CHAPTER

13

SIDES AND ANGLES OF A TRIANGLE

Animation 13.1: Sides and Angles of a Triangle
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Students Learning Outcomes

After studying this unit, the students will be able to:

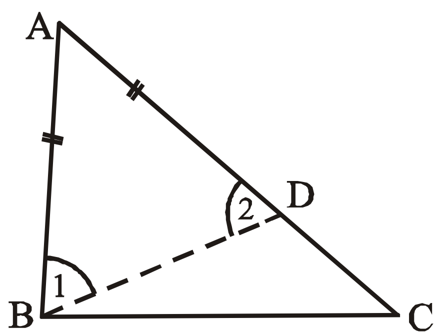
- prove that if two sides of a triangle are unequal in length, the longer side has an angle of greater measure opposite to it.
- prove that if two angles of a triangle are unequal in measure, the side opposite to the greater angle is longer than the side opposite to the smaller angle.
- prove that the sum of the lengths of any two sides of a triangle is greater than the length of the third side.
- prove that from a point, out-side a line, the perpendicular is the shortest distance from the point on the line.

Introduction

Recall that if two sides of a triangle are equal, then the angles apposite to them are also equal and vice-versa. But in this unit we shall study some interesting inequality relations among sides and angles of a triangle.

Theorem 13.1.1

If two sides of a triangle are unequal in length, the longer side has an angle of greater measure opposite to it.



Given

In ΔABC, $m\overline{AC} > m\overline{AB}$

To Prove

$m\angle ABC > m\angle ACB$

Construction

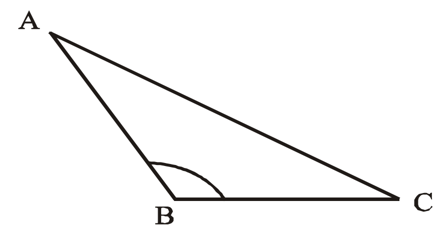
On $\overline{AC}$ take a point D such that $\overline{AD} \cong \overline{AB}$. Join B to D so that ΔADB is an isosceles triangle. Label $\angle 1$ and $\angle 2$ as shown in the given figure.

Proof

Statements	Reasons
In ΔABD	
$m\angle 1 = m\angle 2$ ...... (i)	Angles opposite to congruent sides, (construction)
In ΔBCD, $m\angle ACB < m\angle 2$	
i.e. $m\angle 2 > m\angle ACB$ ...... (ii)	(An exterior angle of a triangle is greater than a non-adjacent interior angle)
$\therefore$ $m\angle 1 > m\angle ACB$ ...... (iii)	By (i) and (ii)
But	
$m\angle ABC = m\angle 1 + m\angle DBC$	Postulate of addition of angles.
$\therefore$ $m\angle ABC > m\angle 1$ ...... (iv)	
$\therefore$ $m\angle ABC > m\angle 1 > m\angle ACB$	By (iii) and (iv)
Hence $m\angle ABC > m\angle ACB$	(Transitive property of inequality of real numbers)

Example 1

Prove that in a scalene triangle, the angle opposite to the largest side is of measure greater than 60°. (i.e., two-third of a right-angle)



Given

In ABC, $m\overline{AC} > m\overline{AB}$ $m\overline{AC}$, $m\overline{AB} > m\overline{BC}$.

To Prove

$m\angle B > 60°$.

Proof

Statements	Reasons
In ΔABC	
$m\angle B > m\angle C$	$m\overline{AC} > m\overline{AB}$ (given)
$m\angle B > m\angle A$	$m\overline{AC} > m\overline{BC}$ (given)
But $m\angle A + m\angle B + m\angle C = 180°$	$\angle A, \angle B, \angle C$ are the angles of ΔABC
$\therefore$ $m\angle B + m\angle B + m\angle B > 180°$	$m\angle B > m\angle C$, $m\angle B > m\angle A$ (proved)
Hence $m\angle B > 60°$	$180°/3 = 60°$

Example 2

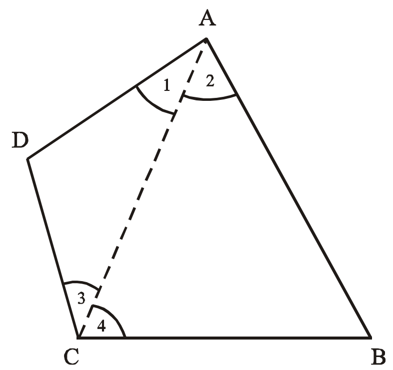
In a quadrilateral ABCD, $\overline{AB}$ is the longest side and $\overline{CD}$ is the shortest side. Prove that $m\angle BCD > m\angle BAD$.

