$

(ii) $\underline{u}.\underline{v} = \underline{v}.\underline{u}$ (commutative property)

(iii) $\underline{u}.(\underline{v} + \underline{w}) = \underline{u}.\underline{v} + \underline{u}.\underline{w}$ (distributive property)

(iv) $(c\,\underline{u}).\underline{v} = c\,(\underline{u}.\underline{v})$, ($c$ is scalar)

The proofs of the properties are left as an exercise for the students.

7.3.4 Analytical Expression of Dot Product $\underline{u}.\underline{v}$ (Dot product of vectors in their components form)

Let $\underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$ and $\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$ be two non-zero vectors.

From distributive Law we can write:

$$\therefore\ \underline{u}.\underline{v} = (a_1\underline{i} + b_1\underline{j} + c_1\underline{k}).(a_2\underline{i} + b_2\underline{j} + c_2\underline{k})$$

$$\begin{aligned} &= a_1a_2(\underline{i}.\underline{i}) + a_1b_2(\underline{i}.\underline{j}) + a_1c_2(\underline{i}.\underline{k}) \\ &+ b_1a_2(\underline{j}.\underline{i}) + b_1b_2(\underline{j}.\underline{j}) + b_1c_2(\underline{j}.\underline{k}) \\ &+ c_1a_2(\underline{k}.\underline{i}) + c_1b_2(\underline{k}.\underline{j}) + c_1c_2(\underline{k}.\underline{k}) \end{aligned} \qquad \left|\begin{aligned} \because\ &\underline{i}.\underline{i} = \underline{j}.\underline{j} = \underline{k}.\underline{k} = 1 \\ &\underline{i}.\underline{j} = \underline{j}.\underline{k} = \underline{k}.\underline{i} = 0 \end{aligned}\right.$$

$$\Rightarrow\ \underline{u}.\underline{v} = a_1a_2 + b_1b_2 + c_1c_2$$

Hence the dot product of two vectors is the sum of the product of their corresponding components.

Equivalence of two definitions of dot product of two vectors has been proved in the following example.

Example 1: (i) If $\underline{v} = [x_1, y_1]$ and $\underline{w} = [x_2, y_2]$ are two vectors in the plane, then

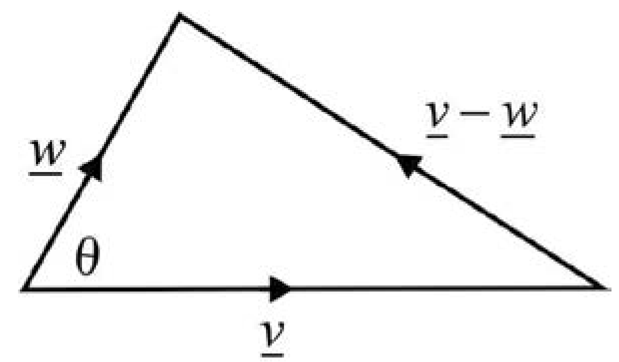
$$\underline{v}.\underline{w} = x_1x_2 + y_1y_2$$

(ii) If $\underline{v}$ and $\underline{w}$ are two non-zero vectors in the plane, then

$$v.w = |\underline{v}||\underline{w}|\cos\theta$$

where θ is the angle between $\underline{v}$ and $\underline{w}$ and $0 \le \theta \le \pi$.

Proof: Let $\underline{v}$ and $\underline{w}$ determine the sides of a triangle then the third side, opposite to the angle θ, has length $|\underline{v} - \underline{w}|$ (by triangle law of addition of vectors)



By law of cosines,

$$|\underline{v} - \underline{w}|^2 = |\underline{v}|^2 + |\underline{w}|^2 - 2|\underline{v}|\ |\underline{w}|\cos\theta \qquad (1)$$

if $\underline{v} = [x_1, y_1]$ and $\underline{w} = [x_2, y_2]$, then

$$\underline{v} - \underline{w} = [x_1 - x_2, y_1 - y_2]$$

So equation (1) becomes:

$$|x_1 - x_2|^2 + |y_1 - y_2|^2 = |x_1^2 + y_1^2| + |x_2^2 + y_2^2| - 2|\underline{v}|\ |\underline{w}|\cos\theta$$

$$-2x_1x_2 - 2y_1y_2 = -2|\underline{v}|\ |\underline{w}|\cos\theta$$

$$\Rightarrow\quad x_1x_2 + y_1y_2 = |\underline{v}|\ |\underline{w}|\cos\theta = \underline{v}.\underline{w}$$

Example 2: If $\underline{u} = 3\underline{i} - \underline{j} - 2\underline{k}$ and $\underline{v} = \underline{i} + 2\underline{j} - \underline{k}$, then

$$\underline{u}.\underline{v} = (-3)(1) + (-1)(2) + (-2)(-1) = 3$$

Example 3: If $\underline{u} = 2\underline{i} - 4\underline{j} + 5\underline{k}$ and $\underline{v} = -4\underline{i} - 3\underline{j} - 4\underline{k}$, then

$$\underline{u}.\underline{v} = (2)(4) + (-4)(-3) + (5)(-4) = 0$$

$\Rightarrow$ $\underline{u}$ and $\underline{v}$ are perpendicular

7.3.5 Angle between two vectors

The angle between two vectors $\underline{u}$ and $\underline{v}$ is determined from the definition of dot product, that is

(a) $\underline{u}.\underline{v} = |\underline{u}||\underline{v}|\cos\theta$, where $0 \le \theta \le \pi$

$$\therefore\ \cos\theta = \frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$$

(b) $\underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$ and $\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$, then

$$\underline{u}.\underline{v} = a_1a_2 + b_1b_2 + c_1c_2$$

$$|\underline{u}| = \sqrt{a_1^2 + b_1^2 + c_1^2} \text{ and } |\underline{v}| = \sqrt{a_2^2 + b_2^2 + c_2^2}$$

$$\because\quad \cos\theta = \frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$$

$$\therefore \quad \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Corollaries:

(i) If θ = 0 or π, the vectors $\underline{u}$ and $\underline{v}$ are collinear.

(ii) If $\theta = \frac{\pi}{2}$, $\cos\theta = 0 \quad \Rightarrow \underline{u}.\underline{v} = 0.$

The vectors $\underline{u}$ and $\underline{v}$ are perpendicular or orthogonal.

Example 4: Find the angle between the vectors

$\underline{u} = 2\underline{i} - \underline{j} + \underline{k}$ and $\underline{v} = -\underline{i} + \underline{j}$

Solution: $\underline{u} \,.\, \underline{v} = (2\underline{i} - \underline{j} + \underline{k}) \,.\, (-\underline{i} + \underline{j} + 0\underline{k})$

$= (2)(-1) + (-1)(1) + (1)(0) = -3$

$\therefore \quad |\underline{u}| = |2\underline{i} - \underline{j} + \underline{k}| = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{6}$

and $\quad |\underline{v}| = |-\underline{i} + \underline{j} + 0\underline{k}| = \sqrt{(-1)^2 + (1)^2 + (0)^2} = \sqrt{2}$

Now $\quad \cos\theta = \frac{\underline{u}.\underline{v}}{|\underline{u}|.|\underline{v}|}$

$\Rightarrow \quad \cos\theta = \frac{-3}{\sqrt{6}\sqrt{2}} = -\frac{\sqrt{3}}{2}$

$\therefore \quad \theta = \frac{5\pi}{6}$

Example 5: Find a scalar α so that the vectors

$2\underline{i} + \alpha\underline{j} + 5\underline{k}$ and $3\underline{i} + \underline{j} + \alpha\underline{k}$ are perpendicular.

Solution:

Let $\quad \underline{u} = 2\underline{i} + \alpha\underline{j} + 5\underline{k}$ and $\underline{v} = 3\underline{i} + \underline{j} + \alpha\underline{k}$

It is given that $\underline{u}$ and $\underline{v}$ are perpendicular

$\therefore \quad \underline{u} \,.\, \underline{v} = 0$

$\Rightarrow \quad (2\underline{i} + \alpha\underline{j} + 5\underline{k}) \,.\, (3\underline{i} + \underline{j} + \alpha\underline{k}) = 0$

$\Rightarrow \quad 6 + \alpha + 5\alpha = 0$

$\therefore \quad \alpha = -1$

Example 6:

Show that the vectors $2\underline{i} - \underline{j} + \underline{k}$, $\underline{i} - 3\underline{j} - 5\underline{k}$ and $3\underline{i} - 4\underline{j} - 4\underline{k}$ form the sides of a right triangle.

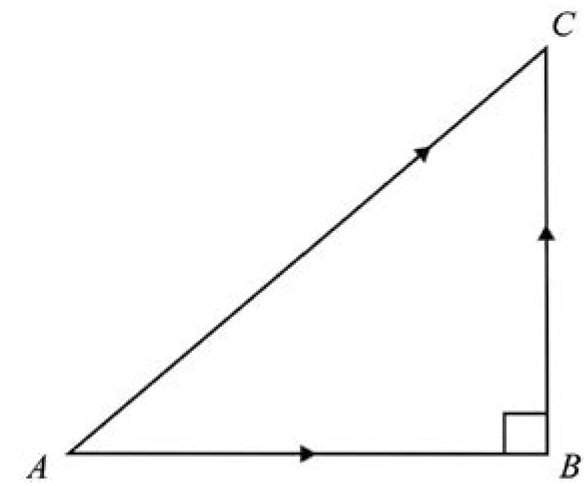
Solution:

Let $\overrightarrow{AB} = 2\underline{i} - \underline{j} + \underline{k}$ and $\overrightarrow{BC} = \underline{i} - 3\underline{j} - 5\underline{k}$

Now $\overrightarrow{AB} + \overrightarrow{BC} = (2\underline{i} - \underline{j} + \underline{k}) + (\underline{i} - 3\underline{j} - 5\underline{k})$

$= 3\underline{i} - 4\underline{j} - 4\underline{k} = \overrightarrow{AC}$ (third side)

$\therefore \quad \overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ form a triangle ABC.



