# Unit - 5 ▶ Polynomials

***After completion of this unit, the students will be able to:***

- Recall constant, variable, literal and algebraic expression.
- Define
  - Polynomial
  - Degree of a polynomial
  - Coefficient of a polynomial
- Recognize polynomial in one, two and more variables.
- Recognize polynomials of various degrees (e.g. linear, quadratic, cubic and biquadratic polynomials).
- Add, subtract and multiply polynomials.
- Divide a polynomial by a linear polynomial.

## 5.1 ALGEBRAIC EXPRESSIONS:

An algebraic expression is made up of symbols and signs of algebra. Algebra helps us to make general formula because algebra is linked with arithmetic. For example, $x^2 + 2x + 1$ and $\sqrt{x} - \frac{1}{\sqrt{x}}$ $x \neq 0$ are algebraic expressions.

Algebra was introduced by Muslim Mathematician named Al Khawaraizmi (780 - 850). He was also considered the "father of modern Algebra".

### 5.1.1 Recall Constant, Variable, Literal and Algebraic Expression

- **Constant:**

A symbol that has a fixed numerical value is called a constant. For example in $5x + 7$, $5, 7$ is a constant term.

- **Variable:**

Variable is a symbol, usually a letter that is used to represent a quantity that may have an infinite number of values are also called unknowns. For example, in $x^2 + y + 3z$; $x$, y and $z$ are variables.

- **Literal:**

The alphabets that are used to represent constants or coefficients are called literals. For example, in $ax^2 + bx + c$; $a, b$ and $c$ are literals whereas $x$ is a variable.

- **Algebraic Expression:**

An expression which connects variables and constants by algebraic operations of addition, subtraction, multiplication and division is called an algebraic expression. A few algebraic expressions are given below:

**(i)** $14$ **(ii)** $x + 2y$ **(iii)** $4x - y + 5$ **(iv)** $\frac{-2}{x} + y$ **(v)** $3y + 7z - \frac{5}{7}$

## 5.2 POLYNOMIAL

### 5.2.1 Definitions

- **Polynomial**

A polynomial expression or simply a polynomial is an algebraic expression consisting of one or more terms in each of which the exponent of the variable is zero or a positive integer.

For example, $13, -x, 5x + 3y, x^2 - 3x + 1$ are all polynomials.

The following algebraic expressions are not polynomials.

$x^{-2}, \frac{1}{y}, \quad x^3 - x^{-3} + 3, \quad x^2 + y^{-4} - 7$ and $\frac{x}{y} + 5x$

- **Degree of a Polynomial**

Degree of a polynomial is the highest power of a part (term) in a polynomial.

Degree of a term in a polynomial is the sum of the exponents on the variables in a single term. The degree of $2x^3y^4$ is $7$ as $3 + 4 = 7$

- **Coefficient of a Polynomial**

In a term the number multiplied by the variable is the coefficient of the variable.

In $4x + 6y$, $4$ is coefficient of $x$ and $6$ is coefficient of y.

## 5.2.2 Recognition of Polynomial in one, two and more Variables

### (a) Polynomials in one Variable

Consider the following Polynomials:

**(i)** $x^4 + 4$ **(ii)** $x^2 - x + 1$ **(iii)** $y^3 + y^2 - y + 1$ **(iv)** $y^2 - y + 8$

In polynomials (i) and (ii) $x$ is the variable and in polynomial (iii) and (iv) $y$ is the variable. All these polynomials are polynomials in one variable.

### (b) Polynomials in two Variables

Consider the following Polynomials:

**(i)** $x^2 + y + 2$ **(ii)** $x^2y + xy + 6$ **(iii)** $x^2z + xz + z$ **(iv)** $x^2z + 8$

In polynomials (i) and (ii) $x$, $y$ are the variables. In polynomials (iii) and (iv) $x$, $z$ are the variables. All these polynomials are in two variables.

### (c) Polynomials in more Variables

Similarly $x^2yz + xy^2z + xy + 7$ is a polynomial in three variables $x$, $y$ and $z$

## 5.2.3 Recognition of polynomials of various degrees

**(e.g., linear, quadratic, cubic and biquadratic polynomials)**

### (a) Linear Polynomials:

Consider the following polynomials:

**(i)** $x + 2$ **(ii)** $x$ **(iii)** $x + 2y$ **(iv)** $x + z$

In all these polynomials the degree of the variables $x$, $y$ or $z$ is one. Such types of polynomials are linear polynomials.

### (b) Quadratic Polynomials:

Let us write a few polynomials in which the highest exponent or sum of exponents is always $2$.