Given

In quad. ABCD, $\overline{AB}$ is the longest side and $\overline{CD}$ is the shortest side.

To Prove

$m\angle BCD > m\angle BAD$



Construction

Joint A to C.

Name the angles $\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$ as shown in the figure.

Proof

Statements	Reasons
In ΔABC, $m\angle 4 > \angle 2$ I	$m\overline{AB} > m\overline{BC}$ (given)
In ΔACD, $m\angle 3 > m\angle 1$ II	$m\overline{AD} > m\overline{CD}$ (given)
$\therefore$ $m\angle 4 + m\angle 3 > m\angle 2 + m\angle 1$	From I and II
Hence $m\angle BCD > m\angle BAD$	$\because \begin{cases} m\angle 4 + m\angle 3 = m\angle BCD \\ m\angle 2 + m\angle 1 = m\angle BAD \end{cases}$

Theorem 13.1.2

(Converse of Theorem 13.1.1)

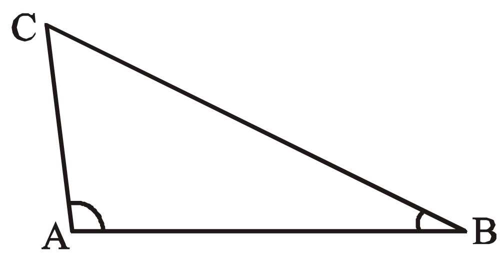
If two angles of a triangle are unequal in measure, the side opposite to the greater angle is longer than the side opposite to the smaller angle.

Given

In ΔABC, $m\angle A > m\angle B$

To Prove

$m\overline{BC} > m\overline{AC}$



Proof

Statements	Reasons
If , $m\overline{BC} \ngtr m\overline{AC}$, then	
either (i) $m\overline{BC} = m\overline{AC}$ or (ii) $m\overline{BC} < m\overline{AC}$	(Trichotomy property of real numbers)
From (i) if $m\overline{BC} = m\overline{AC}$, then $m\angle A = m\angle B$	(Angles opposite to congruent sides are congruent)
which is not possible.	Contrary to the given.
From (ii) if mBC < mAC, then $m\angle A < m\angle B$	(The angle opposite to longer side is greater than angle opposite to smaller side)
This is also not possible.	Contrary to the given.
$\therefore$ $m\overline{BC} \neq m\overline{AC}$	
and $m\overline{BC} \nless m\overline{AC}$	
Thus $m\overline{BC} > m\overline{AC}$	Trichotomy property of real numbers.

Corollaries

(i) The hypotenuse of a right angled triangle is longer than each of the other two sides.

(ii) In an obtuse angled triangle, the side opposite to the obtuse angle is longer than each of the other two sides.

Example

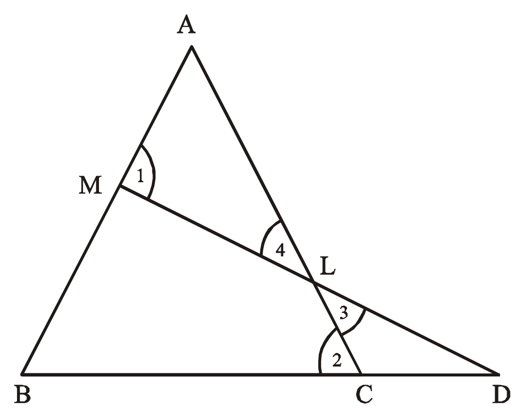
ABC is an isosceles triangle with base $\overline{BC}$. On $\overrightarrow{BC}$ a point D is taken away from C. A line segment through D cuts $\overline{AC}$ at L and $\overline{AB}$ at M. Prove that $m\overline{AL} > m\overline{AM}$.

Given

In ΔABC, $\overline{AB} \cong \overline{AC}$

D is a point on $\overrightarrow{BC}$ away from C.

A line segment through D cuts $\overline{AC}$ at L and $\overline{AB}$ at M.



