Unit - 6 ▶ Factorization, Simultaneous Equations

***After completion of this unit, the students will be able to*:**

- Recall the formulas:
  - $(a + b)^2 = a^2 + 2ab + b^2$
  - $(a - b)^2 = a^2 - 2ab + b^2$
- $a^2 - b^2 = (a - b)(a + b)$ and apply them to solve problems like:
- Evaluate $(102)^2$, $(1.02)^2$, $(98)^2$ and $(0.98)^2$.
- Find $x^2 + \frac{1}{x^2}$ and $x^4 + \frac{1}{x^4}$ when the value of $x \pm \frac{1}{x}$ is given.

Factorize expressions of the following types:

  - $Ka + kb + kc$
  - $ac + ad + bc + bd$
  - $a^2 \pm 2ab + b^2$
  - $a^2 - b^2$
  - $a^2 + 2ab + b^2 - c^2$
- Recognize the formulas:
  - $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
  - $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ and apply them to solve the problems like this
  - $x^3 + \frac{1}{x^3}$ and $x^3 - \frac{1}{x^3}$ when the value of $x \pm \frac{1}{x}$ is given.
- Recognize simultaneous linear equations in one and two variables.
- Give the concept of formation of linear equation in two variables.
- Know that:
  - A single linear equation in two unknown is satisfied by many pair of values as required.
  - Two linear equations in two unknown have only one solution (i.e. one pair of values).
- Solve simultaneous linear equations using
  - Method of equating the coefficients
  - Method of elimination by substitution
  - Method of cross multiplication
  - Solve real life problems involving two simultaneous linear equations in two variables.
- Eliminate a variable from two equations by:
  - Substitution
  - Application of formulas

6.1 BASIC ALGEBRAIC FORMULAS

- $\boldsymbol{(a + b)^2 = a^2 + 2ab + b^2}$

Example 1: Evaluate $(107)^2$ by using formula

Solution: $(107)^2 = (100 + 7)^2$

$= (100)^2 + 2(100 \times 7) + (7)^2$
$= 10000 + 1400 + 49$
$= 11449$

- $\boldsymbol{(a - b)^2 = a^2 - 2ab + b^2}$

Example 2: Using the formula, evaluate $(87)^2$

Solution: $(87)^2 = (90 - 3)^2$

$= (90)^2 - 2(90 \times 3) + (3)^2$
$= 8100 - 540 + 9$
$= 7569$

- $\boldsymbol{a^2 - b^2 = (a + b)(a - b)}$

Example 3: Using the formula, evaluate 107×93

Solution: $107 \times 93 = (100 + 7)(100 - 7)$

$= (100)^2 - (7)^2$
$= 10000 - 49$
$= 9951$

Example 4: Find the value of $x^2 + \frac{1}{x^2}$ and $x^4 + \frac{1}{x^4}$ when $x - \frac{1}{x} = 2$

Solution: Here, $x - \frac{1}{x} = 2$

$$\left(x - \frac{1}{x}\right)^2 = (2)^2 \quad \text{(Taking square of both the sides)}$$

or $$x^2 - 2(x)\left(\frac{1}{x}\right) + \left(\frac{1}{x}\right)^2 = 4$$

or $$x^2 - 2 + \frac{1}{x^2} = 4$$

or $$x^2 + \frac{1}{x} = 4 + 2$$

or $$x^2 + \frac{1}{x^2} = 6$$

or $$\left(x^2 + \frac{1}{x^2}\right) = (6)^2 \quad \text{(Again taking square of both the sides)}$$

or $(x^2)^2 + 2(x^2)\left(\frac{1}{x^2}\right) + \left(\frac{1}{x^2}\right)^2 = 36$

or $x^4 + 2 + \frac{1}{x^4} = 36$

or $x^4 + \frac{1}{x^4} = 36 - 2$

$x^4 + \frac{1}{x^4} = 34$

EXERCISE 6.1

Solve the following questions by using formulas:

1. Evaluate square of each of the following:

(i) 53 **(ii)** 77 **(iii)** 509 **(iv)** 1006

2. Evaluate each of the following:

(i) $(57)^2$ **(ii)** $(95)^2$ **(iii)** $(598)^2$ **(iv)** $(1997)^2$

3. Evaluate:

(i) 46×54 **(ii)** 197×203 **(iii)** 999×1001 **(iv)** 0.96×1.04

4. **(i)** Find the value of $x^2 + \frac{1}{x^2}$, when $x + \frac{1}{x} = 7$

(ii) Find the value of $x^2 + \frac{1}{x^2}$, when $x - \frac{1}{x} = 3$

(iii) Find the value of $x^4 + \frac{1}{x^4}$, when $x - \frac{1}{x} = 1$

6.2 FACTORIZATION

Factors of an expression are the expressions whose product is the given expression.

The process of expressing the given expressions as a product of its factors is called 'Factorization' or 'Factorizing'.

(i) Type $Ka + Kb + Kc$:

Example 1: Factorize $2x - 4y + 6z$

Solution: $2x - 4y + 6z$

$= 2(x - 2y + 3z)$ "2" is a factor common to each term

Example 2: Factorize $x^2 - xy + xz$

Solution: $x^2 - xy + xz$

$= x(x - y + z)$

Example 3: Factorize $3x^2 - 6xy$

Solution: $3x^2 - 6xy$

$= 3x(x - 2y)$

EXERCISE 6.2

Factorize the following:

1. $3x - 9y$
2. $xy + xz$
3. $6ab - 14ac$
4. $3m^3np - 6m^2n$
5. $30x^3 - 45xy$
6. $17x^2y^2 - 51$
7. $4x^3 + 3x^2 + 2x$
8. $2p^2 - 4p^3 + 8p$
9. $x^3y - x^2y + xy^2$
10. $7x^4 - 14x^2y + 21xy^3$
11. $x^2y^2z^2 - xyz^2 + xyz$
12. $4x^3y^2 - 8xy + 4xy^3$
13. $xy^4 - 3xy^3 - 6xy^2$
14. $x^2y^2z + x^2yz^2 + xy^2z^2$
15. $77x^2y - 33xy^2 - 55x^2y^2$
16. $5x^5 + 10x^4 + 15x^3$

(ii) Type $ac + ad + bc + bd$:

Consider the following examples for such cases.