**(i)** $x^2$ **(ii)** $x^2 - 3$ **(iii)** $xy + 1$

In the first two polynomials $x$ is the variable and its degree is $2$. In the third polynomial $x$, $y$ are the variables and sum of their exponents is $1 + 1 = 2$. Its degree is also $2$. Therefore polynomials of the type (i), (ii) and (iii) are quadratic polynomials.

## (c) Cubic Polynomials:

Consider the following polynomials:

**(i)** $5x^3 + x^2 - 4x + 1$ **(ii)** $x^2 y + x y^2 + y - 2$

The degree of each one of the polynomial is 3. These polynomials are called cubic polynomials.

## (d) Biquadratic Polynomials:

Let us take a few polynomials of 4 degrees.

**(i)** $x^4 + x^3y + x^2y^2 + y^3 - 1$ **(ii)** $y^4 + y^3 - y^2 - y + 8$

These are biquadratic polynomials.

## EXERCISE 5.1

**1.** Write the constants given in the expression.

**(i)** $3x + 4$ **(ii)** $2x^3 - 1$ **(iii)** $5y + 2x$ **(iv)** $7y^2 - 8$

**2.** Write the variables taken in the equations.

**(i)** $2x - 1 = 0$ **(ii)** $y + x = 3$

**(iii)** $x^2 - x - 1 = 0$ **(iv)** $7y^2 - 2y + 3 = 0$

**3.** Write the literals used in the equations.

**(i)** $ax^2 + bx + c - y = 0$ **(ii)** $cx^2 + dx = 0$

**(iii)** $bx + d = 0$ **(iv)** $ay^2 + d = 0$

**4.** Separate the polynomial expressions and expressions that are not polynomials

**(i)** $x^2 + x - 1$ **(ii)** $x^2y + xy^2 + 7$

**(iii)** $x^{-2} + y + 7$ **(iv)** $\frac{x}{y^2} + 1 - \frac{y^2}{x}$

**(v)** $x^3 - x^2 + y - 1$ **(vi)** $x^4 + x^2 + 5x + \frac{1}{2}$

**5.** What constants are used in the following expressions?

**(i)** $7x - 6y + 3z$ **(ii)** $5x^2 - 3$

**(iii)** $8x^2 + 2y + 5$ **(iv)** $9y + 3x - 2z$

**6.** Write the degree of the polynomials given below.

**(i)** $x + 1$ **(ii)** $x^2 + x$

**(iii)** $x^3 - xy + 1$ **(iv)** $x^2y^2 + x^3 + y^2 - 1$

**7.** Separate the polynomials as linear, quadratic, cubic and biquadratic.

**(i)** $3x + 1$ **(ii)** $x^2 - 2$ **(iii)** $y^2 - y$

**(iv)** $x + y$ **(v)** $x^3 + x^2 - 2$ **(vi)** $x^4 + x^3 + x^2$

**(vii)** $x^2y^2 + xy$ **(viii)** $x^2 + xy + 8$

# 5.3 OPERATIONS ON POLYNOMIALS

## 5.3.1 Addition, Subtraction and Multiplication of Polynomials

### (i) Addition of algebraic expressions (Polynomials)

If $P(x)$ and $Q(x)$ are two polynomials, then their addition is represented as $P(x) + Q(x)$. In order to add two or more than two polynomials we first write the polynomials in descending or ascending order and like terms each in the form of columns. Finally, we add the coefficients of like terms.

**Example 1:** Add $3x^3 + 5x^2 - 4x$, $x^3 - 6 + 3x^2$ and $6 - x^2 - x$

**Solution:**

$$\begin{array}{rrrrr} & 3x^3 & +5x^2 & -4x & +0 \\ & x^3 & +3x^2 & +0x & -6 \\ & 0x^3 & -x^2 & -x & +6 \\ \hline \text{Sum:} & 4x^3 & +7x^2 & -5x & \\ \hline \end{array}$$

### (ii) Subtraction of polynomials

The subtraction of two polynomials $P$ and $Q$ is represented by $P - Q$ or $[P + (-Q)]$. If the sum of two polynomials is zero then $P$ and $Q$ are called additive inverse of each other.

If $P = x + y$ and $Q = -x - y$,

Then $P + Q = (x + y) + (-x - y) = 0$

Like addition we write the polynomials in descending or ascending order and then change the sign of every term of the polynomial which is to be subtracted.