To Prove

$m\overline{AL} > m\overline{AM}$

Proof

Statements	Reasons
In ΔABC	
$\angle B \cong \angle 2$ I	$\overline{AB} \cong \overline{AC}$ (given)
In ΔMBD	
$m\angle 1 > m\angle B$ II	($\angle 1$ is an ext. $\angle$ and $\angle B$ is its internal opposite $\angle$)
$\therefore$ $m\angle 1 > m\angle 2$ III	From I and II
In ΔLCD,	
$m\angle 2 > m\angle 3$ IV	($\angle 2$ is an ext. $\angle$ and $\angle 3$ is its internal opposite $\angle$)
$\therefore$ $m\angle 1 > m\angle 3$ V	From III and IV
But $\angle 3 \cong \angle 4$ VI	Vertical angles
$\therefore$ $m\angle 1 > m\angle 4$	From V and VI
Hence $m\overline{AL} > m\overline{AM}$	In ΔALM, $m\angle 1 > m\angle 4$ (proved)

Theorem 13.1.3

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

Given

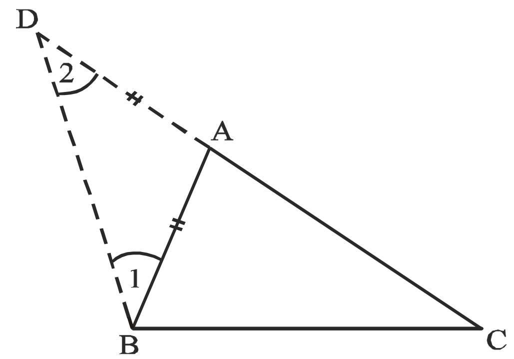
ΔABC

To Prove

(i) $m\overline{AB} + m\overline{AC} > m\overline{BC}$

(ii) $m\overline{AB} + m\overline{BC} > m\overline{AC}$

(iii) $m\overline{BC} + m\overline{CA} > m\overline{AB}$



Construction

Take a point D on $\overrightarrow{CA}$ such that $\overline{AD} \cong \overline{AB}$. Join B to D and name the angles. $\angle 1$, $\angle 2$ as shown in the given figure.

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Proof

Statements	Reasons
In ΔABD,	
$\angle 1 \cong \angle 2$ ....(i)	$\overline{AD} \cong \overline{AB}$ (construction)
$m\angle DBC > m\angle 1$ ....(ii)	$m\angle DBC = m\angle 1 + m\angle ABC$
$\therefore$ $m\angle DBC > m\angle 2$ ....(iii)	From (i) and (ii)
In ΔDBC	
$m\overline{CD} > m\overline{BC}$	By (iii)
i.e., $m\overline{AD} + m\overline{AC} > m\overline{BC}$	$m\overline{CD} = m\overline{AD} + m\overline{AC}$
Hence $m\overline{AB} + m\overline{AC} > m\overline{BC}$	$m\overline{AD} = m\overline{AB}$ (construction)
Similarly,	
$m\overline{AB} + m\overline{BC} > m\overline{AC}$	
and	
$m\overline{BC} + m\overline{CA} > m\overline{AB}$	

Example 1

Which of the following sets of lengths can be the lengths of the sides of a triangle?

(a) 2 cm, 3 cm, 5 cm (b) 3 cm, 4 cm, 5 cm, (c) 2 cm, 4 cm, 7 cm,

(a) $\because$ $2 + 3 = 5$

$\therefore$ This set of lengths cannot be those of the sides of a triangle.

(b) $\because$ $3 + 4 > 5$, $3 + 5 > 4$, $4 + 5 > 3$

$\therefore$ This set can form a triangle

(c) $\because$ $2 + 4 < 7$

$\therefore$ This set of lengths cannot be the sides of a triangle.

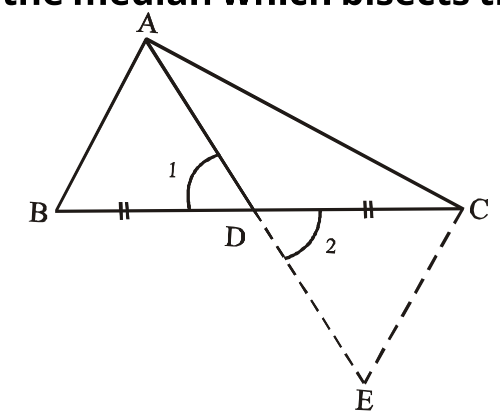
Example 2

Prove that the sum of the measures of two sides of a triangle is greater than twice the measure of the median which bisects the third side.

Given

In ΔABC,

median $\overline{AD}$ bisects side $\overline{BC}$ at D.



To Prove

$m\overline{AB} + m\overline{AC} > 2m\overline{AD}$.

Construction

On $\overrightarrow{AD}$ take a point E, such that $\overline{DE} \cong \overline{AD}$. Join C to E. Name the angles $\angle 1$, $\angle 2$ as shown in the figure.