Example 4: Factorize: $3x + cx + 3c + c^2$

Solution: $3x + cx + 3c + c^2$

$= (3x + cx) + (3c + c^2)$

$= x(3 + c) + c(3 + c)$

$= (3 + c)(x + c)$

Example 5: Factorize: $2x^2y - 2xy + 4y^2x - 4y^2$

Solution: $2x^2y - 2xy + 4y^2x - 4y^2$

$= 2y(x^2 - x + 2yx - 2y)$

$= 2y[x(x - 1) + 2y(x - 1)]$

$= 2y(x - 1)(x + 2y)$

EXERCISE 6.3

Factorize the following:

1. $ax - by + bx - ay$
2. $2ab - 6bc - a + 3c$
3. $x^2 + 2x - 3x - 6$
4. $x^2 + 5x - 2x - 10$
5. $x^2 - 7x + 2x - 14$
6. $x^2 + 3x - 4x - 12$
7. $y^2 - 9y + 3y - 27$
8. $x^2 - 8x - 4x + 32$
9. $x^2 - 7x - 5x + 35$
10. $x^2 - 13x - 2x + 26$
11. $a(x - y) - b(x - y)$
12. $y(y - a) - b(y - a)$
13. $a^2(pq - rs) + b^2(pq - rs)$
14. $ab(x + y) + cd(x + y)$

(iii) Type $a^2 \pm 2ab + b^2$:

Consider the following examples for such cases.

Example 6: Factorize: $9a^2 + 30ab + 25b^2$

Solution:
$9a^2 + 30ab + 25b^2$
$= (3a)^2 + 2\,(3a \times 5b) + (5b)^2$
$= (3a + 5b)^2$

Example 7: Factorize: $16x^2 - 64x + 64$

Solution:
$16x^2 - 64x + 64$
$= 16\,(x^2 - 4x + 4)$
$= 16\,\left[(x)^2 - 2(2)(x) + (2)^2\right]$
$= 16\,(x - 2)^2$

Example 8: Factorize: $8x^3y + 8x^2y^2 + 2xy^3$

Solution:
$8x^3y + 8x^2y^2 + 2xy^3$
$= 2xy\,(4x^2 + 4xy + y^2)$
$= 2xy\,[(2x)^2 + 2(2x)(y) + (y^2)]$
$= 2xy\,(2x + y)^2$

EXERCISE 6.4

Factorize:

1. $x^2 + 14x + 49$
2. $9a^2 + 12ab + 4b^2$
3. $16 + 24a + 9a^2$
4. $25x^2 + 80xy + 64y^2$
5. $7a^4 + 84a^2 + 252$
6. $4a^2 + 120a + 900$
7. $x^2 - 34x + 289$
8. $49x^2 - 84x + 36$
9. $x^2 - 18xy + 81y^2$
10. $a^4 - 26a^2 + 169$
11. $2a^2 - 64a + 512$
12. $1 - 6a^2b^2c + 9a^4b^4c^2$
13. $4x^4 + 20x^3yz + 25x^2y^2z^2$
14. $\frac{9}{16}x^2 + xy + \frac{4}{9}y^2$
15. $\frac{49}{64}x^2 - 2xy + \frac{64}{49}y^2$
16. $\frac{a^2}{b^2}x^2 - \frac{2ac}{bd}xy + \frac{c^2y^2}{d^2}$
17. $16x^6 - 16x^5 + 4x^4$
18. $a^4b^4x^2 - 2a^2b^2c^2d^2xy + c^4d^4y^2$

(iv) Type $a^2 - b^2$:

Consider the following examples for such cases.

Example 9: Factorize: $25x^2 - 64$

Solution:
$25x^2 - 64$
$= (5x)^2 - (8)^2$
$= (5x + 8)\,(5x - 8)$

Example 10: Factorize: $16y^2b - 81bx^2$

Solution: $16y^2b - 81bx^2$

$= b\,(16y^2 - 81x^2)$

$= b\,[(4y)^2 - (9x)^2]$

$= b\,(4y + 9x)\,(4y - 9x)$

Example 11: Factorize: $(3x - 5y)^2 - 49z^2$

Solution: $(3x - 5y)^2 - 49z^2$

$= (3x - 5y)^2 - (7z)^2$

$= (3x - 5y + 7z)\,(3x - 5y - 7z)$

Example 12: Factorize: $36\,(x + y)^2 - 25\,(x - y)^2$

Solution: $36\,(x + y)^2 - 25\,(x - y)^2$

$= [6(x + y)]^2 - [5(x - y)]^2$

$= [6(x + y) + 5(x - y)]\,[6(x + y) - 5(x - y)]$

$= (11x + y)\,(x + 11y)$

Example 13: Use formula to evaluate: $(677)^2 - (323)^2$

Solution: $(677)^2 - (323)^2$

$= (677 + 323)\,(677 - 323)$

$= 1000 \times 354$

$= 354000$

Example 14: Simplify: $\dfrac{0.987 \times 0.987 - 0.643 \times 0.643}{0.987 + 0.643}$

Solution:

$$\frac{0.987 \times 0.987 - 0.643 \times 0.643}{0.987 + 0.643}$$

$$= \frac{(0.987)^2 - (0.643)^2}{0.987 + 0.643}$$

$$= \frac{(0.987 + 0.643)(0.987 - 0.643)}{0.987 + 0.643}$$

$= 0.987 - 0.643$

$= 0.344$

EXERCISE 6.5

Factorize the following expressions:

1. $9 - x^2$

2. $-6 + 6y^2$

3. $16x^2y^2 - 25a^2b^2$

4. $x^3y - xy^3$

5. $16a^2 - 400b^2$

6. $a^2b^3 - 64a^2b$

7. $7xy^2 - 343x$

8. $5x^3 - 45x$

9. $11\,(a + b)^2 - 99c^2$

10. $75 - 3(a - b)^2$

11. $\left(x-\frac{9}{5}\right)^2 - \frac{36}{25}y^2$

12. $25\left(x+\frac{5}{4}\right)^2 - 16\left(x+\frac{7}{4}\right)^2$

13. $16(a+b)^2 - 49(a-b)^2$

14. $36\left(x-\frac{1}{4}\right)^2 - 64\left(x-\frac{5}{4}\right)^2$

Evaluate the following:

15. $(371)^2 - (129)^2$

16. $(674.17)^2 - (325.83)^2$

17. $\dfrac{(0.567)^2 - (0.433)^2}{0.567 - 0.433}$

18. $\dfrac{(0.409)^2 - (0.391)^2}{0.409 - 0.391}$

(v) Type $a^2 \pm 2ab + b^2 - c^2$:

This type can be explained through the following examples.