Further we prove that △*ABC* is a right triangle

$\overrightarrow{AB} \,.\, \overrightarrow{BC} = (2\underline{i} - \underline{j} + \underline{k}).(\underline{i} - 3\underline{j} - 5\underline{k})$

$= (2)(1) + (-1)(-3) + (1)(-5)$

$= 2 + 3 - 5$

$= 0$

$\therefore \quad \overrightarrow{AB} \perp \overrightarrow{BC}$

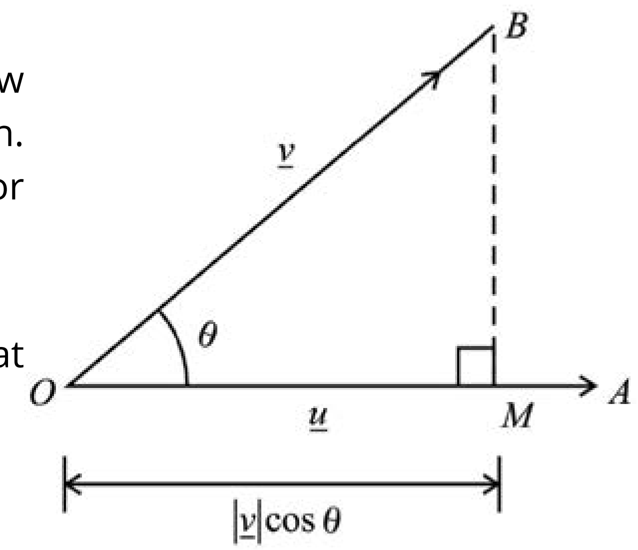
Hence △*ABC* is a right triangle.

7.3.6 Projection of one Vector upon another Vector:

In many physical applications, it is required to know "how much" of a vector is applied along a given direction. For this purpose we find the projection of one vector along the other vector.

Let $\overrightarrow{OA} = \underline{u}$ and $\overrightarrow{OB} = \underline{v}$

Let θ be the angle between them, such that 0 ≤ θ ≤ π.



Draw $\overline{BM} \perp OA$. Then $\overline{OM}$ is called the projection of $\underline{v}$ along $\underline{u}$.

$$\text{Now} \quad \frac{\overline{OM}}{\overline{OB}} = \cos\theta, \text{ that is,}$$

$$\overline{OM} = \left|\overline{OB}\right|\cos\theta = \left|\underline{v}\right|\cos\theta \qquad (1)$$

By definition, $\cos\theta = \dfrac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$ (2)

From (1) and (2), $\overline{OM} = |\underline{v}|.\dfrac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$

$\therefore$ Projection of $\underline{v}$ along $\underline{u} = \dfrac{\underline{u}.\underline{v}}{|\underline{u}|}$

Similarly, projection of $\underline{u}$ along $\underline{v} = \dfrac{\underline{u}.\underline{v}}{|\underline{v}|}$

Example 7: Show that the components of a vector are the projections of that vector along $\underline{i}$, $\underline{j}$ and $\underline{k}$ respectively.

Solution: Let $\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$, then

$$\text{Projection of } \underline{v} \text{ along } \underline{i} = \frac{\underline{v}.\underline{i}}{|\underline{i}|} = (a\underline{i} + b\underline{j} + c\underline{k}).\underline{i} = a$$

$$\text{Projection of } \underline{v} \text{ along } \underline{j} = \frac{\underline{v}.\underline{j}}{|\underline{j}|} = (a\underline{i} + b\underline{j} + c\underline{k}).\underline{j} = b$$

$$\text{Projection of } \underline{v} \text{ along } \underline{k} = \frac{\underline{v}.\underline{k}}{|\underline{k}|} = (a\underline{i} + b\underline{j} + c\underline{k}).\underline{k} = c$$

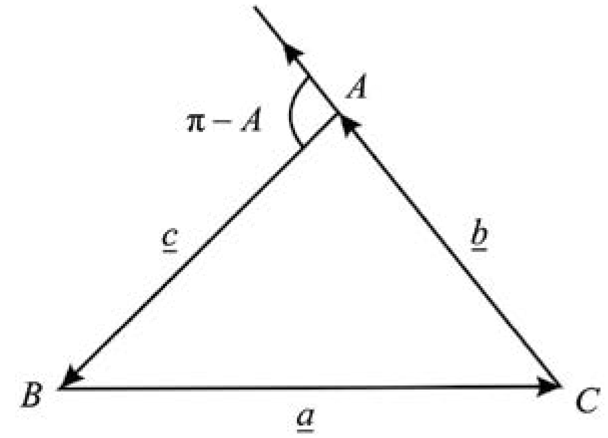
Hence components a, b and c of vector $\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$ are projections of vector $\underline{v}$ along $\underline{i}$, $\underline{j}$ and $\underline{k}$ respectively.

Example 8: Prove that in any triangle ABC

(i) $a^2 = b^2 + c^2 - 2bc\cos A$ (Cosine Law)

(ii) $a = b\cos C + c\cos B$ (Projection Law)

Solution: Let the vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ be along the sides BC, CA and AB of the triangle ABC as shown in the figure.



$\therefore \quad \underline{a} + \underline{b} + \underline{c} = \underline{0}$

$\Rightarrow \quad \underline{a} = -(\underline{b} + \underline{c})$

Now $\underline{a}.\underline{a} = (\underline{b} + \underline{c}).(\underline{b} + \underline{c})$

$\Rightarrow \quad = \underline{b}.\underline{b} + \underline{b}.\underline{c} + \underline{c}.\underline{b} + \underline{c}.\underline{c}$

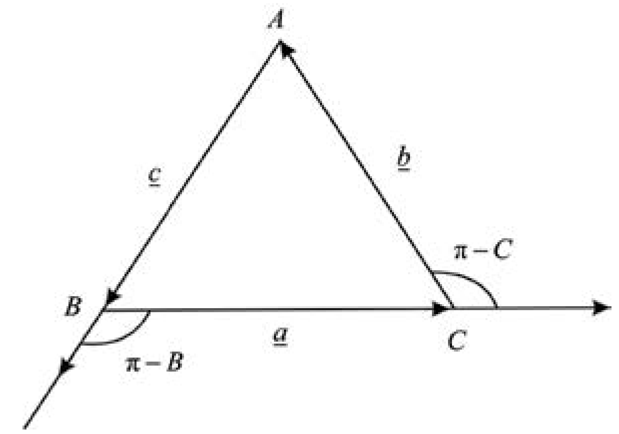
$\Rightarrow \quad a^2 = b^2 + 2\underline{b}.\underline{c} + c^2$ $\quad (\ \underline{b}.\underline{c} \quad \underline{c}.\underline{b})$

$\Rightarrow \quad a^2 = b^2 + c^2 + 2bc.\cos(\pi - A)$

$\therefore \quad a^2 = b^2 + c^2 - 2bc\cos A$

(ii) $\underline{a} + \underline{b} + \underline{c} = \underline{0}$

$\Rightarrow \quad \underline{a} = -\underline{b} - \underline{c}$



Take dot product with $\underline{a}$

$\underline{a}.\underline{a} = -\underline{a}.\underline{b} - \underline{a}.\underline{c}$

$= -ab\cos(\pi - C) - ac\cos(\pi - B)$

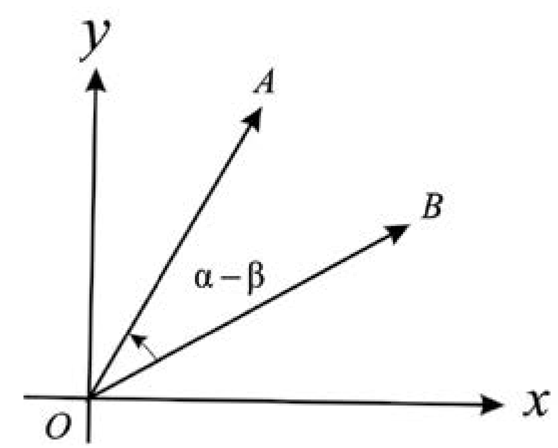
$a^2 = ab\cos C + ac\cos B$

$\Rightarrow a = b\cos C + c\cos B$

Example 9: Prove that: $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Solution: Let $\overline{OA}$ and $\overline{OB}$ be the unit vectors in the xy-plane making angles α and β with the positive x-axis.