Proof

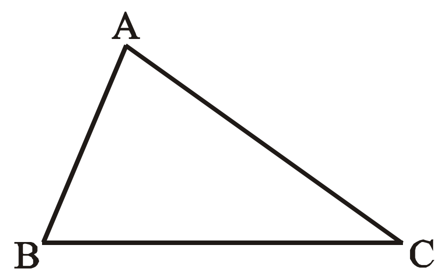
Statements	Reasons
In $\Delta ABD \longleftrightarrow \Delta ECD$	
$\overline{BD} \cong \overline{CD}$	Given
$\angle 1 \cong \angle 2$	Vertical angles
$\overline{AD} \cong \overline{ED}$	Construction
$\Delta ABD \cong \Delta ECD$	S.A.S. Postulate
$\overline{AB} \cong \overline{EC}$ I	Corresponding sides of $\cong \Delta$s
$m\overline{AC} + m\overline{EC} > m\overline{AE}$ II	ACE is a triangle
$m\overline{AC} + m\overline{AB} > m\overline{AE}$	From I and II
Hence $m\overline{AC} + m\overline{AB} > 2m\overline{AD}$	$m\overline{AE} = 2m\overline{AD}$ (construction)

Example 3

Prove that the difference of measures of two sides of a triangle is less than the measure of the third side.

Given

ΔABC



To Prove

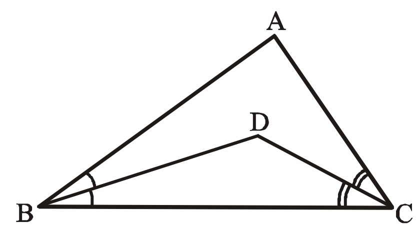
$m\overline{AC} - m\overline{AB} < m\overline{BC}$
$m\overline{BC} - m\overline{AB} < m\overline{AC}$
$m\overline{BC} - m\overline{AC} > m\overline{AB}$

Proof:

Statements	Reasons
$m\overline{AB} + m\overline{BC} > m\overline{AC}$	ABC is a triangle
$(m\overline{AB} + m\overline{BC} - m\overline{AB})$ $>(m\overline{AC} - m\overline{AB})$	Subtracting $m\overline{AB}$ from both sides
$\therefore$ $m\overline{BC}>(m\overline{AC} - m\overline{AB})$	
or $m\overline{AC} - m\overline{AB} < m\overline{BC}$ I	$a > b \Rightarrow b < a$
Similarly	
$m\overline{BC} - m\overline{AB} < m\overline{AC}$	
$m\overline{BC} - m\overline{AC} < m\overline{AB}$	Reason similar to I

EXERCISE 13.1

1. Two sides of a triangle measure 10 cm and 15 cm. Which of the following measure is possible for the third side?
   (a) 5 cm (b) 20 cm (c) 25 cm (d) 30 cm
2. O is an interior point of the ΔABC. Show that $m\overline{OA} + m\overline{OB} + m\overline{OC} > \frac{1}{2}(m\overline{AB} + m\overline{BC} + m\overline{CA})$
3. In the Δ ABC, $m\angle B = 70°$ and $m\angle C = 45°$. Which of the sides of the triangle is longest and which is the shortest?
4. Prove that in a right-angled triangle, the hypotenuse is longer than each of the other two sides.
5. In the triangular figure, $m\overline{AB} > m\overline{AC}$. $\overline{BD}$ and $\overline{CD}$ are the bisectors of B and C respectively. Prove that $m\overline{BD} > m\overline{DC}$.

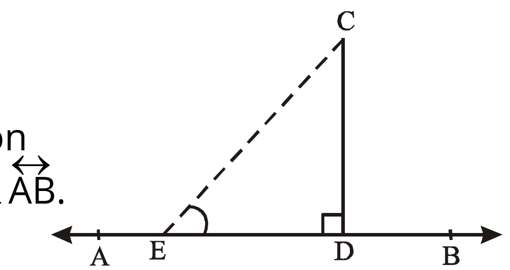


Theorem 13.1.4

From a point, outside a line, the perpendicular is the shortest distance from the point to the line.

Given

A line AB and a point C (not lying on $\overleftrightarrow{AB}$) and a point D on $\overleftrightarrow{AB}$ such that $CD \perp \overleftrightarrow{AB}$.



To Prove

$m\overline{CD}$ is the shortest distance form the point C to $\overleftrightarrow{AB}$.

Construction

Take a point E on $\overleftrightarrow{AB}$. Join C and E to form a ΔCDE.