Example 15: $a^2 - 2ab + b^2 - 4c^2$

Solution:

$$(a^2 - 2ab + b^2) - 4c^2$$
$$= (a-b)^2 - (2c)^2$$
$$= (a-b-2c)(a-b+2c)$$

Example 16: $4a^2 + 4ab + b^2 - 9c^2$

Solution:

$$4a^2 + 4ab + b^2 - 9c^2$$
$$= (2a)^2 + 2(2a)(b) + (b)^2 - 9c^2$$
$$= (2a+b)^2 - (3c)^2$$
$$= (2a+b-3c)(2a+b+3c)$$

EXERCISE 6.6

Factorize:

1. $a^2 + 2ab + b^2 - c^2$

2. $a^2 + 6ab + 9b^2 - 16c^2$

3. $a^2 + b^2 + 2ab - 9a^2b^2$

4. $x^2 - 4xy + 4y^2 - 9x^2y^2$

5. $9a^2 - 6ab + b^2 - 16c^2$

6.3 MANIPULATION OF ALGEBRAIC EXPRESSION

- **Formula $(a+b)^3 = a^3 + 3ab(a+b) + b^3$**

Example 1: Expand $(3a+4b)^3$

Solution:

$$(3a+4b)^3$$
$$= (3a)^3 + 3(3a)(4b)(3a+4b) + (4b)^3$$
$$= 27a^3 + 36ab(3a+4b) + 64b^3$$
$$= 27a^3 + 108a^2b + 144ab^2 + 64b^3$$

- **Formula** $\boldsymbol{(a-b)^3 = a^3 - 3ab\,(a-b) - b^3}$

This type can be explained with the following examples.

Example 2: Expand $(2a-3b)^3$

Solution: $(2a-3b)^3$

$= (2a)^3 - 3\,(2a)(3b)(2a-3b) - (3b)^3$

$= 8a^3 - 18ab\,(2a-3b) - 27b^3$

$= 8a^3 - 36a^2b + 54ab^2 - 27b^3$

Example 3: If $x + \frac{1}{x} = 5$, then find the value of $x^3 + \frac{1}{x^3}$

Solution: We have, $x + \frac{1}{x} = 5$

$$\left(x+\frac{1}{x}\right)^3 = (x)^3 + 3\,(x)\left(\frac{1}{x}\right) \times \left(x+\frac{1}{x}\right) + \left(\frac{1}{x}\right)^3$$

$$\left(x+\frac{1}{x}\right)^3 = x^3 + 3\left(x+\frac{1}{x}\right) + \frac{1}{x^3}$$

$$\left(x+\frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x+\frac{1}{x}\right)$$

$$(5)^3 = x^3 + \frac{1}{x^3} + 3(5) \qquad \therefore \quad \left(x+\frac{1}{x}\right) = 5$$

$$125 = x^3 + \frac{1}{x^3} + 15$$

$$\Rightarrow \quad x^3 + \frac{1}{x^3} = 125 - 15$$

Thus, $x^3 + \frac{1}{x^3} = 110$

EXERCISE 6.7

1. Find the cube of the following:

(i)	$x+4$	**(ii)**	$2m+1$	**(iii)**	$a-2b$
(iv)	$5x-1$	**(v)**	$2a+\mathrm{b}$	**(vi)**	$3x+10$
(vii)	$2m+3n$	**(viii)**	$4-3a$	**(ix)**	$3x+3y$
(x)	$7+2b$	**(xi)**	$4x-2y$	**(xii)**	$5m+4n$

2. If $x + \frac{1}{x} = 8$, then find the value of $x^3 + \frac{1}{x^3}$

3. If $x - \frac{1}{x} = 3$, then find the value of $x^3 - \frac{1}{x^3}$

4. If $x + \frac{1}{x} = 7$, then find the value of $x^3 + \frac{1}{x^3}$

5. If $x - \frac{1}{x} = 2$, then find the value of $x^3 - \frac{1}{x^3}$

6. Find the cube of the following by using formula.

(i) 13 **(ii)** 103 **(iii)** 0.99

6.4 SIMULTANEOUS LINEAR EQUATIONS

If two or more linear equations consisting of same set of variables are satisfied simultaneously by the same values of the variables, then these equations are called simultaneous linear equations.

6.4.1 Recognizing Simultaneous Linear Equations in One and Two Variables

We know that a linear equation is an algebraic equation in which each term is either a constant or a variable or the product of a constant or a variable. The standard form of linear equation consisting of one variable is:

$$ax + b, \qquad \forall\, a, b \in R$$

Similarly, a linear equation in two variables is of the form $ax + by = c$, where a, b and c are constants. Two linear equations considered together, form a system of linear equations. For example

$x + y = 2$ and $x - y = 1$ is a system of two linear equations with two variables x and y. This system of two linear equations is known as the simplest form of linear system which can be written in general form as:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

6.4.2 Concept of Formation of Linear Equation in Two Variables

Statements involving two unknowns can be written in algebraic form as explained in the following examples.

Example 1: Write an equation for each statement.

(i) The price of a book and 3 pencils is 90 rupees.
(ii) Sum of two numbers is 5.
(iii) The weight of Iram is half of the weight of Ali.