So that $\angle AOB = \alpha - \beta$



$$\text{Now} \quad \overrightarrow{OA} = \cos\alpha\underline{i} + \sin\alpha\underline{j}$$

$$\text{and} \quad \overrightarrow{OB} = \cos\beta\underline{i} + \sin\beta\underline{j}$$

$$\therefore \quad \overrightarrow{OA}.\overrightarrow{OB} = (\cos\alpha\underline{i} + \sin\alpha\underline{j}).(\cos\beta\underline{i} + \sin\beta\underline{j})$$

$$\Rightarrow \quad \left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\therefore \quad \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \qquad \left(\because \left|\overrightarrow{OA}\right| = \left|\overrightarrow{OB}\right| = 1\right)$$

EXERCISE 7.3

1. Find the cosine of the angle θ between $\underline{u}$ and $\underline{v}$:

(i) $\underline{u}=3\underline{i}+\underline{j}-\underline{k},\ \underline{v}=2\underline{i}-\underline{j}+\underline{k}$ (ii) $\underline{u}=\underline{i}-3\underline{j}+4\underline{k},\ \underline{v}=4\underline{i}-\underline{j}+3\underline{k}$

(iii) $\underline{u}=[-3,\ 5]\ ,\ \underline{v}=[6,\ -2]$ (iv) $\underline{u}=[2,-3,\ 1]\ ,\ \underline{v}=[2,\ 4,\ 1]$

2. Calculate the projection of $\underline{a}$ along $\underline{b}$ and projection of $\underline{b}$ along $\underline{a}$ when:

(i) $\underline{a}=\underline{i}-\underline{k},\ \underline{b}=\underline{j}+\underline{k}$ (ii) $\underline{a}=3\underline{i}+\underline{j}-\underline{k}\ ,\ \underline{b}=-2\underline{i}-\underline{j}+\underline{k}$

3. Find a real number α so that the vectors $\underline{u}$ and $\underline{v}$ are perpendicular.

(i) $\underline{u}=2\alpha\underline{i}+\underline{j}-\underline{k}\ ,\quad \underline{v}=\underline{i}+\alpha\underline{j}+4\underline{k}$

(ii) $\underline{u}=\alpha\underline{i}+2\alpha\underline{j}+3\underline{k}\ ,\quad \underline{v}=\underline{i}+\alpha\underline{j}+3\underline{k}$

4. Find the number z so that the triangle with vertices *A*(1, –1, 0), *B*(–2, 2, 1) and *C*(0, 2, z) is a right triangle with right angle at *C*.

5. If $\underline{v}$ is a vector for which

$\underline{v}.\underline{i}=0,\ \ \underline{v}.\underline{j}=0,\ \ \underline{v}.\underline{k}=0,\ $ find $\underline{v}.$.

6. (i) Show that the vectors $3\underline{i}-2\underline{j}+\underline{k}\ ,\ \underline{i}-3\underline{j}+5\underline{k}$ and $2\underline{i}+\underline{j}-4\underline{k}$ form a right angle.

(ii) Show that the set of points *P* = (1,3,2), *Q* = (4,1,4) and *P* = (6,5,5) form a right triangle.

7. Show that mid point of hypotenuse a right triangle is equidistant from its vertices.

8. Prove that perpendicular bisectors of the sides of a triangle are concurrent.

9. Prove that the altitudes of a triangle are concurrent.

10. Prove that the angle in a semi circle is a right angle.

11. Prove that cos(α + β) = cos α cos β – sin α sin β

12. Prove that in any triangle *ABC*.

(i) *b* = *c* cos *A* + *a* cos *C* (ii) *c* = *a* cos *B* + *b* cos *A*

(iii) $b^2 = c^2 + a^2 - 2ca\cos B$ (iv) $c^2 = a^2 + b^2 - 2ab\cos C$.

7.4 THE CROSS PRODUCT OR VECTOR PRODUCT OF TWO VECTORS

The vector product of two vectors is widely used in Physics, particularly, Mechanics and Electricity. It Is only defined for vectors in space.

Let $\underline{u}$ and $\underline{v}$ be two non-zero vectors. The **cross** or **vector** product of $\underline{u}$ and $\underline{v}$, written as $\underline{u}$ x $\underline{v}$, is defined by

$$\underline{u}\times\underline{v} = \left(|\underline{u}||\underline{v}|\ \sin\theta\right)\ \underline{\hat{n}}$$

where θ is the angle between the vectors, such that 0 ≤ θ ≤ π and $\underline{\hat{n}}$ is a "unit vector perpendicular to the plane of $\underline{u}$ and $\underline{v}$ with direction given by the right hand rule.

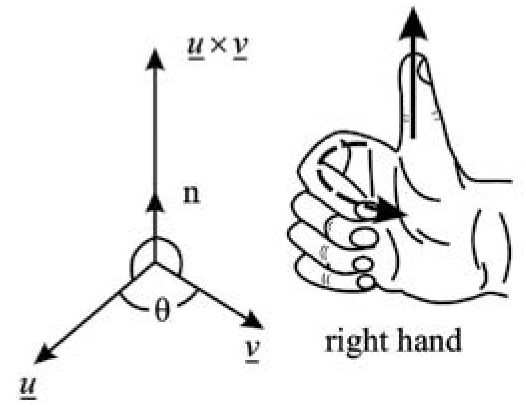


Figure (a)

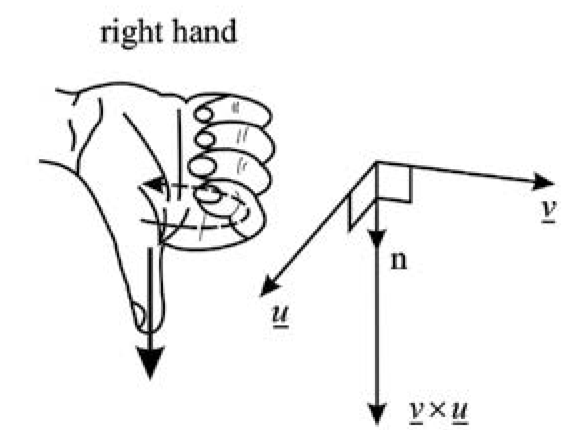


Figure (b)

Right hand rule

(i) If the fingers of the right hand point along the vector $\underline{u}$ and then curl towards the vector $\underline{v}$, then the thumb will give the direction of $\underline{\hat{n}}$ which is $\underline{u}$ x $\underline{v}$. It is shown in the figure (a).

(ii) In figure (b), the right hand rule shows the direction of $\underline{v}$ x $\underline{u}$.

7.4.1 Derivation of useful results of cross products

(a) By applying the definition of cross product to unit vectors $\underline{i},\ \underline{j}$ and $\underline{k}$, we have:

(a) $\underline{i}\times\underline{i}=|\underline{i}||\underline{i}|\sin 0^\circ\ \ \underline{\hat{n}}=0$

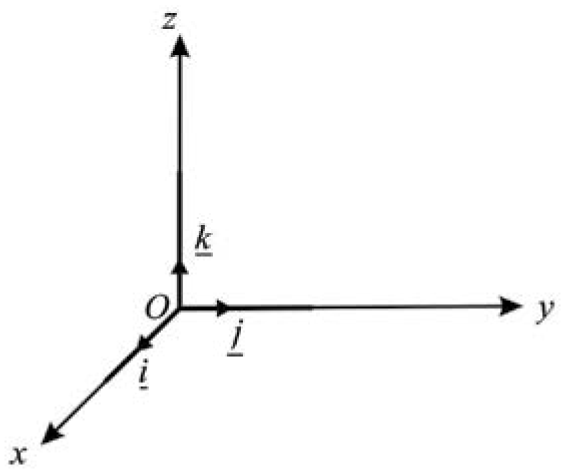
$\underline{j}\times\underline{j}=|\underline{j}||\underline{j}|\sin 0^\circ\ \ \underline{\hat{n}}=0$

$\underline{k}\times\underline{k}=|\underline{k}||\underline{k}|\sin 0^\circ\ \ \underline{\hat{n}}=0$

(b) $\underline{i}\times\underline{j}=|\underline{i}||\underline{j}|\sin 90^\circ\ \ \underline{k}=\underline{k}$

$\underline{j}\times\underline{k}=|\underline{j}||\underline{k}|\sin 90^\circ\ \ \underline{i}=\underline{i}$

$\underline{k}\times\underline{i}=|\underline{k}||\underline{i}|\sin 90^\circ\ \ \underline{j}=\underline{j}$

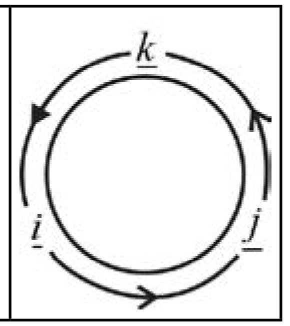


(c) $\underline{u}\times\underline{v}=|\underline{u}||\underline{v}|\ \sin\theta\ \underline{\hat{n}}=|\underline{v}||\underline{u}|\ \sin(-\theta)\ \underline{\hat{n}}=-|\underline{v}||\underline{u}|\ \sin\theta\ \underline{\hat{n}}$

$\Rightarrow\quad \underline{u}\times\underline{v}=-\underline{v}\times\underline{u}$

(d) $\underline{u}\times\underline{u}=|\underline{u}||\underline{u}|\sin 0^\circ\ \ \underline{\hat{n}}=0$

Note: The cross product of $\underline{i}$, $\underline{j}$ and $\underline{k}$ are written in the cyclic pattern. The given figure is helpful in remembering this pattern.



7.4.2 Properties of Cross product

The cross product possesses the following properties:

(i) $\underline{u}\times\underline{v}=\underline{0}$ if $\underline{u}=\underline{0}$ or $\underline{v}=\underline{0}$

(ii) $\underline{u}\times\underline{v}=-\underline{v}\times\underline{u}$

(iii) $\underline{u}\times(\underline{v}+\underline{w})=\underline{u}\times\underline{v}+\underline{u}\times\underline{w}$ (Distributive property)

(iv) $\underline{u}\times(k\underline{v})=(k\underline{u})\times\underline{v}=k(\underline{u}\times\underline{v})$, k is scalar

(v) $\underline{u}\times\underline{u}=\underline{0}$

The proofs of these properties are left as an exercise for the students.

