CHAPTER

2

REAL AND COMPLEX NUMBERS

Animation 2.1:Real And Complex numbers
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Students Learning Outcomes

- After studying this unit , the students will be able to:
- Recall the set of real numbers as a union of sets of rational and irrational numbers.
- Depict real numbers on the number line.
- Demonstrate a number with terminating and non-terminating recurring decimals on the number line.
- Give decimal representation of rational and irrational numbers.
- Know the properties of real numbers.
- Explain the concept of radicals and radicands.
- Differentiate between radical form and exponential form of an expression.
- Transform an expression given in radical form to an exponential form and vice versa.
- Recall base, exponent and value.
- Apply the laws of exponents to simplify expressions with real exponents.
- Define complex number z represented by an expression of the form $z = a + ib$, where a and b are real numbers and $i = \sqrt{-1}$
- Recognize *a* as real part and *b* as imaginary part of *z* = *a* + *ib*.
- Define conjugate of a complex number.
- Know the condition for equality of complex numbers.
- Carry out basic operations (i.e., addition, subtraction, multiplication and division) on complex numbers.

Introduction

The numbers are the foundation of mathematics and we use different kinds of numbers in our daily life. So it is necessary to be familiar with various kinds of numbers In this unit we shall discuss real numbers and complex numbers including their properties. There is a one-one correspondence between real numbers and the points on the real line. The basic operations of addition, subtraction, multiplication and division on complex numbers will also be discussed in this unit.

2. Real and Complex Numbers eLearn.Punjab

2.1 Real Numbers

We recall the following sets before giving the concept of real numbers.

Natural Numbers

The numbers 1, 2, 3, ... which we use for counting certain objects are called natural numbers or positive integers. The set of natural numbers is denoted by N.

i.e., N = {1,2,3, ...}

Whole Numbers

If we include 0 in the set of natural numbers, the resulting set is the set of whole numbers, denoted by W,

i.e., W = {o,1,2,3, ...}

Integers

The set of integers consist of positive integers, 0 and negative integers and is denoted by Z i.e., Z = { ..., –3, –2, –1, 0, 1, 2, 3, ... }

2.1.1 Set of Real Numbers

First we recall about the set of rational and irrational numbers.

Rational Numbers

All numbers of the form p/q where p, q are integers and q is not zero are called rational numbers. The set of rational numbers is denoted by Q,

$$i.e., \quad Q = \left\{ \frac{p}{q} \mid p, q \in Z \wedge q \neq 0 \right\}$$

Irrational Numbers

The numbers which cannot be expressed as quotient of integers are called irrational numbers.

The set of irrational numbers is denoted by Q',

$$Q' = \left\{ x \mid x \neq \frac{p}{q}, p, q \in Z \wedge q \neq 0 \right\}$$

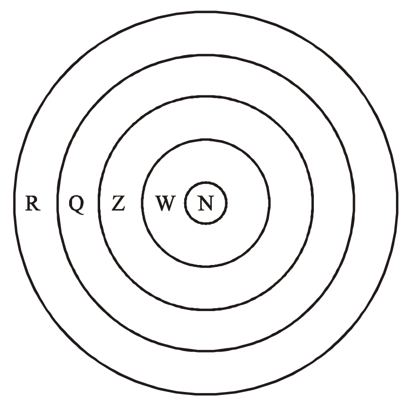
For example, the numbers $\sqrt{2}, \sqrt{3}, \sqrt{5}, \pi$ and e are all irrational numbers. The union of the set of rational numbers and irrational numbers is known as the set of real numbers. It is denoted by R,

i.e., $R = Q \cup Q'$

Here Q and Q′ are both subset of R and $Q \cap Q' = \phi$

Note:

(i) $N \subset W \subset Z \subset Q$
(ii) Q and Q′ are disjoint sets.
(iii) for each prime number p, $\sqrt{p}$ is an irrational number.
(iv) square roots of all positive non- square integers are irrational.

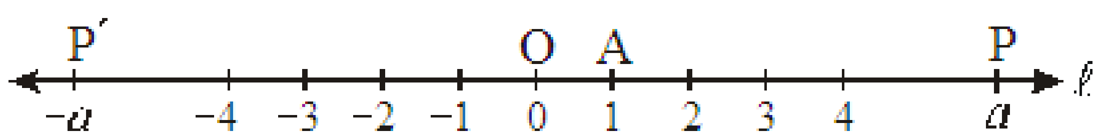


i.e., $N \subset W \subset Z \subset Q \subset R$

2.1.2 Depiction of Real Numbers on Number Line

The real numbers are represented geometrically by points on a number line l such that each real number 'a' corresponds to one and only one point on number line l and to each point P on number line l there corresponds precisely one real number. This type of association or relationship is called a one-to-one correspondence. We establish such correspondence as below.

We first choose an arbitrary point O (the origin) on a horizontal line l and associate with it the real number 0. By convention, numbers to the right of the origin are positive and numbers to the left of the origin are negative. Assign the number 1 to the point A so that the line segment OA represents one unit of length.



The number 'a' associated with a point P on ℓ is called the coordinate of P, and ℓ is called the coordinate line or the real number line. For any real number a, the point P′(– a) corresponding to $-a$ lies at the same distance from O as the point P (a) corresponding to a but in the opposite direction.

2.1.3 Demonstration of a Number with Terminating and Non-Terminating decimals on the Number Line

First we give the following concepts of rational and irrational numbers.

(a) Rational Numbers

The decimal representations of rational numbers are of two types, terminating and recurring.

(i) Terminating Decimal Fractions

The decimal fraction in which there are finite number of digits in its decimal part is called a terminating decimal fraction. For example

$$\frac{2}{5} = 0.4 \text{ and } \frac{3}{8} = 0.375$$

(ii) Recurring and Non-terminating Decimal Fractions

The decimal fraction (non-terminating) in which some digits are repeated again and again in the same order in its decimal part is called a recurring decimal fraction. For example

$$\frac{2}{9} = 0.2222 \text{ and } \frac{4}{11} = 0.363636...$$

(b) Irrational Numbers

It may be noted that the decimal representations for irrational numbers are neither terminating nor repeating in blocks. The decimal form of an irrational number would continue forever and never begin to repeat the same block of digits.

e.g., $\sqrt{2}$ = 1.414213562..., π= 3.141592654..., e= 2.718281829..., etc.