**Example 2:** Subtract $2x^3 - 4x^2 + 8 - x$ from $5x^4 + x - 3x^2 - 9$

**Solution:** Arrange the terms of the polynomials in descending order.

$$\begin{array}{rl} & 5x^4 + 0x^3 - 3x^2 + x - 9 \\ & \pm 0x^4 \pm 2x^3 \mp 4x^2 \mp x \pm 8 \\ \hline \text{Difference:} & 5x^4 - 2x^3 + x^2 + 2x - 17 \\ \hline \end{array}$$

### (iii) Multiplication of polynomials

Multiplication of polynomials is explained through examples:

**Example 3:** Find the product of $4x^2$ and $5x^3$

**Solution:**

$$\begin{aligned} (4x^2)(5x^3) &= 4 \times 5(x^2 \times x^3) && \text{(Associative Law)} \\ &= (20)(x^2 \times x^3) \\ &= 20\,x^{2+3} && \text{(Law of exponents)} \\ &= 20\,x^5 \end{aligned}$$

**Example 4:** Find the product of $3x^2 + 2x - 4$ and $5x^2 - 3x + 3$

**Solution:** Horizontal Method

$(3x^2 + 2x - 4)(5x^2 - 3x + 3)$

$= 3x^2(5x^2 - 3x + 3) + 2x(5x^2 - 3x + 3) - 4(5x^2 - 3x + 3)$

$= 15x^4 - 9x^3 + 9x^2 + 10x^3 - 6x^2 + 6x - 20x^2 + 12x - 12$

$= 15x^4 + (10 - 9)x^3 + (9 - 6 - 20)x^2 + (6 + 12)x - 12$

$= 15x^4 + x^3 - 17x^2 + 18x - 12$

**Example 5:** Multiply $2x - 3$ with $5x + 6$

**Solution:** Vertical Method

$$\begin{array}{r} 5x + 6 \\ \times\ 2x - 3 \\ \hline 10x^2 + 12x \quad\quad \\ -15x - 18 \\ \hline 10x^2 - 3x - 18 \\ \hline \end{array}$$

**Note:** The product of two polynomials is also a polynomial whose degree is equal to the sum of the degrees of the two polynomials.

## 5.3.2 Division of Polynomials

Division is the reverse process of multiplication.

The method of division of polynomials is explained through examples.

**Example 6:** Divide $(-8x^5)$ by $(-4x^3)$

**Solution:** $(-8x^5) \div (-4x^3) = (-8x^5) \times \dfrac{1}{-4x^3}$

$= 2x^{5-3}$

$= 2x^2$

**Example 7:** Divide $x^3 - 2x + 4$ by $x + 2$

**Solution:**

$$\begin{array}{r|l} & x^2 - 2x + 2 \\ \hline x + 2 & x^3 + 0x^2 - 2x + 4 \\ & \pm x^3 \pm 2x^2 \\ \hline & -2x^2 - 2x \\ & \mp 2x^2 \mp 4x \\ \hline & 2x + 4 \\ & \pm 2x \pm 4 \\ \hline & 0 \end{array}$$

**Note:** If a polynomial is exactly divisible by another polynomial then the remainder is zero.

## EXERCISE 5.2

**1.** Add:

**(i)** $1 + 2x + 3x^2, 3x - 4 - 2x^2, x^2 - 5x + 4$

**(ii)** $a^3 + 2a^2 - 6a + 7, a^3 + 2a + 5, 2a^3 + 2a - a^2 - 8$

**(iii)** $a^3 - 2a^2b + b^3, 4a^3 + 2ab^2 + 6a^2b, 2b^3 - 5a^3 - 4a^2b$

**2.** Subtract $P$ from $Q$ when,

**(i)** $P = 3x^4 + 5x^3 + 2x^2 - x$ ; $Q = 4x^4 + 2x^2 + x^3 - x + 1$

**(ii)** $P = 2x + 3y - 4z - 1$ ; $Q = 2y + 3x - 4z + 1$

**(iii)** $P = a^3 + 2a^2b + 3ab^2 + b^3$ ; $Q = a^3 - 3a^2b + 3ab^2 - b^3$

**3.** Find the value of $x - 2y + 3z$ where $x = 2a^2 - a^3 + 3a + 4$, $y = 2a^3 - 3a^2 + 2 - 2a$ and $z = a^4 + 3a^3 - 6 - 5a^2$

**4.** The sum of two polynomials is $x^2 + 2x - y^2$. If one polynomial is $x^2 - 2xy + 3$, then find the other polynomial.