7.4.3 Analytical Expression of $\underline{u}$ x $\underline{v}$ (Determinant formula for $\underline{u}$ x $\underline{v}$)

Let $\underline{u}=a_1\underline{i}+b_1\underline{j}+c_1\underline{k}$ and $\underline{v}=a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$, then

$$\underline{u}\times\underline{v}=(a_1\underline{i}+b_1\underline{j}+c_1\underline{k})\times(a_2\underline{i}+b_2\underline{j}+c_2\underline{k})$$

$$=a_1a_2(\underline{i}\times\underline{i})+a_1b_2(\underline{i}\times\underline{j})+a_1c_2(\underline{i}\times\underline{k}) \quad \text{(by distributive property)}$$

$$+b_1a_2(\underline{j}\times\underline{i})+b_1b_2(\underline{j}\times\underline{j})+b_1c_2(\underline{j}\times\underline{k}) \quad \left|\ \therefore\ \underline{i}\times\underline{j}=\underline{k}=-\underline{j}\times\underline{i}\right.$$

$$+c_1a_2(\underline{k}\times\underline{i})+c_1b_2(\underline{k}\times\underline{j})+c_1c_2(\underline{k}\times\underline{k}) \quad \left|\ \underline{i}\times\underline{i}=\underline{j}\times\underline{j}=\underline{k}\times\underline{k}=0\right.$$

$$=a_1b_2\underline{k}-a_1c_2\underline{j}-b_1a_2\underline{k}+b_1c_2\underline{i}+c_1a_2\underline{j}-c_1b_2\underline{i}$$

$$\Rightarrow\ \underline{u}\times\underline{v}=(b_1c_2-c_1b_2)\underline{i}-(a_1c_2-c_1a_2)\underline{j}+(a_1b_2-b_1a_2)\underline{k} \qquad \text{(i)}$$

The expansion of 3 x 3 determinant

$$\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}=(b_1c_2-c_1b_2)\underline{i}-(a_1c_2-c_1a_2)\underline{j}+(a_1b_2-b_1a_2)\underline{k}$$

The terms on R.H.S of equation (i) are the same as the terms in the expansion of the above determinant

Hence $\underline{u}\times\underline{v}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}$ (ii)

which is known as determinant formula for $\underline{u}$ x $\underline{v}$.

Note: The expression on R.H.S. of equation (ii) is not an actual determinant, since its entries are not all scalars. It is simply a way of remembering the complicated expression on R.H.S. of equation (i).

7.4.4 Parallel Vectors

If $\underline{u}$ and $\underline{v}$ are parallel vectors, ($\theta=0 \ \Rightarrow\ \sin 0^\circ=0$), then

$$\underline{u}\times\underline{v}\ =\ |\underline{u}|\ |\underline{v}|\ \sin\theta\ \hat{\text{n}}$$

$$\underline{u}\times\underline{v}\ =\underline{0} \quad \text{or} \quad \underline{v}\times\underline{u}\ =0$$

And if $\underline{u}\times\underline{v}=\underline{0}$. then

either $\sin\theta=0$ or $|\underline{u}|=0$ or $|\underline{v}|=0$

(i) If $\sin\theta=0 \ \Rightarrow\ \theta=0^\circ$ or 180° , which shows that the vectors $\underline{u}$ and $\underline{v}$ are parallel.

(ii) If $\underline{u}$ = 0 or $\underline{v}$ = 0, then since the zero vector has no specific direction, we adopt the convention that the zero vector is parallel to every vector.

Note: Zero vector is both parallel and perpendicular to every vector. This apparent contradiction will cause no trouble, since the angle between two vectors is never applied when one of them is zero vector.

Example 1: Find a vector perpendicular to each of the vectors

$\underline{a}=2\underline{i}+\underline{j}+\underline{k}$ and $\underline{b}=4\underline{i}+2\underline{j}-\underline{k}$

Solution: A vector perpendicular to both the vectors $\underline{a}$ and $\underline{b}$ is $\underline{a}$ x $\underline{b}$

$$\therefore \quad \underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ 4 & 2 & -1 \end{vmatrix} = -\underline{i} + 6\underline{j} + 8\underline{k}$$

Verification:

$$\underline{a}.\underline{a}\times\underline{b} = (2\underline{i} + \underline{j} + \underline{k}).(-\underline{i} + 6\underline{j} + 8\underline{k}) \quad = (2)(-1) + (-1)(6) + (1)(8) = 0$$

and $$\underline{b}.\underline{a}\times\underline{b} = (4\underline{i} + 2\underline{j} - \underline{k}).(-\underline{i} + 6\underline{j} + 8\underline{k}) \quad = (4)(-1) + (2)(6) + (-1)(8) = 0$$

Hence $\underline{a}$ x $\underline{b}$ is perpendicular to both the vectors $\underline{a}$ and $\underline{b}$.

Example 2: If $\underline{a} = 4\underline{i} + 3\underline{j} + \underline{k}$ and $\underline{b} = 2\underline{i} - \underline{j} + 2\underline{k}$. Find a unit vector perpendicular to both $\underline{a}$ and $\underline{b}$. Also find the sine of the angle between the vectors $\underline{a}$ and $\underline{b}$.

Solution:

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix} = 7\underline{i} - 6\underline{j} - 10\underline{k}$$

and $$|\underline{a}\times\underline{b}| = \sqrt{(7)^2 + (-6)^2 + (10)^2} = \sqrt{185}$$

$\therefore$ A unit vector $\hat{\underline{n}}$ perpendicular to $\underline{a}$ and $\underline{b}$ $= \dfrac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}$

$$= \frac{1}{\sqrt{185}}(7\underline{i} - 6\underline{j} - 10\underline{k})$$

Now $$|\underline{a}| = \sqrt{(4)^2 + (3)^2 + (1)^2} = \sqrt{26}$$

$$|\underline{b}| = \sqrt{(2)^2 + (-1)^2 + (2)^2} = 3$$

If θ is the angle between $\underline{a}$ and $\underline{b}$, then $|\underline{a}\times\underline{b}| = |\underline{a}||\underline{b}| \ \sin\theta$

$$\Rightarrow \quad \sin\theta = \frac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|} = \frac{\sqrt{185}}{3\sqrt{26}}$$

Example 3: Prove that sin(α + β) = sin α cos β +cos α sin β

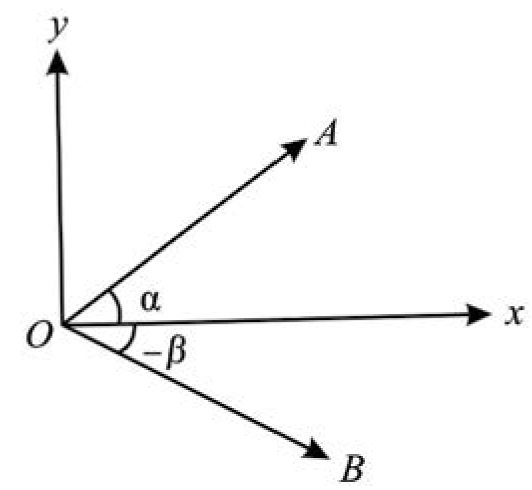
Proof: Let $\overline{OA}$ and $\overline{OB}$ be unit vectors in the *xy*-plane making angles α and –β with the positive *x*-axis respectively

So that $\angle AOB = \alpha + \beta$

Now $\overline{OA} = \cos\alpha\,\underline{i} + \sin\alpha\,\underline{j}$

and $\overline{OB} = \cos(-\beta)\underline{i} + \sin(-\beta)\underline{j}$

$= \cos\beta\,\underline{i} - \sin\beta\,\underline{j}$



$$\therefore \quad \overline{OB}\times\overline{OA} = (\cos\beta\,\underline{i} - \sin\beta\,\underline{j})\times(\cos\alpha\,\underline{i} + \sin\alpha\,\underline{j})$$

$$\Rightarrow \quad |\overline{OB}||\overline{OA}|\sin(\alpha+\beta)\underline{k} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\beta & -\sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$$\Rightarrow \quad \sin(\alpha+\beta)\underline{k} = (\sin\alpha\cos\beta + \cos\alpha\sin\beta)\underline{k}$$

$$\therefore \quad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Example 4: In any triangle *ABC*, prove that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \text{(Law of Sines)}$$

Proof: Suppose vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ are along the sides *BC*, *CA* and *AB* respectively of the triangle *ABC*.