Obviously these decimal representations are neither terminating nor recurring.

We consider the following example.

Example

Express the following decimals in the form $\frac{p}{q}$ where p, q $\in$ Z and q $\neq$ 0

(a) $0.\bar{3} = 0.333...$ (b) $0.\overline{23} = 0.232323...$

Solution

(a) Let $x = 0.\bar{3}$ which can be rewritten as

x = 0.3333... ...... (i)

Note that we have only one digit 3 repeating indefinitely.

So, we multiply both sides of (i) by 10, and obtain

10x = (0.3333...) x 10

or 10x = 3.3333... ...... (ii)

Subtracting (i) from (ii), we have

10x – x = (3.3333...) – (0.3333...)

or 9x = 3 $\Rightarrow x = \frac{1}{3}$

Hence $0.\bar{3} = \frac{1}{3}$

(b) Let $x = 0.\overline{23} = 0.23\ 23\ 23...$

Since two digit block 23 is repeating itself indefinitely, so we multiply both sides by 100 .

Then $100x = 23.\overline{23}$

$100x = 23 + 0.\overline{23} = 23 + x$

$\Rightarrow 100x - x = 23$

$\Rightarrow 99x = 23$

$\Rightarrow x = \frac{23}{99}$

Thus $0.\overline{23} = \frac{23}{99}$ is a rational number.

2.1.4 Representation of Rational and Irrational Numbers on Number Line

In order to locate a number with terminating and non-terminating recurring decimal on the number line, the points associated with the rational numbers $\frac{m}{n}$ and $-\frac{m}{n}$ where *m, n* are positive integers, we subdivide each unit length into *n* equal parts. Then the *m*th point of division to the right of the origin represents $\frac{m}{n}$ and that to the left of the origin at the same distance represents $-\frac{m}{n}$

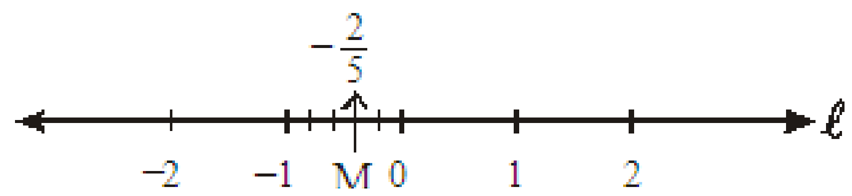
Example

Represent the following numbers on the number line.

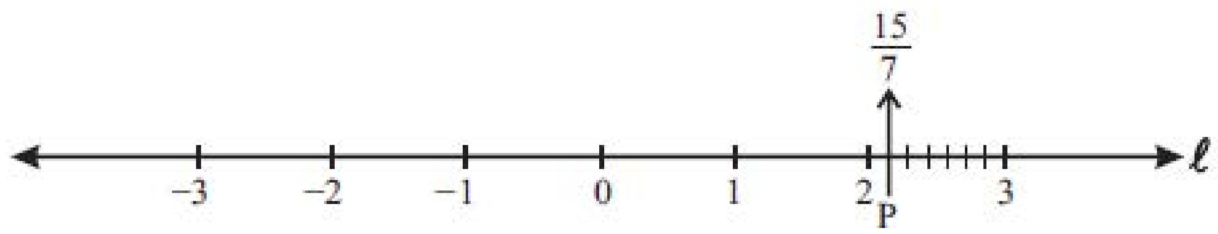
(*i*) $-\frac{2}{5}$ (*ii*) $\frac{15}{5}$ (*iii*) $-1\frac{7}{9}$

Solution

(i) For representing the rational number $-\frac{2}{5}$ on the number line ℓ, divide the unit length between 0 and –1 into five equal parts and take the end of the second part from 0 to its left side. The point M in the following figure represents the rational number $-\frac{2}{5}$

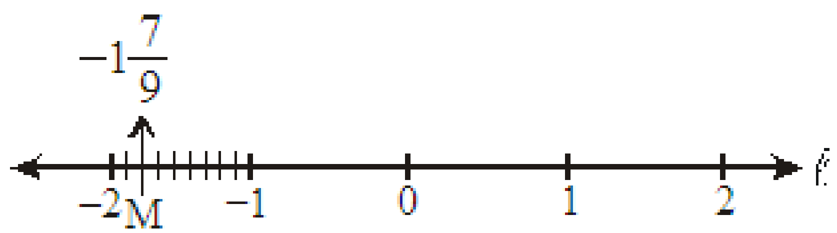


(ii) $\frac{15}{7} = 2 + \frac{1}{7}$: it lies between 2 and 3.



Divide the distance between 2 and 3 into seven equal parts. The point P represents the number $\frac{15}{7} = 2\frac{1}{7}$.

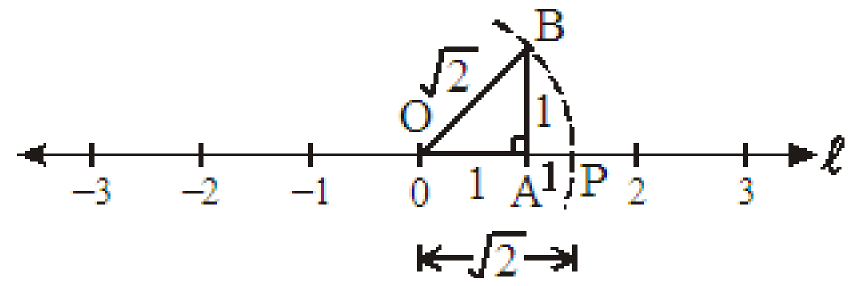
(iii) For representing the rational number, $-1\frac{7}{9}$. divide the unit length between –1 and –2 into nine equal parts. Take the end of the 7th part from –1. The point M in the following figure represents the rational number, $-1\frac{7}{9}$.