Solution: **(i)** Price of a book and 3 pencils = $Rs.90$
Let the price of one book = x
The price of one pencil = y
$\therefore$ The equation can be written as $x + 3y = 90$

(ii) Sum of two numbers = 5
Let the first number = x
The second number = y
$\therefore$ The equation can be written as $x + y = 5$

(iii) Let the weight of Iram = x
The weight of Ali = y
$\therefore$ The equation can be written as $x = \frac{y}{2}$

6.4.3 Solution of a Linear Equation in Two Unknowns

The solution of linear equation $ax + by = c$ in two variables "x" and "y" is an ordered pair of "x" and "y" that satisfies $ax + by = c$. Since a linear equation represents a straight line, hence an equation may have so many solutions.

Example 2: Find four solutions for the equation $3x + y = 2$.

Solution: $3x + y = 2$

Put the value of $x = 0$
$3(0) + y = 2$
$\Rightarrow 0 + y = 2$
$\Rightarrow y = 2$

Put the value of $x = 1$
$3(1) + y = 2$
$\Rightarrow 3 + y = 2$
$\Rightarrow y = 2 - 3 = -1$

Put the value of $x = 2$
$3(2) + y = 2$
$\Rightarrow 6 + y = 2$
$\Rightarrow y = 2 - 6$
$\Rightarrow y = -4$

Put the value of $x = 3$
$3(3) + y = 2$
$\Rightarrow 9 + y = 2$
$\Rightarrow y = 2 - 9$
$\Rightarrow y = -7$

Thus, the solutions of the given equation are infinite $(0, 2), (1, -1), (2, -4), (3, -7), \ldots\ldots$

- **Solution of two Linear Equations in two Unknowns**

A pair of linear equations in two variables is said to form a system of simultaneous linear equations. A pair of values of x and y which satisfy each one of the given equations in x and y is called solution of the system of simultaneous linear equations.

For example, two linear equations $x + y = 5$ and $x - y = 3$ have solution $x = 4$ and $y = 1$ i.e.,

$$\begin{aligned} x + y &= 5 \\ \text{L.H.S} &= x + y \\ &= (4) + (1) \\ &= 5 = \text{R.H.S} \end{aligned} \qquad \begin{aligned} x - y &= 3 \\ \text{L.H.S} &= x - y \\ &= (4) - (1) \\ &= 3 = \text{R.H.S} \end{aligned}$$

Thus, $x = 4$ and $y = 1$ is a solution of the given equations.

EXERCISE 6.8

1. Write equations for the following statements.

(i) The difference between father's age and daughter's age is 26 years.

(ii) The price of 6 biscuits is equal to the price of one chocolate.

(iii) If a number is added to three times of another number, the sum is 25

(iv) The division of sum of two numbers by their difference is equal to 1 (2^{nd} number is less than 1^{st})

(v) Twice of any age increased by 7 years becomes y years.

2. Find two solutions for the equation $2x + y = 3$

3. Find three solutions for the equations $x + y = 2$

4. Find four solutions for the equations $y = 2x$

5. Is $(1, 2)$ a solution set of $x + y = 3$ and $2x + 7y = 16$?

6. Which one of $(3, 1)$ and $(0, 3)$ is a solution of $2x + 5y = 15$ and $y - x = 3$?

6.5 SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

The solution of simultaneous linear equations means finding values for the variables that make them true sentences. Let us learn how to find the solution of simultaneous linear equations.

6.5.1 Solve Simultaneous Linear Equations

There are many methods of solving simultaneous linear equations but here we shall confine ourselves to the following three methods.

- Method of equating the coefficients.
- Method of elimination by substitution.
- Method of cross Multiplication.

• Method of Equating the Coefficients

Example 1: Find the solution with the method of equating the coefficients.

$$9x + 8y = 1$$
$$5x - y = 6$$

Solution:

$$9x + 8y = 1 \qquad \text{......... (i)}$$
$$5x - y = 6 \qquad \text{......... (ii)}$$

Step 1: Convert the given equation into an equivalent equation in such a way that the coefficient of one variable must be same. Multiply both sides of equation (ii) by 8, we have

$$8(5x - y) = 8\,(6)$$
$$40x - 8y = 48 \qquad \ldots\ldots\ldots \text{(iii)}$$

Step 2: Add equations (i) and (iii) to find the value of one variable.

$$\begin{aligned} 9x + 8y &= 1 \\ 40x - 8y &= 48 \\ \hline 49x &= 49 \\ \hline x = \frac{49}{49} &= 1 \end{aligned}$$

Step 3: Put the value of "x" in equation (i) or (ii) to find the value of "y".

$$\begin{aligned} 5x - y &= 6 \qquad \ldots\ldots\ldots\ldots \text{(ii)} \\ 5(1) - y &= 6 \\ 5 - y &= 6 \\ y = 5 - 6 &= -1 \end{aligned}$$

Thus, $x = 1$ and $y = -1$ is the required solution.

Step 4: Check the answer by placing the values of "x" and "y" in any equation.

$$9x + 8y = 1$$
$$\begin{aligned} \text{L.H.S} &= 9x + 8y \\ &= 9(1) + 8(-1) \\ &= 9 - 8 = 1 = \text{R.H.S} \end{aligned}$$

- **Method of Elimination by Substitution**

Example 2: Find the solution set with the method of elimination by substitution.

$$3x + 5y = 5$$
$$x + 2y = 1$$

Solution:

$$3x + 5y = 5 \qquad \ldots\ldots\ldots\ldots \text{(i)}$$
$$x + 2y = 1 \qquad \ldots\ldots\ldots\ldots \text{(ii)}$$

Step 1: Find the value of "x" or "y" from any of the given equations.

From equation (ii)

$$x + 2y = 1 \Rightarrow x = 1 - 2y \qquad \ldots\ldots\ldots\ldots \text{(iii)}$$

Step 2: Substitute the value of "x" in equation (i)

$$\begin{aligned} 3x + 5y &= 5 \\ \Rightarrow \quad 3(1 - 2y) + 5y &= 5 \\ \Rightarrow \quad 3 - 6y + 5y &= 5 \\ \Rightarrow \quad 3 - y &= 5 \\ \Rightarrow \quad y = 3 - 5 &= -2 \end{aligned}$$

Step 3: Put the value of "y" in equation (iii) to find the value of "x".