**5.** Subtract $4x + 6 - 2x^2$ from the sum of $x^3 + x^2 - 2x$ and $2x^3 + 3x - 7$

**6.** Find the product of the following polynomials.

**(i)** $(x + 3)(x^2 - 3x + 9)$ **(ii)** $(3x^2 - 7x + 5)(4x^2 - 2x + 1)$

**(iii)** $(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

**7.** If $P = x^2 - yz$, $Q = y^2 - xz$ and $R = z^2 - xy$, then find $PQ$, $QR$, $PR$ and $PQR$.

**8.** Simplify:

**(i)** $(x^2 + x - 6) \div (x - 2)$ **(ii)** $(x^3 - 19x - 30) \div (x + 3)$

**(iii)** $(x^5 - y^5) \div (x - y)$ **(iv)** $(x^3 + x^2 - 14x - 24) \div (x + 2)$

**(v)** $(16a^5 + 4a^3 - 4a^2 + 3a - 1) \div (4a^2 - 2a + 1)$

**(vi)** $(x^4 - 3x^2y^2 + y^4) \div (x^2 + xy - y^2)$

**9.** What should be added to $4x^3 - 10x^2 + 12x + 6$ so that it becomes exactly divisible by $2x + 1$?

**10.** The product of two polynomials $6y^3 - 11y^2 + 6y - 1$. If one polynomial is $3y^2 - 4y + 1$, then find the other polynomial.

**11.** For what value of $p$ the polynomial $3x^3 - 7x^2 - 9x + p$ becomes exactly divisible by $x - 3$?

## REVIEW EXERCISE 5

**1.** Four options are given against each statement. Encircle the correct one.

**i.** $4x + 2y + 3z$ is an algebraic:

**(a)** expression **(b)** equation **(c)** inequality **(d)** symbol

**ii.** The alphabet which can assume different values is called

**(a)** constant **(b)** variable **(c)** term **(d)** number

**iii.** In $2x - 3y + 4z$, there are variables:

**(a)** 2 **(b)** 3 **(c)** 4 **(d)** 5

**iv.** To represent variable we use:

**(a)** Constants **(b)** Numbers **(c)** Alphabets **(d)** Literals

**v.** An algebraic expression can contain:

**(a)** numbers, variables and operations **(b)** number and operations

**(c)** variables only **(d)** operations only

**vi.** $2x^{-2}$ is:

**(a)** a polynomial **(b)** not a polynomial

**(c)** a constant term **(d)** an inequality

**vii.** In $3x^2 + 2x - 1$, the degree of the polynomial is:

**(a)** 1 **(b)** 2 **(c)** 3 **(d)** 4

**viii.** In a polynomial the number multiplied with the variable is called a:

**(a)** number **(b)** coefficient **(c)** index **(d)** constant

**ix.** Polynomial $3y^2$ is:

**(a)** linear **(b)** quadratic **(c)** cubic **(d)** biquadratic

**x.** Biquadratic algebraic expression is a polynomial of degree:

**(a)** one **(b)** two **(c)** three **(d)** four

**2.** Indicate polynomial and their degree in the following table.

| Sr. # | Algebraic Expression | Polynomial | Degree of polynomial |
|---|---|---|---|
| **i.** | $2.3 + 1.2x$ | | |
| **ii.** | $k^2 + 5k^{-1} + 6$ | | |
| **iii.** | $-9$ | | |
| **iv.** | $2c^4 + 5b + \frac{6}{7}$ | | |

**3.** Find the sum of the following polynomials.

**i.** $2a + 3b + c$, $3a - b - c$, $4b + 5c$, $-2a + 3c$ and $-b + c$

**ii.** $9z + 3y^2 - 5x^3$, $-z - 2y^2 - 4x^3$, $z - x^3$ and $-2z + 3y^2$

**4.** Solve:

**i.** $(-2x^2 + 5y^2 - 3z^2) - (5x^2 - 3y^2 - 6z^2)$

**ii.** $(6x^3 + x^2 - 26) - (9 + 3x^2 - 5x^3)$

**iii.** $(y^2 - 5)(-y^2 + 5)$

**iv.** $(3a + 2b)(4a^2 - 7b + 5)$

**v.** $(x^4 + x - 2) \div (x - 1)$

## SUMMARY

- An expression which connects variables and constants by algebraic operations of addition subtraction, multiplication and division is called an algebraic expression.
- Constants are algebraic symbols that have a fixed value and do not change.
- A symbol in algebra which can assume different numerical values (numbers) is called a variable.
- A literal is a value that is expressed as itself. For example, the number 25 or the word "speed" are both literals.
- An algebraic expression which has finite number of terms and the exponents of variables are whole numbers, is called polynomial.
- A polynomial is either zero or can be written as the sum of a finite number of non-zero terms.
- In a polynomial coefficient is a number or symbol multiplied with a variable in an algebraic term.
- The polynomials of degree one are called linear polynomials.
- The polynomials of degree two are called quadratic polynomials.
- The polynomials of degree three are called cubic polynomials.
- The polynomials of degree four are called biquadratic polynomial.