$\therefore \quad \underline{a} + \underline{b} + \underline{c} = 0$

$\Rightarrow \quad \underline{b} + \underline{c} = -\underline{a} \qquad$ (i)

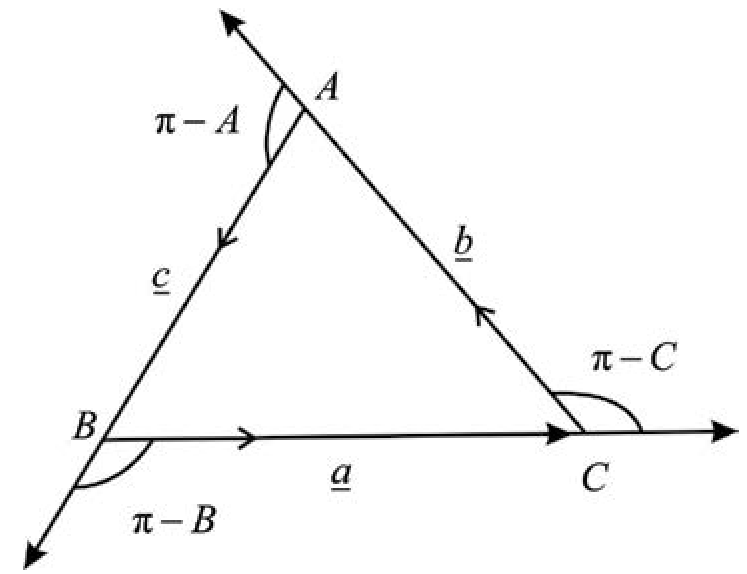
Take cross product with $\underline{c}$

$\underline{b}\times\underline{c} + \underline{c}\times\underline{c} = -\underline{a}\times\underline{c}$

$\underline{b}\times\underline{c} = \underline{c}\times\underline{a} \qquad (\because \underline{c}\times\underline{c} = 0)$

$\Rightarrow \quad |\underline{b}\times\underline{c}| = |\underline{c}\times\underline{a}|$

$|\underline{b}||\underline{c}| \ \sin(\pi - A) = |\underline{c}||\underline{a}| \ \sin(\pi - B)$



$\Rightarrow \quad bc\sin A = ca\sin B \ \Rightarrow \ b\sin A = a\sin B$

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} \qquad \text{(ii)}$$

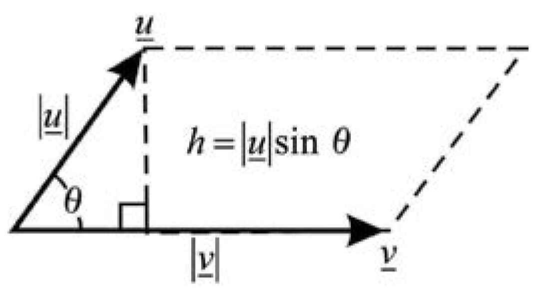
similarly by taking cross product of (i) with $\underline{b}$, we have

$$\frac{a}{\sin A}=\frac{c}{\sin C} \qquad \text{(iii)}$$

From (ii) and (iii), we get $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

7.4.5 Area of Parallelogram

If $\underline{u}$ and $\underline{v}$ are two non-zero vectors and θ is the angle between $\underline{u}$ and $\underline{v}$, then $|\underline{u}|$ and $|\underline{v}|$ represent the lengths of the adjacent sides of a parallelogram, (see figure)



We know that:

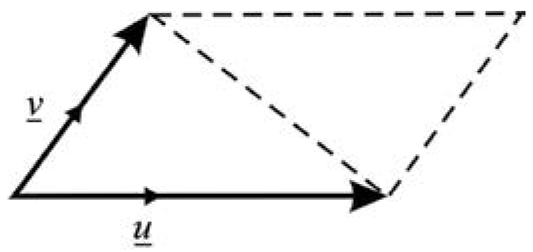
Area of parallelogram = base x height

$$= (\text{base})\ (h) \ = \ |\underline{u}||\underline{v}|\ \sin\theta$$

$\therefore$ Area of parallelogram $= |\underline{u}\times\underline{v}|$

7.4.6 Area of Triangle

From figure it is clear that



$$\text{Area of triangle}=\frac{1}{2}(\text{Area of parallelogram})$$

$$\therefore \quad \text{Area of triangle}=\frac{1}{2}\ |\underline{u}\times\underline{v}|$$

where $\underline{u}$ and $\underline{v}$ are vectors along two adjacent sides of the triangle.

Example 5: Find the area of the triangle with vertices *A*(1, –1, 1), *B*(2, 1, –1) and *C*(–1, 1, 2)
Also find a unit vector perpendicular to the plane ABC.

Solution: $\overrightarrow{AB}=(2-1)\underline{i}+(1+1)\underline{j}+(-1-1)\underline{k}=\underline{i}+2\underline{j}-2\underline{k}$

$\overrightarrow{AC}=(-1-1)\underline{i}+(1+1)\underline{j}+(2-1)\underline{k}=-2\underline{i}+2\underline{j}+\underline{k}$

$$\text{Now} \quad \overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 1 & 2 & -2\\ -2 & 2 & 1\end{vmatrix}=(2+4)\underline{i}-(1-4)\underline{j}+(2+4)\underline{k}=6\underline{i}+3\underline{j}+6\underline{k}$$

The area of the parallelogram with adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is given by

$$|\overrightarrow{AB}\times\overrightarrow{AC}|=|6\underline{i}+3\underline{j}+6\underline{k}|=\sqrt{36+9+36}=\sqrt{81}=9$$

$$\therefore \quad \text{Area of triangle}=\frac{1}{2}\ |\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}|6\underline{i}+3\underline{j}+6\underline{k}|=\frac{9}{2}$$

$$\text{A unit vector} \perp \text{to the plane } ABC=\frac{\overrightarrow{AB}\times\overrightarrow{AC}}{|\overrightarrow{AB}\times\overrightarrow{AC}|}=\frac{1}{9}(6\underline{i}+3\underline{j}+6\underline{k})=\frac{1}{3}(2\underline{i}+\underline{j}+2\underline{k})$$

Example 6: Find area of the parallelogram whose vertices are *P*(0, 0, 0), *Q*(–1, 2, 4), *R*(2, –1, 4) and *S*(1, 1, 8).

Solution: Area of parallelogram = $|\underline{u}\times\underline{v}|$

where $\underline{u}$ and $\underline{v}$ are two adjacent sides of the parallelogram

$$\overrightarrow{PQ}=(-1-0)\underline{i}+(-2-0)\underline{j}+(4-0)\underline{k}=-\underline{i}+2\underline{j}+4\underline{k}$$

$$\text{and} \quad \overrightarrow{PR}=(2-0)\underline{i}+(-1-0)\underline{j}+(4-0)\underline{k}=2\underline{i}-\underline{j}+4\underline{k}$$

$$\text{Now} \quad \overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ -1 & 2 & 4\\ 2 & -1 & 4\end{vmatrix}=(8+4)\underline{i}-(-4-8)\underline{j}+(1-4)\underline{k}$$

$$\therefore \quad \text{Area of parallelogram}=|\overrightarrow{PQ}\times\overrightarrow{PR}|=|12\underline{i}+12\underline{j}-3\underline{k}|$$

$$=\sqrt{144+144+9}$$

$$=\sqrt{297}$$

Be careful!:
Not all pairs of vertices give a side e.g. $\overrightarrow{PS}$ is not a side, it is diagonal since $\overrightarrow{PQ}+\overrightarrow{PR}=\overrightarrow{PS}$

Example7: If $\underline{u}=2\underline{i}-\underline{j}+\underline{k}$ and $\underline{v}=4\underline{i}+2\underline{j}-\underline{k}$, find by determinant formula

(i) $\underline{u}$ x $\underline{u}$ (ii) $\underline{u}$ x $\underline{v}$ (iii) $\underline{v}$ x $\underline{u}$

Solution: $\underline{u}=2\underline{i}-\underline{j}+\underline{k}$ and $\underline{v}=4\underline{i}+2\underline{j}-\underline{k}$

By determinant formula

(i) $\underline{u}\times\underline{u}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 2 & -1 & 1\\ 2 & -1 & 1\end{vmatrix}=0$ $(\therefore$ Two rows are same)

(ii) $\underline{u}\times\underline{v}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 2 & -1 & 1\\ 4 & 2 & -1\end{vmatrix}=(1-2)\underline{i}-(-2-4)\underline{j}+(4+4)\underline{k}=-\underline{i}+6\underline{j}+8\underline{k}$

(iii) $\underline{v}\times\underline{u}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 4 & 2 & -1\\ 2 & -1 & 1\end{vmatrix}=(2-1)\underline{i}-(4+2)\underline{j}+(-4-4)\underline{k}=\underline{i}-6\underline{j}-8\underline{k}$

EXERCISE 7.4

1. Compute the cross product $\underline{a}$ x $\underline{b}$ and $\underline{b}$ x $\underline{a}$. Check your answer by showing that each $\underline{a}$ and $\underline{b}$ is perpendicular to $\underline{a}$ x $\underline{b}$ and $\underline{b}$ x $\underline{a}$.

(i) $\underline{a}=2\underline{i}+\underline{j}-\underline{k}$, $\underline{b}=\underline{i}-\underline{j}+\underline{k}$ (ii) $\underline{a}=\underline{i}+\underline{j}$, $\underline{b}=\underline{i}-\underline{j}$

(iii) $\underline{a}=3\underline{i}-2\underline{j}+\underline{k}$, $\underline{b}=\underline{i}+\underline{j}$ (iv) $\underline{a}=-4\underline{i}+\underline{j}-2\underline{k}$, $\underline{b}=2\underline{i}+\underline{j}+\underline{k}$

2. Find a unit vector perpendicular to the plane containing $\underline{a}$ and $\underline{b}$. Also find sine of the angle between them.