$x = 1 - 2y$ (from (iii))

$x = 1 - 2(-2) = 1 + 4$

$x = 5$

Hence, $x = 5$ and $y = -2$ is the required solution.

Step 4: Check the answer by putting the values in any equation i.e., in (i) or (ii).

$3x + 5y = 5$ from (i)

L.H.S $= 3(5) + 5(-2)$

$= 15 - 10$

$= 5 =$ R.H.S

Also check by putting the values in equation (ii) $x + 2y = 1$

L.H.S $= (5) + 2(-2)$

$= 5 - 4$

$= 1 =$ R.H.S

- **Method of cross Multiplication**

Let the two equations be

$a_1x + b_1y + c_1 = 0$ .... (i)

$a_2x + b_2y + c_2 = 0$ .... (ii)

Multiplying (i) by b_2 and (ii) by b_1, we have

$a_1b_2x + b_1b_2y + b_2c_1 = 0$ .... (iii)

$a_2b_1x + b_1b_2y + b_1c_2 = 0$ .... (iv)

Subtracting (iii) from (iv)

$$\begin{array}{r} a_2b_1x + b_1b_2y + b_1c_2 = 0 \\ \pm a_1b_2x \pm b_1b_2y \pm b_2c_1 = 0 \\ \hline a_2b_1x - a_1b_2x + b_1c_2 - b_2c_1 = 0 \end{array}$$

$\Rightarrow$ $x(a_2b_1 - a_1b_2) = b_2c_1 - b_1c_2$ $\Rightarrow$ $\dfrac{x(a_1b_2 - a_2b_1)}{1} = \dfrac{b_1c_2 - b_2c_1}{1}$

$\Rightarrow$ $x = \dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$ $\Rightarrow$ $\dfrac{x}{b_1c_2 - b_2c_1} = \dfrac{1}{a_1b_2 - a_2b_1}$

Now, multiplying (i) by a_2 and (ii) by a_1, we have

$a_1a_2x + a_2b_1y + a_2c_1 = 0$ .... (v)

$a_1a_2x + a_1b_2y + a_1c_2 = 0$ .... (vi)

Subtracting (v) from (vi)

$$\begin{array}{r} a_1a_2x + a_1b_2y + a_1c_2 = 0 \\ \pm a_1a_2x \pm a_2b_1y \pm a_2c_1 = 0 \\ \hline a_1b_2y - a_2b_1y + a_1c_2 - a_2c_1 = 0 \end{array}$$

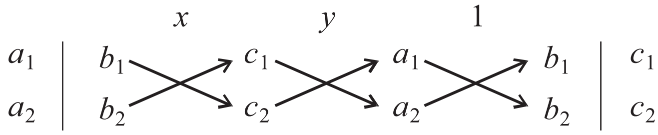
$$\Rightarrow \quad y\,(a_1b_2 - a_2b_1) = a_2c_1 - a_1c_2$$

$$\Rightarrow \quad y = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$$

$$\Rightarrow \quad \frac{y}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\therefore \quad \frac{x}{b_1c_2 - b_2c_1} = \frac{y}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

The following diagram helps in remembering and writing the above solution.



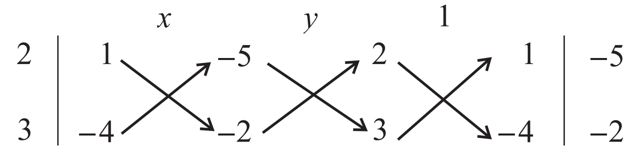
The arrows between two numbers indicate that they are to be multiplied and the second product is to be subtracted from the first.

Example 3: Find the solution set with the method of cross multiplication.

$$2x + y = 5$$
$$3x - 4y = 2$$

Solution: Rewrite the given equation to have zero on the right hand side.

$$2x + y = 5 \qquad \text{............ (i)}$$
$$3x - 4y = 2 \qquad \text{............ (ii)}$$
$$2x + y - 5 = 0$$
$$3x - 4y - 2 = 0$$



Now, we can immediately write down the solution.

$$\frac{x}{(1)(-2)-(-4)(-5)} = \frac{y}{(-5)(3)-(-2)(2)} = \frac{1}{(2)\,(-4) - (3)\,(1)}$$

$$\frac{x}{-2-20} = \frac{y}{-15+4} = \frac{1}{-8-3}$$

$$\Rightarrow \quad \frac{x}{-22} = \frac{y}{-11} = \frac{1}{-11}$$

$$\Rightarrow \quad x = \frac{-22}{-11} = 2 \quad \text{and} \quad y = \frac{-11}{-11} = 1$$

Thus, $x = 2$ and $y = 1$ is the required solution.

Step 4: Check the answer by putting the values of $x = 2$ and $y = 1$ in the equation

$$2x + y = 5 \text{ from (i)}$$
$$\text{L.H.S} = 2(2) + (1)$$
$$= 4 + 1 = 5 = \text{R.H.S}$$

EXERCISE 6.9

1. Find the solution set by using the method of equating the coefficients.

(i) $2x + 5y = -1$, $x - 2y = 4$ **(ii)** $x + y = 2$, $x - y = 0$

(iii) $2x + 3y = 3$, $x + 5y = 5$ **(iv)** $x - 4y = 4$, $4x - y = 16$

(v) $2x - 3y = 6$, $3x + 5y = 0$ **(vi)** $3x - 4y = 7$, $5x + y = 27$

2. Find the solution set by using the method of elimination by substitution.

(i) $2x + 2y = 5$, $x - 2y = 3$ **(ii)** $5x + 2y = 15$, $-2x + y = 4$

(iii) $6x + y = 2$, $x - 4y = 15$ **(iv)** $2x + 7y = 10$, $3x + y = 3$

(v) $2x - 4y = -10$, $y - 5x = -5$ **(vi)** $x + 8y = 15$, $3x - y = 0$

3. Find the solution set by using the method of cross multiplication.

(i) $2x - 7y = 11$, $5x - 10y = 10$ **(ii)** $11x + 12y = 15$, $12x + 11y = -23$

(iii) $2x - 9y + 10 = 0$, $3x - 5y - 10 = 0$ **(iv)** $5x + y - 56 = 0$, $x + 18y - 29 = 0$

(v) $9x - 11y - 15 = 0$, $7x - 13y - 25 = 0$ **(vi)** $2y - 10x - 86 = 0$, $2x + 5y - 11 = 0$

6.5.2 Solving Real Life Problems Involving Two Simultaneous Linear Equations in two Variables

Example 4: A number is half of another number. The sum of 3 times of 1^{st} number and 4 times of 2^{nd} number is 22. Find the numbers.