Proof

Statements	Reasons
In ΔCDE	
$m\angle CDB > m\angle CED$	(An exterior angle of a triangle is greater than non adjacent interior angle).
But $m\angle CDB = m\angle CDE$	Supplement of right angle.
$\therefore$ $m\angle CDE > m\angle CED$	
or $m\angle CED < m\angle CDE$	$a > b \Rightarrow b < a$
or $m\overline{CD} < m\overline{CE}$	Side opposite to greater angle is greater.
But E is any point on $\overleftrightarrow{AB}$ Hence $m\overline{CD}$ is the shortest distance from C to $\overleftrightarrow{AB}$.	

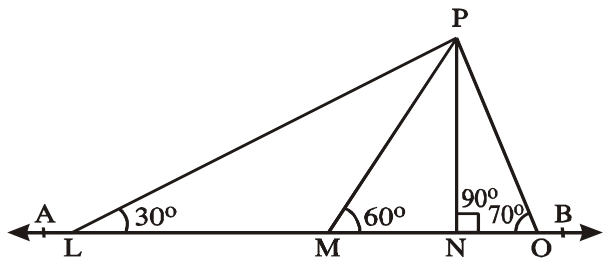
Note:

(i) The distance between a line and a point not on it, is the length of the perpendicular line segment from the point to the line.

(ii) The distance between a line and a point lying on it is zero.

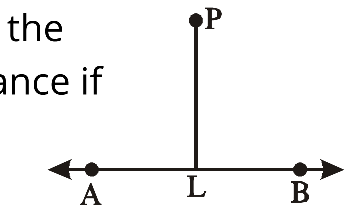
EXERCISE 13.2

1. In the figure, P is any point and AB is a line. Which of the following is the shortest distance between the point P and the line AB?

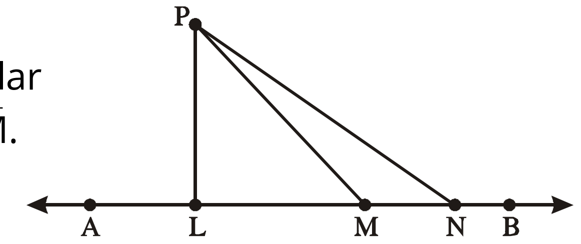


(a) $m\overline{PL}$ (b) $m\overline{PM}$ (c) $m\overline{NP}$ (d) $m\overline{PO}$

2. In the figure, P is any point lying away from the line AB. Then $m\overline{PL}$ will be the shortest distance if
(a) $m\angle PLA = 80°$ (b) $m\angle PLB = 100°$
(c) $m\angle PLA = 90°$



3. In the figure, $\overline{PL}$ is prependicular to the line $\overline{AB}$ and $m\overline{LN} > m\overline{LM}$. Prove that $m\overline{PN} > m\overline{PM}$.



REVIEW EXERCISE 13

1. Which of the following are true and which are false?
   - (i) The angle opposite to the longer side is greater. ......
   - (ii) In a right-angled triangle greater angle is of 60°. ......
   - (iii) In an isosceles right-angled triangle, angles other than right angle are each of 45°. ......
   - (iv) A triangle having two congruent sides is called equilateral triangle. ......
   - (v) A perpendicular from a point to line is shortest distance. ...
   - (vi) Perpendicular to line form an angle of 90°. ......
   - (vii) A point out side the line is collinear. ......
   - (viii) Sum of two sides of triangle is greater than the third. ......
   - (ix) The distance between a line and a point on it is zero. ......
   - (x) Triangle can be formed of lengths 2 cm, 3 cm and 5 cm. ...
2. What will be angle for shortest distance from an outside point to the line?
3. If 13 cm, 12 cm, and 5 cm are the lengths of a triangle, then verify that difference of measures of any two sides of a triangle is less than the measure of the third side.
4. If 10 cm, 6 cm and 8 cm are the lengths of a triangle, then verify that sum of measures of two sides of a triangle is greater than the third side.
5. 3 cm, 4 cm and 7 cm are not the lengths of the triangle. Give the reason.

6. If 3 cm and 4 cm are lengths of two sides of a right angle triangle, then what should be the third length of the triangle.

SUMMARY

In this unit we stated and proved the following theorems:

- If two sides of a triangle are unequal in length, the longer side has an angle of greater measure opposite to it.
- If two angles of a triangle are unequal in measure, the side opposite to the greater angle is longer than the side opposite to the smaller angle.
- The sum of the lengths of any two sides of a triangle is greater than the length of the third side.
- From a point, outside a line, the perpendicular is the shortest distance from the point to the line.