(i) $\underline{a}=2\underline{i}-6\underline{j}-3\underline{k}$, $\underline{b}=4\underline{i}+3\underline{j}-\underline{k}$ (ii) $\underline{a}=-\underline{i}-\underline{j}-\underline{k}$, $\underline{b}=2\underline{i}-3\underline{j}+4\underline{k}$

(iii) $\underline{a}=2\underline{i}-2\underline{j}+4\underline{k}$, $\underline{b}=-\underline{i}+\underline{j}-2\underline{k}$ (iv) $\underline{a}=\underline{i}+\underline{j}$, $\underline{b}=\underline{i}-\underline{j}$

3. Find the area of the triangle, determined by the point $\underline{P}$, $\underline{Q}$ and $\underline{R}$.

(i) $P(0,\ 0,\ 0)$; $Q(2,\ 3,\ 2)$; $R(-1,\ 1,\ 4)$

(ii) $P(1,\ -1,\ -1)$; $Q(2,\ 0,\ -1)$; $R(0,\ 2,\ 1)$

4. find the area of parallelogram, whose vertices are:

(i) $A(0,\ 0,\ 0)$; $B(1,\ 2,\ 3)$; $C(2,\ -1,\ 1)$; $D(3,\ 1,\ 4)$

(ii) $A(1,\ 2,\ -1)$; $B(4,\ 2,\ -3)$; $C(6,\ -5,\ 2)$; $D(9,\ -5,\ 0)$

(iii) $A(-1,\ 1,\ 1)$; $B(-1,\ 2,\ 2)$; $C(-3,\ 4,\ -5)$; $D(-3,\ 5,\ -4)$

5. Which vectors, if any, are perpendicular or parallel

(i) $\underline{u}=5\underline{i}-\underline{j}+\underline{k}$; $\underline{v}=\underline{j}-5\underline{k}$; $\underline{w}=-15\underline{i}+3\underline{j}-3\underline{k}$

(ii) $\underline{u}=\underline{i}+2\underline{j}-\underline{k}$; $\underline{v}=-\underline{i}+\underline{j}+\underline{k}$; $\underline{w}=-\frac{\pi}{2}\underline{i}-\pi\underline{j}+\frac{\pi}{2}\underline{k}$

6. Prove that: $\underline{a}$ x ($\underline{b}$ + $\underline{c}$) + $\underline{b}$ x ($\underline{c}$ + $\underline{a}$) + $\underline{c}$ x ($\underline{a}$ + $\underline{b}$) = 0

7. If $\underline{a}$ + $\underline{b}$ + $\underline{c}$ = 0, then prove that $\underline{a}$ x $\underline{b}$ = $\underline{b}$ x $\underline{c}$ = $\underline{c}$ x $\underline{a}$

8. Prove that: sin(α – β) = sin α cos β + cos α sin β.

9. If $\underline{a}$ x $\underline{b}$ = 0 and $\underline{a}.\underline{b}$ = 0, what conclusion can be drawn about $\underline{a}$ or $\underline{b}$?

7.5 SCALAR TRIPLE PRODUCT OF VECTORS

There are two types of triple product of vectors:

(a) **Scalar Triple Product:** $(\underline{u}\times\underline{v}).\underline{w}$ or $\underline{u}.(\underline{v}\times\underline{w})$

(b) **Vector Triple product:** $\underline{u}\times(\underline{v}\times\underline{w})$

In this section we shall study the scalar triple product only

Definition

Let $\underline{u}=a_1\underline{i}+b_1\underline{j}+c_1\underline{k}$, $\underline{v}=a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$ and $\underline{w}=a_3\underline{i}+b_3\underline{j}+c_3\underline{k}$ be three vectors

The scalar triple product of vectors $\underline{u}$, $\underline{v}$ and $\underline{w}$ is defined by

$\underline{u}.(\underline{v}$ x $\underline{w})$ or $\underline{v}.(\underline{w}$ x $\underline{u})$ or $\underline{w}.(\underline{u}$ x $\underline{v})$

The scalar triple product $\underline{u}.(\underline{v}$ x $\underline{w})$ is written as

$\underline{u}.(\underline{v}$ x $\underline{w})$ = $[\underline{u}\ \underline{v}\ \underline{w}]$

7.5.1 Analytical Expression of $\underline{u}.(\underline{v}$ x $\underline{w})$

Let $\underline{u}=a_1\underline{i}+b_1\underline{j}+c_1\underline{k}$, $\underline{v}=a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$ and $\underline{w}=a_3\underline{i}+b_3\underline{j}+c_3\underline{k}$

Now $\underline{v}\times\underline{w}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$

$$\Rightarrow \quad \underline{v}\times\underline{w} = (b_2c_3 - b_3c_2)\underline{i} - (a_2c_3 - a_3c_2)\underline{j} + (a_2b_3 - a_3b_2)\underline{k}$$

$$\therefore \quad \underline{u}.(\underline{v}\times\underline{w}) = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

$$\Rightarrow \quad \underline{u}.(\underline{v}\times\underline{w}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

which is called the **determinant formula** for scalar triple product of $\underline{u}$, $\underline{v}$ and $\underline{w}$ in component form.

$$\text{Now} \quad \underline{u}.(\underline{v}\times\underline{w}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{Interchanging } R_1 \text{ and } R_2$$

$$= \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \end{vmatrix} \quad \text{Interchanging } R_2 \text{ and } R_3$$

$$\therefore \quad \underline{u}.(\underline{v}\times\underline{w}) = \underline{v}.(\underline{w}\times\underline{u})$$

$$\text{Now} \quad \underline{v}.(\underline{w}\times\underline{u}) = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

$$= -\begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} \quad \text{Interchanging } R_1 \text{ and } R_2$$

$$= \begin{vmatrix} a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \quad \text{Interchanging } R_2 \text{ and } R_3$$

$$\therefore \quad \underline{v}.(\underline{w}\times\underline{u}) = \underline{w}.(\underline{u}\times\underline{v})$$

$$\text{Hence} \quad \underline{u}.(\underline{v}\times\underline{w}) = \underline{v}.(\underline{w}\times\underline{u}) = \underline{w}.(\underline{u}\times\underline{v})$$

Note: (i) The value of the triple scalar product depends upon the cycle order of the vectors, but is independent of the position of the dot and cross. So the dot and cross, may be interchanged without altering the value i.e;

(ii) $(\underline{u}\times\underline{v}) \,.\, \underline{w} = \underline{u} \,.\, (\underline{v}\times\underline{w}) = [\underline{u}\ \underline{v}\ \underline{w}]$

$(\underline{v}\times\underline{w}) \,.\, \underline{u} = \underline{v} \,.\, (\underline{w}\times\underline{u}) = [\underline{v}\ \underline{w}\ \underline{u}]$

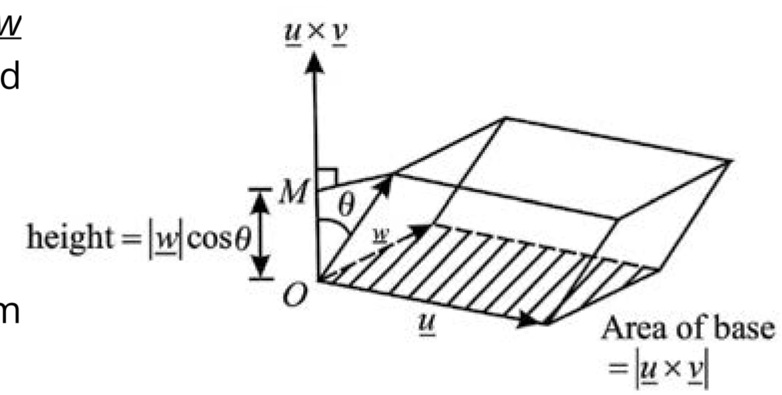
$(\underline{w}\times\underline{u}) \,.\, \underline{v} = \underline{w} \,.\, (\underline{u}\times\underline{v}) = [\underline{w}\ \underline{u}\ \underline{v}]$

(iii) The value of the product changes if the order is non-cyclic.

(iv) $\underline{u}.\underline{v}.\underline{w}$ and $\underline{u}$ x ($\underline{v}.\underline{w}$) are meaningless.

7.5.2 The Volume of the Parallelepiped

The triple scalar product ($\underline{u}$ x $\underline{v}$).$\underline{w}$ represents the volume of the parallelepiped having $\underline{u}$, $\underline{v}$ and $\underline{w}$ as its conterminous edges.



As it is seen from the formula that:

$$(\underline{u}\times\underline{v}).\underline{w} = |\underline{u}\times\underline{v}||\underline{w}|\cos\theta$$

Hence (i) $|\underline{u}\times\underline{v}|$ = area of the parallelogram with two adjacent sides, $\underline{u}$ and $\underline{v}$.