Solution: Suppose that the numbers are x and y. Then according to given condition.

$$x = \frac{y}{2} \qquad \text{............ (i)}$$
$$3x + 4y = 22 \qquad \text{............ (ii)}$$

From equation (i) we get,

$$x = \frac{y}{2} \Rightarrow y = 2x \qquad .... \text{(iii)}$$

Put the value of "y" in equation (ii)

$$3x + 4(2x) = 22 \Rightarrow 3x + 8x = 22 \Rightarrow 11x = 22 \Rightarrow x = \frac{22}{11} = 2$$

Put the value of "x" in equation (iii)

$$y = 2x \Rightarrow y = 2\ (2) = 4$$

Thus, the numbers are 2 and 4.

Example 5: 11 years ago Ali's age was 5 times of Waleed's age. But after 7 years Ali's age will be 2 times of Waleed's age. Find their ages.

Solution: Suppose that Ali's age is "x" years and Waleed's age is "y" years.

Before 11 years their ages were:

Ali's age $= (x - 11)$ years, Waleed's age $= (y - 11)$ years

Then according to the given condition,

Ali's age $= 5$ (Waleed's age)

$$\Rightarrow \quad x - 11 = 5\ (y - 11)$$
$$\Rightarrow \quad x - 11 = 5y - 55$$
$$\Rightarrow \quad x - 5y = -55 + 11$$
$$\Rightarrow \quad x - 5y = -44 \qquad .... \text{(i)}$$

After 7 years their ages will be:

Ali's age $= (x + 7)$ years, Waleed's age $= (y + 7)$ years

Then according to the given condition,

Ali's age $= 2$ (Waleed's age)

$$\Rightarrow \quad x + 7 = 2\ (y + 7)$$
$$\Rightarrow \quad x + 7 = 2y + 14$$
$$\Rightarrow \quad x - 2y = 14 - 7$$
$$\Rightarrow \quad x - 2y = 7 \qquad .... \text{(ii)}$$

By solving equation (i) and (ii).

$$x - 5y = -44 \qquad .... \text{(i)}$$
$$\pm x \mp 2y = \pm 7 \qquad .... \text{(ii)}$$

$$-3y = -51 \text{ (By subtracting)}$$

$$\Rightarrow \quad y = 17$$

Put the value of "y" in equation (ii)

$$x - 2y = 7$$
$$\Rightarrow \quad x - 2\ (17) = 7$$
$$\Rightarrow \quad x - 34 = 7$$
$$\Rightarrow \quad x = 34 + 7 = 41$$

Thus, Ali's age $= 41$ years and Waleed's age $= 17$ years.

Example 6: If numerator and denominator of a fraction are increased by 5, the fraction becomes $\frac{1}{2}$ and if numerator and denominator are decreased by 3, the fraction becomes $\frac{2}{5}$. Find the fraction.

Solution: Suppose the numerator is x and denominator is y, therefore the fraction is $\frac{x}{y}$. Then, according to the given condition.

$$\frac{x+5}{y+5} = \frac{1}{2} \Rightarrow 2(x+5) = y+5 \Rightarrow 2x+10 = y+5 \Rightarrow 2x - y = -5$$

$$\Rightarrow \quad y = 2x + 5 \quad \text{...........} (i)$$

Then, by the second condition.

$$\frac{x-3}{y-3} = \frac{2}{5}$$

$$\Rightarrow \quad 5(x-3) = 2(y-3)$$

$$\Rightarrow \quad 5x - 15 = 2y - 6$$

$$\Rightarrow \quad 5x - 2y = 15 - 6$$

$$5x - 2y = 9 \quad \text{...........} (ii)$$

Put the value of "y" from equation (i), in equation (ii) we have,

$$5x - 2(2x+5) = 9$$

$$\Rightarrow \quad 5x - 4x - 10 = 9$$

$$\Rightarrow \quad x - 10 = 9$$

$$\Rightarrow \quad x = 10 + 9 = 19$$

Put the value of "x" in equation (i),

$$y = 2x + 5 \quad \text{...........} (iii)$$

$$\Rightarrow \quad y = 2(19) + 5$$

$$\Rightarrow \quad y = 38 + 5$$

$$\Rightarrow \quad y = 43$$

Thus, the required fraction is $\frac{19}{43}$

EXERCISE 6.10

1. Ahmad added 5 in the twice of a number. Then he subtracted half of the number from the result. Finally, he got the answer 8. Find the number.

2. If we add 3 in the half of a number, we get the same result as we subtract 1 from the quarter of the number. Find the number.

3. The sum of two numbers is 5 and their difference is 1. Find the numbers.

4. The difference of two numbers is 4. The sum of twice of one number and 3 times of the other number is 43. Find the numbers.

5. Adnan is 7 years older than Adeel. Find their ages when $\frac{1}{4}$ of Adnan's age is equal to the $\frac{1}{2}$ of Adeel's age.

6. 5 years ago Ahsan's age was 7 times of Shakeel's age but after 3 years Ahsan's age will be 4 times of Shakeel's age. Calculate their ages.

7. The denominator of a fraction is 5 more than the numerator. But if we subtract 2 from the numerator and the denominator of the fraction, we get $\frac{1}{6}$. Find the fraction.

8. Fida bought $3kg$ melons and 4kg mangoes for $Rs.470$. Anam bought $5kg$ melons and 6kg mangoes for Rs.730. Calculate the price of melons and mangoes per kg.