Irrational numbers such as $\sqrt{2}$, $\sqrt{5}$ etc. can be located on the line ℓ by geometric construction. For example, the point corresponding to $\sqrt{2}$ may be constructed by forming a right ΔOAB with sides (containing the right angle) each of length 1 as shown in the figure. By Pythagoras Theorem,

$$OB = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

By drawing an arc with centre at O and radius OB = $\sqrt{2}$, we get the point P representing $\sqrt{2}$ on the number line.



EXERCISE 2.1

1. Identify which of the following are rational and irrational numbers.

   (i) $\sqrt{3}$ (ii) $\frac{1}{6}$ (iii) π (iv) $\frac{15}{2}$ (v) 7.25 (vi) $\sqrt{29}$

2. Convert the following fractions into decimal fractions.

   (i) $\frac{17}{25}$ (ii) $\frac{19}{4}$ (iii) $\frac{57}{8}$ (iv) $\frac{205}{18}$ (v) $\frac{5}{8}$ (vi) $\frac{25}{38}$

3. Which of the following statements are true and which are false?

   (i) $\frac{2}{3}$ is an irrational number. (ii) π is an irrational number.

   (iii) $\frac{1}{9}$ is a terminating fraction. (iv) $\frac{3}{4}$ is a terminating fraction.

   (v) $\frac{4}{5}$ is a recurring fraction.

4. Represent the following numbers on the number line.

   (i) $\frac{2}{3}$ (ii) $-\frac{4}{5}$ (iii) $1\frac{3}{4}$ (iv) $-2\frac{5}{8}$ (v) $2\frac{3}{4}$ (vi) $\sqrt{5}$

5. Give a rational number between $\frac{3}{4}$ and $\frac{5}{9}$.
6. Express the following recurring decimals as the rational number $\frac{p}{q}$ where p, q are integers and $q \neq 0$ (i) $0.\overline{5}$ (ii) $0.\overline{13}$ (iii) $0.\overline{67}$

2.2 Properties of Real Numbers

If a, b are real numbers, their sum is written as $a + b$ and their product as ab or $a \times b$ or $a \,.\, b$ or $(a)\,(b)$.

(a) Properties of Real numbers with respect to Addition and Multiplication Properties of real numbers under addition are as follows:

(i) Closure Property

$$a + b \in R, \quad \forall \; a, b \in R$$

e. g., if -3 and $5 \in R$,
then $-3 + 5 = 2 \in R$

(ii) Commutative Property

$$a + b = b + a, \quad \forall \; a, b \in R$$

e.g., if $2, 3 \in R$,
then $2 + 3 = 3 + 2$
or $5 = 5$

(iii) Associative Property

$$(a + b) + c = a + (b + c), \quad \forall \; a, b, c \in R$$

e.g., if $5, 7, 3 \in R$,
then $(5 + 7) + 3 = 5 + (7 + 3)$
or $12 + 3 = 5 + 10$
or $15 = 15$

(iv) Additive Identity

There exists a unique real number 0, called additive identity, such that

$$a + 0 = a = 0 + a, \quad \forall \; a \in R$$

(v) Additive Inverse

For every $a \in R$, there exists a unique real number $-a$, called the additive inverse of a, such that

$$a + (-a) = 0 = (-a) + a$$

e.g., additive inverse of 3 is -3 since $3 + (-3) = 0 = (-3) + (3)$

Properties of real numbers under multiplication are as follows:

(i) Closure Property

$$ab \in R, \quad \forall \; a, b \in R$$

e.g., if $-3, 5 \in R$,
then $(-3)(5) \in R$
or $-15 \in R$

(ii) Commutative Property

$$ab = ba, \quad \forall \; a, b \in R$$

e.g., if $\frac{1}{3}, \frac{3}{2} \in R$
then $\left(\frac{1}{3}\right)\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)\left(\frac{1}{3}\right)$
or $\frac{1}{2} = \frac{1}{2}$

(i) Associative Property

$$(ab)c = a(bc), \quad \forall \; a, b, c \in R$$

e.g., if $2, 3, 5 \in R$,
then $(2 \times 3) \times 5 = 2 \times (3 \times 5)$
or $6 \times 5 = 2 \times 15$
or $30 = 30$

(ii) Multiplicative Identity

There exists a unique real number 1, called the multiplicative identity, such that

$a \cdot 1 = a = 1 \cdot a, \quad \forall \; a \in R$

(iii) Multiplicative Inverse

For every non-zero real number, there exists a unique real number a^{-1} or $\frac{1}{a}$, called multiplicative inverse of a, such that

$$aa^{-1} = 1 = a^{-1}a$$

or $$a \times \frac{1}{a} = 1 = \frac{1}{a} \times a$$

e.g., if $5 \in R$, then $\frac{1}{5} \in R$

such that

$$5 \times \frac{1}{5} = 1 = \frac{1}{5} \times 5$$

So, 5 and $\frac{1}{5}$ are multiplicative inverse of each other.