(ii) $|\underline{w}|\cos\theta$ = height of the parallelepiped

$$(\underline{u}\times\underline{v}).\underline{w} = |\underline{u}\times\underline{v}||\underline{w}|\cos\theta = (\text{Area of parallelogram})(\text{height})$$

= Volume of the parallelepiped

Similarly, by taking the base plane formed by $\underline{v}$ and $\underline{w}$, we have

The volume of the parallelepiped = ($\underline{v}$ x $\underline{w}$).$\underline{u}$

And by taking the base plane formed by $\underline{w}$ and $\underline{u}$, we have

The volume of the parallelepiped = ($\underline{w}$ x $\underline{u}$).$\underline{v}$

So, we have: ($\underline{u}$ x $\underline{v}$).$\underline{w}$ = ($\underline{v}$ x $\underline{w}$).$\underline{u}$ = ($\underline{w}$ x $\underline{u}$).$\underline{v}$

7.5.3 The Volume of the Tetrahedron:

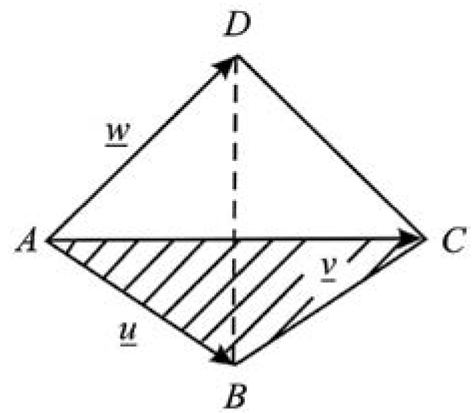
Volume of the tetrahedron *ABCD*

$$= \frac{1}{3}\,(\Delta ABC)\ (\text{height of } D \text{ above the place } ABC)$$

$$=\frac{1}{3}\cdot\frac{1}{2}\,|\underline{u}\times\underline{v}|\;(h)$$

$$=\frac{1}{6}\text{ (Area of parallelogram with } AB \text{ and } AC \text{ as adjacent sides) } (h)$$

$$=\frac{1}{6}\text{(Volume of the parallelepiped with } \underline{u},\ \underline{v},\ \underline{w} \text{ as edges)}$$



Thus Volume $= \frac{1}{6}(\underline{u}\times\underline{v}).\underline{w} \quad = \frac{1}{6}[\underline{u}\ \underline{v}\ \underline{w}]$

Properties of triple scalar Product:

1. If $\underline{u}$, $\underline{v}$ and $\underline{w}$ are coplanar, then the volume of the parallelepiped so formed is zero i.e; the vectors $\underline{u}$, $\underline{v}$, $\underline{w}$ are coplanar $\Leftrightarrow$ $(\underline{u}\times\underline{v}).\underline{w}=0$
2. If any two vectors of triple scalar product are equal, then its value is zero i.e;
   $[\underline{u}\ \underline{u}\ \underline{w}] = [\underline{u}\ \underline{v}\ \underline{v}] = 0$

Example 1: Find the volume of the parallelepiped determined by

$\underline{u}=\underline{i}+2\underline{j}-\underline{k}$, $\underline{v}=\underline{i}-\underline{j}+3\underline{k}$, $\underline{w}=\underline{i}-7\underline{j}-4\underline{k}$

Solution: Volume of the parallelepiped $=\underline{u}.\underline{v}\times\underline{w}=\begin{vmatrix}1 & 2 & -1\\ 1 & -2 & 3\\ 1 & -7 & -4\end{vmatrix}$

$\Rightarrow$ Volume = 1 (8 + 21) – 2(–4 – 3) –1 (–7 + 2)

= 29+ 14 + 5 = 48

Example 2: Prove that four points

$A(-3, 5, -4)$, $B(-1, 1, 1)$, $C(-1, 2, 2)$ and $D(-3, 4, -5)$ are coplaner.

Solution: $\overrightarrow{AB}=(-1+3)\underline{i}+(1-5)\underline{j}+(1+4)\underline{k} \quad =2\underline{i}-4\underline{j}+5\underline{k}$

$\overrightarrow{AC}=(-1+3)\underline{i}+(2-5)\underline{j}+(2+4)\underline{k} \quad =2\underline{i}-3\underline{j}+6\underline{k}$

$\overrightarrow{AD}=(3-3)\underline{i}+(4-5)\underline{j}+(-5+4)\underline{k} \quad =0\underline{i}-\underline{j}-\underline{k}=-\underline{j}-\underline{k}$

Volume of the parallelepiped formed by $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ is

$$\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right]=\begin{vmatrix}2 & -4 & 5\\ 2 & -3 & 6\\ 0 & -1 & -1\end{vmatrix}=2(3+6)+4(-2-0)+5(-2-0)$$

= 18 – 8 – 10 = 0

As the volume is zero, so the points A, B, C and D are coplaner.

Example 3: Find the volume of the tetrahedron whose vertices are $A(2, 1, 8)$, $B(3, 2, 9)$, $C(2, 1, 4)$ and $D(3, 3, 0)$

Solution: $\overrightarrow{AB}=(3-2)\underline{i}+(2-1)\underline{j}+(9-8)\underline{k} \quad =\underline{i}+\underline{j}+\underline{k}$

$\overrightarrow{AC}=(2-2)\underline{i}+(1-1)\underline{j}+(4-8)\underline{k} \quad =0\underline{i}-0\underline{j}-4\underline{k}$

$\overrightarrow{AD}=(3-2)\underline{i}+(3-1)\underline{j}+(0-8)\underline{k} \quad =\underline{i}+2\underline{j}-8\underline{k}$

$\therefore$ Volume of the tetrahedron $=\frac{1}{6}\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right]$

$$=\frac{1}{6}\begin{vmatrix}1 & 1 & 1\\ 0 & 0 & -4\\ 1 & 2 & -8\end{vmatrix}=\frac{1}{6}[4(2-1)]=\frac{4}{6}=\frac{2}{3}$$

Example 4: Find the value of α, so that $\alpha\underline{i}+\underline{j}$, $\underline{i}+\underline{j}+3\underline{k}$ and $2\underline{i}+\underline{j}-2\underline{k}$ are coplaner.

Solution: Let $\underline{u}=\alpha\underline{i}+\underline{j}$, $\underline{v}=\underline{i}+\underline{j}+3\underline{k}$ and $\underline{w}=2\underline{i}+\underline{j}-2\underline{k}$

Triple scalar product

$$[\underline{u}\ \underline{v}\ \underline{w}]=\begin{vmatrix}\alpha & 1 & 0\\ 1 & 1 & 3\\ 2 & 1 & -2\end{vmatrix}=\alpha(-2-3)-1(-2-6)+0(1-2)$$

$= -5\alpha + 8$

The vectors will be coplaner if $-5\alpha+8=0 \quad \Rightarrow \quad \alpha=\frac{8}{5}$

Example 5: Prove that the points whose position vectors are $A(-6\underline{i}+3\underline{j}+2\underline{k})$, $B(3\underline{i}-2\underline{j}+4\underline{k})$, $C(5\underline{i}+7\underline{j}+3\underline{k})$, $D(-13\underline{i}+17\underline{j}-\underline{k})$ are coplaner.

Solution: Let O be the origin.

$\therefore \quad \overrightarrow{OA}=-6\underline{i}+3\underline{j}+2\underline{k} \;;\; \overrightarrow{OB}=3\underline{i}-2\underline{j}+4\underline{k}$

$\therefore \quad \overrightarrow{OC}=5\underline{i}+7\underline{j}+3\underline{k} \;;\; \overrightarrow{OD}=-13\underline{i}+17\underline{j}-\underline{k}$

$\therefore \quad \overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(3\underline{i}-2\underline{j}+4\underline{k})-(-6\underline{i}+3\underline{j}+2\underline{k})$

$\therefore \quad =9\underline{i}-5\underline{j}+2\underline{k}$

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(5\underline{i}+7\underline{j}+3\underline{k})-(-6\underline{i}+3\underline{j}+2\underline{k})$

$\therefore \quad =11\underline{i}+4\underline{j}+\underline{k}$

$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=(-13\underline{i}+17\underline{j}-\underline{k})-(-6\underline{i}+3\underline{j}+2\underline{k})$

$\therefore \quad =-7\underline{i}+14\underline{j}-3\underline{k}$

$$\text{Now} \quad \overrightarrow{AB}.(\overrightarrow{AC}\times\overrightarrow{AD})=\begin{vmatrix} 9 & -5 & 2 \\ 11 & 4 & 1 \\ -7 & 14 & -3 \end{vmatrix}$$

$$=9(-12-14)+5(-33+7)+2(154+28)$$

$$=-234-130+364 \;=0$$

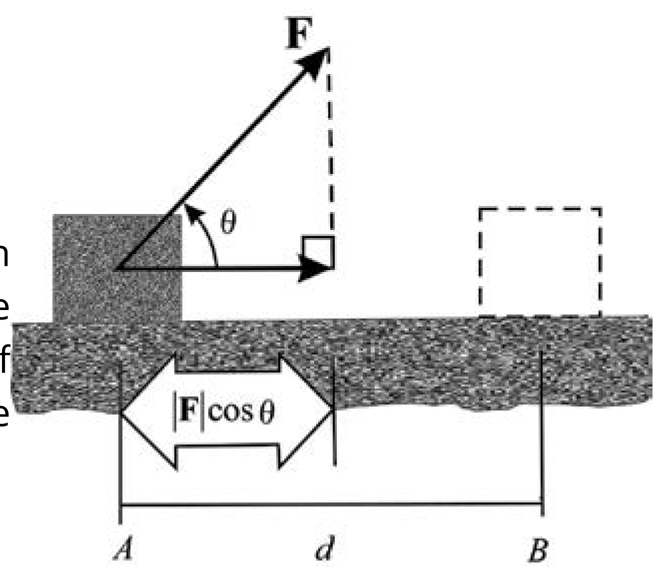
$\therefore \quad \overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}$ are coplaner

$\Rightarrow$ The points A, B, C and D are coplaner.

7.5.4 Application of Vectors in Physics and Engineering

(a) Work done.

If a constant force $\underline{F}$, applied to a body, acts at an angle θ to the direction of motion, then the work done by $\underline{F}$ is defined to be the product of the component of $\underline{F}$ in the direction of the displacement and the distance that the body moves.



In figure, a constant force $\underline{F}$ acting on a body, displaces it from A to B.

$\therefore$ Work done = (component of $\underline{F}$ along AB) (displacement)

$$=(F\cos\theta)(AB) \;=\underline{F}\,.\,\overrightarrow{AB}$$

Example 6: Find the work done by a constant force $\underline{F}=2\underline{i}+4\underline{j}$, if its points of application to a body moves it from $A(1, 1)$ to $B(4, 6)$.
(Assume that $|\underline{F}|$ is measured in Newton and $|\underline{d}|$ in meters.)