9. The cost of 2 footballs and 10 basketballs is $Rs.2300$ and the cost of 7 footballs and 5 basketballs is $Rs.2650$. Calculate the price of each football and basketball.

10. If the numerator and denominator of a fraction are increased by 1, the fraction becomes $\frac{2}{3}$ and if the numerator and denominator of same fraction are decreased by 2, it becomes $\frac{1}{3}$. Find the fraction.

11. If the numerator and denominator of a fraction are decreased by 1, the fraction becomes $\frac{1}{2}$. If the numerator and denominator of the same fraction are decreased by 3, it becomes $\frac{1}{4}$. Find the fraction.

6.6 ELIMINATION

Look at the following simultaneous linear equations.

$x + 5 = 8$ .... (i)

$x - 1 = 1$ .... (ii)

From above it can be seen that the equation (i) is true for $x = 3$ and the equation (ii) is true for $x = 2$, but both equations are not true for a unique value of x.

Now observe the following simultaneous linear equations.

$x + a = 5$ .... (iii)

$x + b = 4$ .... (iv)

Here the equation (iii) is true for $x = 5 - a$ and the equation (iv) is true for $x = 4 - b$. While finding a single value of x for which both the equations are true, we put

$$5 - a = 4 - b$$
$$\Rightarrow \quad a - b = 5 - 4$$
$$\Rightarrow \quad a - b = 1 \qquad \text{............ (v)}$$

It can be noted that a new relation (v) is established here which is independent of x. This process is called elimination and the relation $a - b = 1$ is called eliminated.

6.6.1 ELIMINATION OF A VARIABLE FROM TWO EQUATIONS

At least two equations are required for elimination of one variable. There are different methods of elimination, but we learn here only two methods through examples.

(a) Elimination of Variable from two Equations by Substitution

Example 1: Eliminate "x" from the following equations by substitution method.

$$ax - b = 0$$
$$cx - d = 0$$

Solution: Given:

$$ax - b = 0 \qquad \text{............ (i)}$$
$$cx - d = 0 \qquad \text{............ (ii)}$$

From equation (i), we have

$$ax = b \qquad \text{or} \qquad x = \frac{b}{a}$$

Put the value of x in equation (ii), we get

$$c\left(\frac{b}{a}\right) - d = 0$$

$\Rightarrow \quad bc - ad = 0 \quad \Rightarrow \quad bc = ad$ Here "x" is eliminated.

Example 2: Eliminate "x" from $ax^2 + bx + c = 0$ and $\ell x + m = 0$ by substitution method.

Solution:

$$ax^2 + bx + c = 0 \qquad \text{............ (i)}$$
$$\ell x + m = 0 \qquad \text{............ (ii)}$$

From equation (ii), we have,

$$\ell x + m = 0 \Rightarrow x = \frac{-m}{\ell}$$

Put the value of x in equation (i)

$$a\left(\frac{-m}{\ell}\right)^2 + b\left(\frac{-m}{\ell}\right) + c = 0$$

$$\Rightarrow \quad a\frac{m^2}{\ell^2} - b\frac{m}{\ell} + c = 0$$

$$\Rightarrow \quad \frac{am^2}{\ell^2} - \frac{bm}{\ell} + c = 0 \qquad \text{(Multiply equation by } \ell^2\text{)}$$

$$\Rightarrow \quad am^2 - b\ell m + c\ell^2 = 0$$

This is the required result.

EXERCISE 6.11

1. Eliminate "x" from the following equations by substitution method.

(i) $ax - b = 0$
$cx - d = 0$

(ii) $2x + 3y = 5$
$x - y = 2$

(iii) $x + a = b$
$x^2 + a^2 = b^2$

(iv) $a - b = 2x$
$a^2 + b^2 = 3x^2$

(v) $x - m = \ell$
$(\ell - m)x + a = 0$

2. Eliminate v_i from the following equations.

(i) $v_f = v_i + at$
$S = v_i t + \frac{1}{2}at^2$

(ii) $v_f = v_i + at$
$2aS = v_f^2 - v_i^2$

(iii) $v_f = v_i - gt$
$S = v_i t + \frac{1}{2}gt^2$

(b) Elimination of a Variable from two Equations by Application of Formulas

Example 3: Elimination of "x" from the following equations by using the formula.

$$x + \frac{1}{x} = \ell; \qquad x^2 + \frac{1}{x^2} = m^2$$

Solution: $x + \frac{1}{x} = \ell$ .... (i)

and $x^2 + \frac{1}{x^2} = m^2$ .... (ii)

Taking square of both the sides of (i), we have

$$\left(x + \frac{1}{x}\right)^2 = (\ell)^2$$

or $\quad x^2 + \frac{1}{x^2} + 2 = \ell^2$

or $\quad x^2 + \frac{1}{x^2} = \ell^2 - 2$ .... (iii)

Compare equations (ii) and (iii), we get

$$\ell^2 - 2 = m^2$$

This is the required relation.

Example 4: Eliminate "t" from the following equations.

$$x = \frac{2at}{1+t^2} \quad , \quad y = \frac{b\left(1-t^2\right)}{1+t^2}$$

Solution: $x = \frac{2at}{1+t^2}$ .... (i) , $y = \frac{b\left(1-t^2\right)}{1+t^2}$ .... (ii)

Equation (i) gives

$$\frac{x}{a} = \frac{2t}{1+t^2}$$

or $\quad \left(\frac{x}{a}\right)^2 = \left(\frac{2t}{1+t^2}\right)^2$ (Taking square of both the sides)

or $\quad \frac{x^2}{a^2} = \frac{4t^2}{1+2t^2+t^4}$ .... (iii)

Equation (ii) gives

$$\frac{y}{b} = \frac{1-t^2}{1+t^2}$$

or $\quad \left(\frac{y}{b}\right)^2 = \left(\frac{1-t^2}{1+t^2}\right)^2$ (Taking square of both the sides)

or $\quad \frac{y^2}{b^2} = \frac{1-2t^2+t^4}{1+2t^2+t^4}$ .... (iv)

By adding equations (iii) and (iv),

we have, $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{4t^2}{1+2t^2+t^4} + \frac{1-2t^2+t^4}{1+2t^2+t^4}$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{4t^2+1-2t^2+t^4}{1+2t^2+t^4}=\frac{1+2t^2+t^4}{1+2t^2+t^4}=1$$

Thus, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is the required solution.