(vi) Multiplication is Distributive over Addition and Subtraction

For all $a, b, c \in R$

$a(b + c) = ab + ac$ (Left distributive law)

$(a + b)c = ac + bc$ (Right distributive law)

e.g., if $2, 3, 5 \in R$, then

$2(3 + 5) = 2 \times 3 + 2 \times 5$

or $2 \times 8 = 6 + 10$

or $16 = 16$

And for all $a, b, c \in R$

$a(b - c) = ab - ac$ (Left distributive law)

$(a - b)c = ac - bc$ (Right distributive law)

e.g., if $2, 5, 3 \in R$, then

$2(5 - 3) = 2 \times 5 - 2 \times 3$

or $2 \times 2 = 10 - 6$

or $4 = 4$

Note:

(i) The symbol $\forall$ means “for all”,

(ii) a is the multiplicative inverse of a^{-1}, i.e., $a = (a^{-1})^{-1}$

(b) Properties of Equality of Real Numbers

Properties of equality of real numbers are as follows:

(i) Reflexive Property

$a = a, \quad \forall \; a \in R$

(ii) Symmetric Property

If $a = b$, then $b = a$, $\forall \; a, b \in R$

(iii) Transitive Property

If $a = b$ and $b = c$, then $a = c$, $\forall \; a, b, c \in R$

(iv) Additive Property

If $a = b$, then $a + c = b + c$, $\forall \; a, b, c \in R$

(v) Multiplicative Property

If $a = b$, then $ac = bc$, $\forall \; a, b, c \in R$

(vi) Cancellation Property for Addition

If $a + c = b + c$, then $a = b$, $\forall \; a, b, c \in R$

(vii) Cancellation Property for Multiplication

If $ac = bc, c \neq 0$ then $a = b$, $\forall \; a, b, c \in R$

(c) Properties of Inequalities of Real Numbers

Properties of inequalities of real numbers are as follows:

(i) Trichotomy Property

$\forall\ a, b \in R$

$a < b$ or $a = b$ or $a > b$

(ii) Transitive Property

$\forall\ a, b, c \in R$

(a) $a < b$ and $b < c \Rightarrow a < c$ (b) $a > b$ and $b > c \Rightarrow a > c$

(iii) Additive Property

$\forall\ a, b, c \in R$

$a < b \Rightarrow a + c < b + c$
$a < b \Rightarrow c + a < c + b$

and (b)

$a > b \Rightarrow a + c > b + c$
$a > b \Rightarrow c + a > c + b$

(iv) Multiplicative Property

(a) $\forall\ a, b, c \in R$ and $c > 0$

(i) $a > b \Rightarrow ac > bc$, $a > b \Rightarrow ca > cb$ (ii) $a < b \Rightarrow ac < bc$, $a < b \Rightarrow ca < cb$

(b) $\forall\ a, b, c \in R$ and $c < 0$

(i) $a > b \Rightarrow ac < bc$, $a > b \Rightarrow ca < cb$ (ii) $a < b \Rightarrow ac > bc$, $a < b \Rightarrow ca > cb$

(v) Multiplicative Inverse Property

$\forall\ a, b \in R$ and $a \neq 0, b \neq 0$

(a) $a < b \Leftrightarrow \frac{1}{a} > \frac{1}{b}$ (b) $a > b \Leftrightarrow \frac{1}{a} < \frac{1}{b}$

EXERCISE 2.2

1. Identify the property used in the following

(i) $a + b = b + a$ (ii) $(ab)c = a(bc)$
(iii) $7 \times 1 = 7$ (iv) $x > y$ or $x = y$ or $x < y$
(v) $ab = ba$ (vi) $a + c = b + c \Rightarrow a = b$
(vii) $5 + (-5) = 0$ (viii) $7 \times \frac{1}{7} = 1$
(ix) $a > b \Rightarrow ac > bc \quad (c > 0)$

2. Fill in the following blanks by stating the properties of real numbers used.

$3x + 3(y - x)$
$= 3x + 3y - 3x,$ ....
$= 3x - 3x + 3y,$ ....
$= 0 + 3y,$ ....
$= 3y$ ....

3. Give the name of property used in the following.

(i) $\sqrt{24} + 0 = \sqrt{24}$
(ii) $-\frac{2}{3}\left(5 + \frac{7}{2}\right) = \left(-\frac{2}{3}\right)(5) + \left(-\frac{2}{3}\right)\left(\frac{7}{2}\right)$
(iii) $\pi + (-\pi) = 0$
(iv) $\sqrt{3} \cdot \sqrt{3}$ is a real number
(v) $\left(-\frac{5}{8}\right)\left(-\frac{8}{5}\right) = 1$

2.3 Radicals and Radicands

2.3.1 Concept of Radicals and Radicands

If n is a positive integer greater than 1 and a is a real number, then any real number x such that $x^n = a$ is called the nth root of a, and in symbols is written as

$$x = \sqrt[n]{a}, \quad or \quad x = (a)^{1/n},$$

In the radical $\sqrt[n]{a}$, the symbol $\sqrt{\ }$ is called the radical sign, n is called the index of the radical and the real number a under the radical sign is called the radicand or base.

Note:

$\sqrt[2]{a}$ is usually written as $\sqrt{a}$.

2.3.2 Difference between Radical form and Exponential form

In radical form, radical sign is used

e.g., $x = \sqrt[n]{a}$ is a radical form.

$\sqrt[3]{x}$, $\sqrt[5]{x^2}$ are examples of radical form.

In exponential form, exponential is used in place of radicals,
e.g., $x = (a)^{1/n}$ is exponential form.
$x^{3/2}$, $z^{2/7}$ are examples of exponential form.

Properties of Radicals

Let a, b d R and m, n be positive integers. Then,

(i) $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ (ii) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

(iii) $\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a}$ (iv) $\sqrt[n]{a^m} = (\sqrt[n]{a})^m$ (v) $\sqrt[n]{a^n} = a$

2.3.3 Transformation of an Expression given in Radical form to Exponential form and vice versa

The method of transforming expression in radical form to exponential form and vice versa is explained in the following examples.

Example 1

Write each radical expression in exponential notation and each exponential expression in radical notation. Do not simplify.