Solution: The constant force $\underline{F}=2\underline{i}+4\underline{j}$,

The displacement of the body = $\underline{d}=\overrightarrow{AB}$

$$=(4-1)\underline{i}+(6-1)\underline{j} \;=3\underline{i}+5\underline{j}$$

$\therefore$ work done = $\underline{F}\,.\,\underline{d}$

$$=(2\underline{i}+4\underline{j})\,.\,(3\underline{i}+5\underline{j})$$

$$=(2)(3)+(4)(5) \;=26\ nt.\,m$$

Example 7: The constant forces $2\underline{i}+5\underline{j}+6\underline{k}$ and $-\underline{i}+2\underline{j}+\underline{k}$ act on a body, which is displaced from position $P(4,-3,-2)$ to $Q(6,1,-3)$. Find the total work done.

Solution: Total force = $(2\underline{i}+5\underline{j}+6\underline{k})+(-\underline{i}+2\underline{j}+\underline{k})$

$\Rightarrow \quad \underline{F}=\underline{i}+3\underline{j}+5\underline{k}$

The displacement of the body $=\overrightarrow{PQ}=(6-4)\underline{i}+(1+3)\underline{j}+(-3+2)\underline{k}$

$\Rightarrow \quad \underline{d}=2\underline{i}+4\underline{j}-\underline{k}$

$\therefore$ work done = $\underline{F}\,.\,\underline{d}$

$$=(\underline{i}+3\underline{j}+5\underline{k})\,.\,(2\underline{i}+4\underline{j}-\underline{k})$$

$$=2+12-5 \;=9\ nt.\,m$$

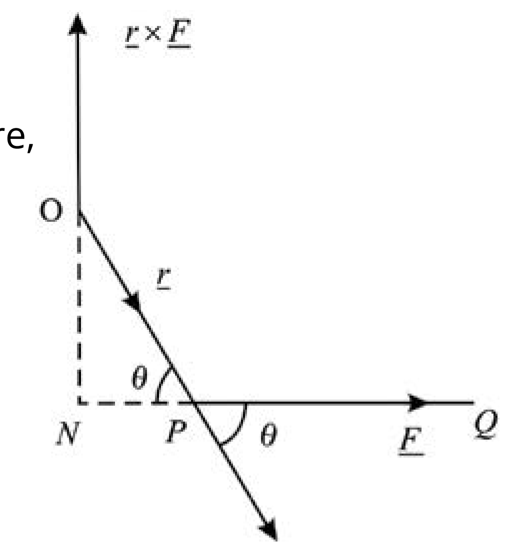
(b) Moment of Force

Let a force $\underline{F}\ (\overrightarrow{PQ})$ act at a point *P* as shown in the figure, then moment of $\underline{F}$ about *O*.

$= $ product of force $\underline{F}$ and perpendicular $ON.\ \underline{\hat{n}}$

$= (PQ)(ON)(\underline{\hat{n}}) \ = \ (PQ)(OP)\sin\theta\ .\ \underline{\hat{n}}$

$= \overrightarrow{OP} \times \overrightarrow{PQ} \ = \ \underline{r} \times \underline{F}$



Example 8: Find the moment about the point *M*(–2 , 4, –6) of the force represented by $\overrightarrow{AB}$, where coordinates of points *A* and *B* are (1, 2, –3) and (3, –4, 2) respectively.

Solution:

$$\overrightarrow{AB} = (3-1)\underline{i} + (-4-2)\underline{j} + (2+3)\underline{k} = 2\underline{i} - 6\underline{j} + 5\underline{k}$$

$$\overrightarrow{MA} = (1+2)\underline{i} + (2-4)\underline{j} + (-3+6)\underline{k} = 3\underline{i} - 2\underline{j} + 3\underline{k}$$

$$\text{Moment of } \overrightarrow{AB} \text{ about } (-2, 4, -6) = \underline{r} \times \underline{F} = \overrightarrow{MA} \times \overrightarrow{AB}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2 & 3 \\ 2 & -6 & 5 \end{vmatrix}$$

$$= (-10+18)\underline{i} - (15-6)\underline{j} + (-18+4)\underline{k}$$

$$= 8\underline{i} - 9\underline{j} - 14\underline{k}$$

$$\text{Magnitude of the moment} = \sqrt{(8)^2 + (-9)^2 + (-14)^2} = \sqrt{341}$$

EXERCISE 7.5

1. Find the volume of the parallelepiped for which the given vectors are three edges.

   (i) $\underline{u} = 3\underline{i} + 2\underline{k}$; $\quad \underline{v} = \underline{i} + 2\underline{j} + \underline{k}$; $\quad \underline{w} = -\underline{j} + 4\underline{k}$

   (ii) $\underline{u} = \underline{i} - 4\underline{j} - \underline{k}$; $\quad \underline{v} = \underline{i} - \underline{j} - 2\underline{k}$; $\quad \underline{w} = 2\underline{i} - 3\underline{j} + \underline{k}$

   (iii) $\underline{u} = \underline{i} - 2\underline{j} - 3\underline{k}$; $\quad \underline{v} = 2\underline{i} - \underline{j} - \underline{k}$; $\quad \underline{w} = \underline{j} + \underline{k}$

2. Verify that

   $\underline{a}\ .\ \underline{b} \times \underline{c} \ = \ \underline{b}\ .\ \underline{c} \times \underline{a} \ = \ \underline{c}\ .\ \underline{a} \times \underline{b}$

   if $\underline{a} = 3\underline{i} - \underline{j} + 5\underline{k}$, $\quad \underline{b} = 4\underline{i} + 3\underline{j} - 2\underline{k}$, and $\underline{c} = 2\underline{i} + 5\underline{j} + \underline{k}$

3. Prove that the vectors $\underline{i} - 2\underline{j} + 3\underline{k}$, $-2\underline{i} + 3\underline{j} - 4\underline{k}$ and $\underline{i} - 3\underline{j} + 5\underline{k}$ are coplanar

4. Find the constant α such that the vectors are coplanar.

   (i) $\underline{i} - \underline{j} + \underline{k}$, $\quad \underline{i} - 2\underline{j} - 3\underline{k}$ and $3\underline{i} - \alpha\underline{j} + 5\underline{k}$.

   (ii) $\underline{i} - 2\alpha\underline{j} - \underline{k}$, $\quad \underline{i} - \underline{j} + 2\underline{k}$ and $\alpha\underline{i} - \underline{j} + \underline{k}$

5. (a) Find the value of:

   (i) $2\underline{i} \times 2\underline{j}.\underline{k}$ (ii) $3\underline{j}.\underline{k} \times \underline{i}$ (iii) $[\underline{k}\ \ \underline{i}\ \ \underline{j}]$ (iv) $[\underline{i}\ \ \underline{i}\ \ \underline{k}]$

   (b) Prove that $\underline{u}.(\underline{v} \times \underline{w}) + \underline{v}.(\underline{w} \times \underline{u}) + \underline{w}.(\underline{u} \times \underline{v}) = 3\ \underline{u}.(\underline{v} \times \underline{w})$

6. Find volume of the Tetrahedron with the vertices

   (i) (0, 1, 2), (3, 2, 1), (1, 2, 1) and (5, 5, 6)

   (ii) (2, 1, 8), (3, 2, 9), (2, 1, 4) and (3, 3, 10).

7. Find the work done, if the point at which the constant force $\underline{F} = 4\underline{i} + 3\underline{j} + 5\underline{k}$ is applied to an object, moves from $P_1(3,1,-2)$ to $P_2(2,4,6)$.

8. A particle, acted by constant forces $4\underline{i} + \underline{j} - 3\underline{k}$ and $3\underline{i} - \underline{j} - \underline{k}$, is displaced from *A*(1, 2, 3) to *B*(5, 4, 1). Find the work done.

9. A particle is displaced from the point *A*(5, –5, –7) to the point *B*(6, 2, –2) under the action of constant forces defined by $10\underline{i} - \underline{j} + 11\underline{k}$, $4\underline{i} + 5\underline{j} + 9\underline{k}$ and $-2\underline{i} + \underline{j} - 9\underline{k}$. Show that the total work done by the forces is 102 units.

10. A force of magnitude 6 units acting parallel to $2\underline{i} - 2\underline{j} + \underline{k}$ displaces, the point of application from (1, 2, 3) to (5, 3, 7). Find the work done.

11. A force $\underline{F} = 3\underline{i} + 2\underline{j} - 4\underline{k}$ is applied at the point (1, –1, 2). Find the moment of the force about the point (2, –1, 3).

12. A force $\underline{F} = 4\underline{i} - 3\underline{k}$, passes through the point *A*(2,–2,5). Find the moment of $\underline{F}$ about the point *B*(1,–3,1).

13. Give a force $\underline{F} = 2\underline{i} + \underline{j} - 3\underline{k}$ acting at a point *A*(1, –2, 1). Find the moment of $\underline{F}$ about the point *B*(2, 0, –2).

14. Find the moment about *A*(1, 1, 1) of each of the concurrent forces $\underline{i} - 2\underline{j}$, $3\underline{i} + 2\underline{j} - \underline{k}$, $5\underline{j} + 2\underline{k}$, where *P*(2,0,1) is their point of concurrency.

15. A force $\underline{F} = 7\underline{i} + 4\underline{j} - 3\underline{k}$ is applied at *P*(1,–2,3). Find its moment about the point *Q*(2,1,1).