EXERCISE 6.12

1. Eliminate "x" from the following equations by using appropriate formula.

(i) $x-\frac{1}{x}=m\ ;\ x^2+\frac{1}{x^2}=n^2$ **(ii)** $x-\frac{1}{x}=\frac{a}{2}\ ;\ x^2+\frac{1}{x^2}=b^2$

(iii) $\frac{x^2}{\ell^2}+\frac{\ell^2}{x^2}=b^2;\ \frac{\ell}{x}-\frac{x}{\ell}=a$ **(iv)** $\frac{x}{c}+\frac{c}{x}=2a\ ;\ \frac{x}{c}-\frac{c}{x}=3b$

(v) $x-\frac{1}{x}=\ell\ ;x^3-\frac{1}{x^3}=m^3$ **(vi)** $x-\frac{1}{x}=p\ ;\ x^2+\frac{1}{x^2}=2q^2$

(vii) $x^2+\frac{1}{x^2}=3m^2;\ x^4+\frac{1}{x^4}=n^4$ **(viii)** $x-\frac{1}{x}=a\ ;\ x^4+\frac{1}{x^4}=a^4$

2. Eliminate "t" from the following equations.

(i) $at^2=x$, $bt^3=y$ **(ii)** $x-y=2t$, $x^2+y^2=3t^2$

REVIEW EXERCISE 6

1. Four options are given against each statement. Encircle the correct one.

i. The square of 99 by formula is:

(a) $(100)^2-2(100)\,(1)+(1)^2$ **(b)** $(100)^2+2(100)(1)+(1)^2$

(c) $(100)^2+2(100)\,(1)-(1)^2$ **(d)** $(100)^2-2(100)(1)-(1)^2$

ii. If $x+\frac{1}{x}=9$, then $x^2+\frac{1}{x^2}=$?

(a) 81 **(b)** 18 **(c)** 27 **(d)** 79

iii. The correct factorization of $5y\,(y-3)+4\,(y-3)$ is:

(a) $(5y+y)\,(4-3)$ **(b)** $(5y-3)\,(y-r)$

(c) $(5y+4)\,(y-3)$ **(d)** $(y+3)\,(5y+4)$

iv. The factorization of $4x^2-12xy+9y^2$ is:

(a) $(2x+3y)\,(2x-3y)$ **(b)** $(2x-3y)\,(2x-3y)$

(c) $(2x+3y)\,(2x+3y)$ **(d)** $(2x-3y)\,(2x+3y)$

v. If $x - \frac{1}{x} = 3$, then $x^3 - \frac{1}{x^3} =$?

(a) 27 **(b)** 18 **(c)** 30 **(d)** 36

vi. If $x + y = 6, x - y = 2$, then $y =$?

(a) 4 **(b)** 2 **(c)** 6 **(d)** 8

vii. After eliminating "x" from $ax^2 = b$ and $cx^2 = d$, we get:

(a) $bc = ad$ **(b)** $bd = ac$ **(c)** $\frac{a}{b} = \frac{c}{d}$ **(d)** $abc = d$

viii. After eliminating x from $x + \frac{1}{x} = b, x^2 + \frac{1}{x^2} = a^2$, we get:

(a) $a^2 = b^2 + 2$ **(b)** $a^2 + b^2 = 2$

(c) $a^2 - b^2 = -2$ **(d)** $a^2 + b^2 = -2$

2. Answer the following questions.

i. What are the simultaneous linear equations?

ii. Write any three methods for solving simultaneous linear equations.

iii. How many equations are required for elimination of one variable?

3. Find the value of $x^4 + \frac{1}{x^4}$, when $x + \frac{1}{x} = 7$.

4. Factorize the following:

i. $3xy + 6x^2y^2 + 9xz$ **ii.** $y^4 - 12y^2 + 36$ **iii.** $x^8 - y^8$

5. Find the cube of the following:

i. 13 **ii.** $2x - 3y$ **iii.** $7a - b$

6. If $x + \frac{1}{x} = 5$, then find the value of $x^3 + \frac{1}{x^3}$.

7. Eliminate "x" by substitution method from the following equations.

i. $ax - b = 0, \ cx^2 + mx = 0$ **ii.** $\ell x - n = 0, sx^2 + tx + u = 0$

8. Eliminate "x" from the following equations by using formula.

i. $x + \frac{1}{x} = \frac{a}{3}, x^2 + \frac{1}{x^2} = b^2$ **ii.** $x + \frac{1}{x} = 3b, x^3 + \frac{1}{x^3} = a^3$

iii. $x - \frac{1}{x} = a, x^4 + \frac{1}{x^4} = b^4$

9. If the numerator and denominator of a fraction are increased by 1, the fraction becomes $\frac{3}{4}$ and if the numerator and denominator of same fraction are decreased by 1, it becomes $\frac{2}{3}$. Find the fraction.

10. Eliminate *"t"* from the following equations.

(i) $x=\frac{1+t^2}{1-t^2}$, $y=\frac{2at}{1-t^2}$ **(ii)** $x=\frac{1+t^2}{2at}$, $y=\frac{b(1-t^2)}{1+t^2}$

SUMMARY

- Three basic algebraic formulas are:
  i. $(a+b)^2 = a^2 + 2ab + b^2$
  ii. $(a-b)^2 = a^2 - 2ab + b^2$
  iii. $a^2 - b^2 = (a+b)(a-b)$
- Expressing polynomials as product of two or more polynomials that cannot be further expressed as product of factors is called Factorization.
- The cubic formulas are:
  i. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
  ii. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
- If a and b are real numbers (and if a and b are not both equal to 0) then $ax+by=r$ is called a linear equation in two variables x and y, a and b are coefficients and r is constant of the equation.
- Simultaneous linear equations mean a collection of linear equations all of which are satisfied by the same values of the variables.
- The general form of the simultaneous linear system of equations in two variables is:

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$