(i) $\sqrt[5]{-8}$ (ii) $\sqrt[3]{x^5}$ (iii) $y^{3/4}$ (iv) $x^{-3/2}$

Solution

(i) $\sqrt[5]{-8} = (-8)^{1/5}$ (ii) $\sqrt[3]{x^5} = x^{5/3}$

(iii) $y^{3/4} = \sqrt[4]{y^3}$ or $(\sqrt[4]{y})^3$ (iv) $x^{-3/2} = \sqrt{x^{-3}}$ or $(\sqrt{x})^{-3}$

Example 2

Simplify $\sqrt[3]{16x^4y^5}$

Solution

$$\begin{aligned}
\sqrt[3]{16x^4y^5} &= \sqrt[3]{(2)(8)(x)(x^3)(y^2)(y^3)}, && \text{...... (factorizing)}\\
&= \sqrt[3]{2xy^2\,(2^3)(x^3)(y^3)}, && \text{...... (arranging perfect cubes)}\\
&= \sqrt[3]{2xy^2}\,\sqrt[3]{(2^3)(x^3)(y^3)}, && \text{...... property (i)}\\
&= \sqrt[3]{2xy^2}\,\sqrt[3]{2^3}\,\sqrt[3]{x^3}\,\sqrt[3]{y^3}, && \text{...... property (i)}\\
&= 2xy\sqrt[3]{2xy^2}, && \text{...... property (v)}
\end{aligned}$$

EXERCISE 2.3

1. Write each radical expression in exponential notation and each exponential expression in radical notation. Do not simplify.

   (i) $\sqrt[3]{-64}$ (ii) $2^{3/5}$ (iii) $-7^{1/3}$ (iv) $y^{-2/3}$

2. Tell whether the following statements are true or false?

   (i) $5^{1/5} = \sqrt{5}$ (ii) $2^{2/3} = \sqrt[3]{4}$ (iii) $\sqrt{49} = \sqrt{7}$ (iv) $\sqrt[3]{x^{27}} = x^3$

3. Simplify the following radical expressions.

(i) $\sqrt[3]{-125}$ (ii) $\sqrt[4]{32}$ (iii) $\sqrt[5]{\frac{3}{32}}$ (iv) $\sqrt[3]{-\frac{8}{27}}$

2.4 Laws of Exponents / Indices

2.4.1 Base and Exponent

In the exponential notation a^n (read as a to the nth power) we call 'a' as the base and 'n' as the exponent or the power to which the base is raised.

From this definition, recall that, we have the following laws of exponents.

If $a, b \in$ R and m, n are positive integers, then

I	$a^m \cdot a^n = a^{m+n}$	II	$(a^m)^n = a^{mn}$
III	$(ab)^n = a^n b^n$	IV	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$
V	$= a^m/a^n$, a^{m-n} , $a \neq 0$	VI	$a^0 = 1$, where $a \neq 0$
VII	$a^{-n} = \frac{1}{a^n}$, where $a \neq 0$		

2.4.2 Applications of Laws of Exponents

The method of applying the laws of indices to simplify algebraic expressions is explained in the following examples.

Example 1

Use rules of exponents to simplify each expression and write the answer in terms of positive exponents.

(i) $\frac{x^{-2}x^{-3}y^7}{x^{-3}y^4}$ (ii) $\left(\frac{4a^3b^0}{9a^{-5}}\right)^{-2}$

Solution

(i) $\frac{x^{-2}x^{-3}y^7}{x^{-3}y^4} = \frac{x^{-5}y^7}{x^{-3}y^4}$ $(a^m a^n = a^{m+n})$

$= \frac{y^{7-4}}{x^{-3+5}} = \frac{y^3}{x^2}$ $\left(\frac{a^m}{a^n} = a^{m-n}\right)$

(ii) $\left(\frac{4a^3b^0}{9a^{-5}}\right)^{-2} = \left(\frac{4a^{3+5}\times 1}{9}\right)^{-2}$ $\left(\frac{a^m}{a^n} = a^{m-n}, b^0 = 1\right)$

$= \left(\frac{4a^8}{9}\right)^{-2} = \left(\frac{9}{4a^8}\right)^{+2}$ $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

$= \frac{81}{16a^{16}}$ $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Example 2

Simplify the following by using laws of indices:

(i) $\left(\frac{8}{125}\right)^{-4/3}$ (ii) $\frac{4(3)^n}{3^{n+1}-3^n}$

Solution

Using Laws of Indices,

(i) $\left(\frac{8}{125}\right)^{-4/3} = \left(\frac{125}{8}\right)^{4/3} = \frac{(125)^{4/3}}{(8)^{4/3}} = \frac{(5^3)^{4/3}}{(2^3)^{4/3}} = \frac{5^4}{2^4} = \frac{625}{16}$

(ii) $\frac{4(3)^n}{3^{n+1}-3^n} = \frac{4(3)^n}{3^n[3-1]} = \frac{4(3)^n}{2(3^n)} = \frac{4}{2} = 2$

EXERCISE 2.4

1. Use laws of exponents to simplify:

(i) $\frac{(243)^{-2/3}(32)^{-1/5}}{\sqrt{(196)^{-1}}}$ (ii) $(2x^5y^{-4})(-8x^{-3}y^2)$

(iii) $\left(\dfrac{x^{-2}y^{-1}z^{-4}}{x^{4}y^{-3}z^{0}}\right)^{-4/3}$ (iv) $\dfrac{(81)^{n}.3^{5}-(3)^{4n-1}(243)}{(9^{2n})(3^{3})}$

2. Show that

$$\left(\frac{x^a}{x^b}\right)^{a+b}\times\left(\frac{x^b}{x^c}\right)^{b+c}\times\left(\frac{x^c}{x^a}\right)^{c+a}=1$$

3. Simplify

(i) $\dfrac{2^{1/3}\times(27)^{1/3}\times(60)^{1/2}}{(180)^{1/2}\times(4)^{-1/3}\times(9)^{1/4}}$ (ii) $\sqrt{\dfrac{(216)^{2/3}\times(25)^{1/2}}{(.04)^{-1/2}}}$

(iii) $5^{2^3}\div(5^2)^3$ (iv) $(x^3)^2\div x^{3^2},\ x\neq 0$

2.5 Complex Numbers

We recall that the square of a real number is non-negative. So the solution of the equation $x^2 + 1 = 0$ or $x^2 = -1$ does not exist in R. To overcome this inadequacy of real numbers, we need a number whose square is –1. Thus the mathematicians were tempted to introduce a larger set of numbers called the set of complex numbers which contains R and every number whose square is negative. They invented a new number –1, called the imaginary unit, and denoted it by the letter i (iota) having the property that $i^2=-1$. Obviously i is not a real number. It is a new mathematical entity that enables us to enlarge the number system to contain solution of every algebraic equation of the form $x^2 = -a$, where $a > 0$. By taking new number $i=\sqrt{-1}$, the solution set of $x^2 + 1 = 0$ is

$$\{\sqrt{-1},-\sqrt{-1}\}\quad\text{or}\quad\{i,-i\}$$

Note:

The Swiss mathematician Leonard Euler (1707 – 1783) was the first to use the symbol i for the number $\sqrt{-1}$
Numbers like $\sqrt{-1},\sqrt{-5}$ etc. are called pure imaginary numbers.

Integral Powers of i

By using $i=\sqrt{-1}$,we can easily calculate the integral powers of i.

e.g., $i^2=-1,\ i^3=i^2\times i=-i,\ i^4=i^2\times i^2=(-1)(-1)=1,\ i^8=(i^2)^4=(-1)^4=1,$ $i^{10}=(i^2)^5=(-1)^5=-1$, etc.

A pure imaginary number is the square root of a negative real number.

2.5.1 Definition of a Complex Number

A number of the form z = a + bi where a and b are real numbers and $i=\sqrt{-1}$,is called a complex number and is represented by z i.e., z = a + ib

2.5.2 Set of Complex Numbers

The set of all complex numbers is denoted by C, and

$$C=\{z\,|\,z=a+bi,\ \text{where } a,b\in R \text{ and } i=\sqrt{-1}\}$$

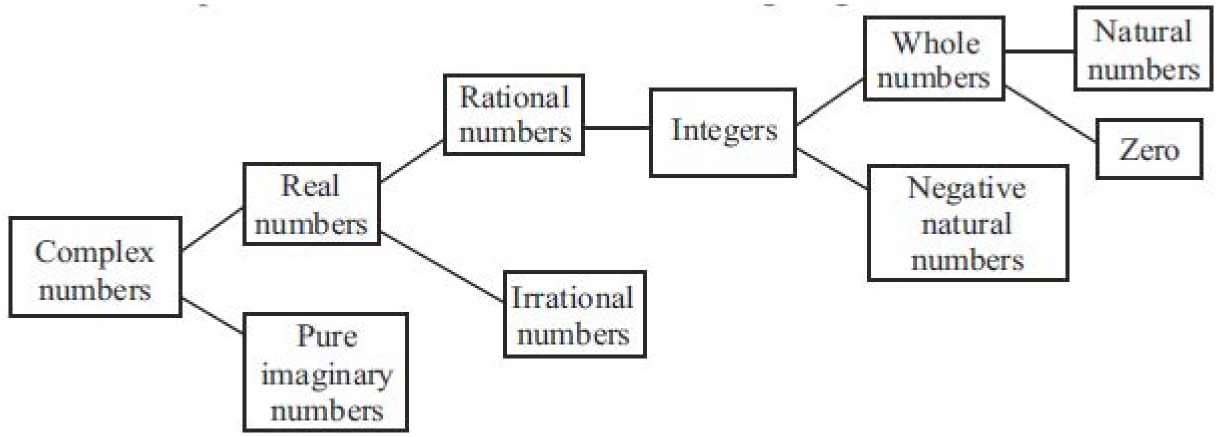
The numbers *a* and *b*, called the real and imaginary parts of *z*, are denoted as *a* = R(*z*) and *b* = Im(*z*).

Observe that:

(i) Every a ∈ R may be identified with complex numbers of the form a + Oi taking b = 0. Therefore, every real number is also a complex number. Thus R ⊂ C. Note that every complex number is not a real number.

(ii) If a = 0, then a + bi reduces to a purely imaginary number bi. The set of purely imaginary numbers is also contained in C.

(iii) If a= b = 0, then z = 0 + i0 is called the complex number 0.
The set of complex numbers is shown in the following diagram



2.5.3 Conjugate of a Complex Number

If we change i to –i in z = a + bi, we obtain another complex number a – bi called the complex conjugate of z and is denoted by z (read z bar).

Thus, if $z = -1 - i$, then $\overline{z} = -1 + i$.

The numbers a + bi and a — bi are called conjugates of each other.

Note that:

(i) $\overline{\overline{z}} = z$
(ii) The conjugate of a real number $z = a + oi$ coincides with the number itself, since $\overline{z} = \overline{a + 0i} = a - 0i$.
(iii) conjugate of a real number is the same real number.

2.5.4 Equality of Complex Numbers and its Properties

For all a, b, c, d $\in$ R,
a + bi = c + di if and only if a= c and b = d.
e.g., $2x + y^2 i = 4 + 9i$ if and only if

$2x = 4$ and $y^2 = 9$, i.e., $x = 2$ and $y = \pm 3$

Properties of real numbers R are also valid for the set of complex numbers.

(i) $z_1 = z_1$ (Reflexive law)
(ii) If $z_1 = z_2$, then $z_2 = z_1$ (Symmetric law)
(iii) If $z_1 = z_2$ and $z_2 = z_3$, then $z_1 = z_3$ (Transitive law)

EXERCISE 2.5

1. Evaluate
   (i) i^7 (ii) i^{50} (iii) i^{12}
   (iv) $(-)^8$ (v) $(-i)^5$ (vi) i^{27}
2. Write the conjugate of the following numbers.
   (i) 2 + 3i (ii) 3 – 5i (iii) –i
   (iv) –3 + 4i (v) –4 – i (vi) i – 3
3. Write the real and imaginary part of the following numbers.
   (i) 1 + i (ii) –1 + 2i (iii) –3i + 2
   (iv) –2 – 2i (v) –3i (vi) 2 + 0i
4. Find the value of x and y if $x + iy + 1 = 4 - 3i$.

2.6 Basic Operations on Complex Numbers

(i) **Addition**

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers and a, b, c, d $\in$ R.

The sum of two complex numbers is given by

$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i$

i.e., the sum of two complex numbers is the sum of the corresponding real and the imaginary parts.

e.g., (3 –8i) + (5 + 2i) = (3 + 5) + (–8 + 2)i = 8 – 6i

(i) **Multiplication**

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers.
The products are found as

(i) If $k \in R$, $kz_1 = k(a + bi) = ka + kbi$.
(Multiplication of a complex number with a scalar)

(ii) $z_1z_2 = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$
(Multiplication of two complex numbers)

The multiplication of any two complex numbers $(a + bi)$ and $(c + di)$ is explained as

$$z_1z_2 = (a + bi)(c + di) = a(c + di) + bi(c + di)$$
$$= ac + adi + bci + bdi^2$$
$$= ac + adi + bci + bd(-1) \quad \text{(since } i^2 = -1\text{)}$$
$$= (ac - bd) + (ad + bc)i \quad \text{(combining like terms)}$$

e.g., $(2 - 3i)(4 + 5i) = 8 + 10i - 12i - 15i^2 = 23 - 2i$. (since $i^2 = -1$)

(iii) Subtraction

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers.
The difference between two complex numbers is given by

$$z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i$$

e.g., $(-2 + 3i) - (2 + i) = (-2 - 2) + (3 - 1)i = -4 + 2i$

i.e., the difference of two complex numbers is the difference of the corresponding real and imaginary parts.

(iv) Division

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers such that $z_2 \neq 0$.

The division of $a + bi$ by $c + di$ is given by

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} = \frac{a+bi}{c+di} \times \frac{c-di}{c-di},$$ (Multiplying the numerator and denominator by $c - di$, the complex conjugateof $c + di$).

$$= \frac{ac + bci - adi - bdi^2}{c^2 - (di)^2}$$
$$= \frac{ac + bci - adi + bd}{c^2 + d^2}, \text{ since } i^2 = -1$$
$$= \frac{(ac+bd) + (bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \left(\frac{bc-ad}{c^2+d^2}\right)i$$

Operations are explained with the help of following examples.

Example 1

Separate the real and imaginary parts of $(-1 + \sqrt{-2})^2$

Solution

Let $z = -1 + \sqrt{-2}$, then

$$z^2 = (-1 + \sqrt{-2})^2 = (-1 + i\sqrt{2})^2, \text{ changing to } i\text{-form}$$
$$= (-1 + i\sqrt{2})(-1 + i\sqrt{2}) = (-1)(-1 + i\sqrt{2}) + i\sqrt{2}(-1 + i\sqrt{2})$$
$$= 1 - i\sqrt{2} - i\sqrt{2} + 2i^2 = -1 - 2\sqrt{2}\,i$$

Hence $\text{Re}(z^2) = -1$ and $\text{Im}(z^2) = -2\sqrt{2}$

Example 2

Express $\frac{1}{1+2i}$ in the standard form $a + bi$.

Solution

We have $\frac{1}{1+2i} = \frac{1}{1+2i} \times \frac{1-2i}{1-2i}$

(multiplying the numerator and denominator by $\overline{1+2i}$)

$$= \frac{1-2i}{1-(2i)^2} = \frac{1-2i}{1-4i^2}, \quad \text{(simplifying)}$$
$$= \frac{1-2i}{5}, \quad \text{(since } i^2 = -1\text{)}$$
$$= \frac{1}{5} - \frac{2}{5}i, \quad \text{which is of the form } a + bi$$

Example 3

Express $\frac{4+5i}{4-5i}$ in the standard form $a + bi$.

Solution

$\frac{4+5i}{4-5i} = (4+5i).\frac{1}{4-5i} \times \frac{4+5i}{4+5i}$ (multiplying and dividing by the conjugate of $(4-5)i)$

$= \frac{(4+5i)^2}{(4)^2-(5i)^2} = \frac{16+40i+25i^2}{16-25i^2}$ (simplifying)

$= \frac{16+40i+25}{16-25},$ (since $i^2=-1$)

$= \frac{-9+40i}{41} = -\frac{9}{41} + \frac{40}{41}i$

Example 4

Solve (3 – 4i) (x + yi) = 1 + 0 . i for real numbers x and y, where $i=\sqrt{-1}$.

Solution

We have (3 – 4i) (x + yi) = 1 + 0 . i

or $3x + 3iy - 4ix - 4i^2y = 1 + 0 . i$

or $3x + 4y + (3y - 4x)i = 1 + 0 . i$

Equating the real and imaginary parts, we obtain

$3x + 4y = 1$ and $3y - 4x = 0$

olving these two equations simultaneously, we have $x=\frac{3}{25}$ and $y=\frac{4}{25}$

EXERCISE 2.6

1. Identify the following statements as true or false.
   (i) $\sqrt{-3}\sqrt{-3}=3$ (ii) $i^{73} = -i$ (iii) $i^{10} = -1$
   (iv) Complex conjugate of $(-6i + i^2)$ is $(-1 + 6i)$
   (v) Difference of a complex number z = a + bi and its conjugate is a real number.
   (vi) If (a – 1) – (b +3)i = 5 + 8i, then a = 6 and b = –11.
   (vii) Product of a complex number and its conjugate is always a non-negative real number.
2. Express each complex number in the standard form a + bi, where a and b are real numbers.

(i) (2 + 3i) + (7 - 2i) (ii) 2(5 + 4i) -3 (7 + 4i)
(iii) -(-3 + 5i) - (4 + 9i) (iv) $2i^2 + 6i^3 + 3i^{16} - 6i^{19} + 4i^{25}$

3. Simplify and write your answer in the form a+ bi.
   (i) $(-7+3i)(-3+2i)$ (ii) $(2-\sqrt{-4})(3-\sqrt{-4})$
   (iii) $(\sqrt{5}-3i)^2$ (iv) $(2-3i)(\overline{3-2i})$

4. Simplify and write your answer in the form a + bi.
   (i) $\frac{-2}{1+i}$ (ii) $\frac{2+3i}{4-i}$ (iii) $\frac{9-7i}{3+i}$
   (iv) $\frac{2-6i}{3+i} - \frac{4+i}{3+i}$ (v) $\left(\frac{1+i}{1-i}\right)^2$ (vi) $\frac{1}{(2+3i)(1-i)}$

5. Calculate (a) $\bar{z}$ (b) $z+\bar{z}$ (c) $z-\bar{z}$ (d) $z\bar{z}$, for each of the following
   (i) z = -i (ii) z = 2 + i
   (iii) $z=\frac{1+i}{1-i}$ (iv) $z=\frac{4-3i}{2+4i}$

6. If z = 2 + 3i and w = 5 - 4i, show that
   (i) $\overline{z+w} = \bar{z} + \bar{w}$ (ii) $\overline{z-w} = \bar{z} - \bar{w}$
   (iii) $\overline{zw} = \bar{z}\,\bar{w}$ (iv) $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$, where $w \neq 0$.
   (v) $\frac{1}{2}(z+\bar{z})$ is the real part of z. (vi) $\frac{1}{2i}(z-\bar{z})$ is the imaginary part of z.

7. Solve the following equations for real x and y.
   (i) (2 – 3i) (x + yi) = 4 + i
   (ii) (3 – 2i) (x + yi) = 2(x – 2yi) + 2i – 1
   (iii) $(3 + 4i)^2 - 2(x - yi) = x + yi$

REVIEW EXERCISE 2

1. **Multiple Choice Questions. Choose the correct answer.**

2. **True or false? Identify.**
   (i) Division is not an associative operation. ....
   (ii) Every whole number is a natural number. ....
   (iii) Multiplicative inverse of 0.02 is 50. ....
   (iv) π is a rational number. ....
   (v) Every integer is a rational number. ....
   (vi) Subtraction is a commutative operation. ....
   (vii) Every real number is a rational number. ....
   (viii) Decimal representation of a rational number is either terminating or recurring. ....

3. Simplify the following:

   (i) $\sqrt[4]{81y^{-12}x^{-8}}$ (ii) $\sqrt{25x^{10n}y^{8m}}$

   (iii) $\left(\dfrac{x^3y^4z^5}{x^{-2}y^{-1}z^{-5}}\right)^{1/5}$ (iv) $\left(\dfrac{32x^{-6}y^{-4}z}{625x^4y\,z^{-4}}\right)^{2/5}$

4. Simplify $\sqrt{\dfrac{(216)^{2/3}\times(25)^{1/2}}{(0.04)^{-3/2}}}$

5. Simplify

$$\left(\frac{a^p}{a^q}\right)^{p+q}\cdot\left(\frac{a^q}{a^r}\right)^{q+r}\div 5(a^p.\,a^r)^{p-r}, a\neq 0$$

6. Simplify $\left(\dfrac{a^{2l}}{a^{l+m}}\right)\left(\dfrac{a^{2m}}{a^{m+n}}\right)\left(\dfrac{a^{2n}}{a^{n+l}}\right)$

7. Simplify $\sqrt[3]{\dfrac{a^l}{a^m}}\times\sqrt[3]{\dfrac{a^m}{a^n}}\times\sqrt[3]{\dfrac{a^n}{a^l}}$

SUMMARY

* Set of real numbers is expressed as R = Q U Q′ where

$Q=\left\{\dfrac{p}{q}\,|\,p,q\in Z\wedge q\neq 0\right\}$, $Q'=\{x\,|\,x$ is not rational$\}$.

* Properties of real numbers w.r.t. addition and multiplication:

Closure: $a+b\in R$, $ab\in R$, $\forall\ a,b\in R$

Associative:

$(a+b)+c=a+(b+c)$, $(ab)c=a(bc)$, $\forall\ a,b,c\in R$

Commutative:

$a+b=b+a$, $ab=ba$, $\forall\ a,b\in R$

Additive Identity:

$a+0=a=0+a$, $\forall\ a\in R$

Multiplicative Identity:

$a.1=a=1.a$, $\forall\ a\in R$

Additive Inverse:

$a+(-a)=0=(-a)+a$, $\forall\ a\in R$

Multiplicative Inverse:

$a\cdot\dfrac{1}{a}=1=\dfrac{1}{a}\cdot a,\ a\neq 0$

Multiplication is distributive over addition and subtraction:

$a(b+c)=ab+ac$, $\forall\ a,b,c\in R$
$(b+c)a=ba+ca$ $\forall\ a,b,c\in R$
$a(b-c)=ab-ac$ $\forall\ a,b,c\in R$
$(a-b)c=ac-bc$ $\forall\ a,b,c\in R$

* Properties of equality in R

Reflexive: $a = a, \ \forall\ a \in R$

Symmetric: $a = b \Rightarrow b = a, \forall\ a, b \in R$

Transitive: $a = b, b = c \Rightarrow a = c, \quad \forall\ a, b, c \in R$

Additive property: If a = b, then $a + c = b + c, \ \forall\ a, b, c \in R$

Multiplicative property: If $a = b$, then $ac = bc, \forall\ a, b, c \in R$

Cancellation property: If $ac = be, c \neq 0$, then $a = b, \forall\ a, b, c \in R$

* In the radical $\sqrt[n]{x}$, $\sqrt{\ }$ is radical sign, x is radicand or base and n is index of radical.

* Indices and laws of indices:

$\forall a, b, c \in R$ and $m, n \in z$,

$(a^m)^n = a^{mn}$, $(ab)^n = a^n b^n$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$$

$a^m a^n = a^{m+n}$

$$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$$

$$a^{-n} = \frac{1}{a^n}, a \neq 0$$

$$a^0 = 1$$

* Complex number z = a + bi is defined using imaginary unit $i = \sqrt{-1}$. where $a, b \in R$ and a = Re (z), b = Im (z)

* Conjugate of $z = a + bi$ is defined as $z = a - bi$