CHAPTER

4

ALGEBRAIC EXPRESSIONS AND ALGEBRAIB FORMULAS

Animation 4.1: Algebraic Expressions and Algebraic Formulas
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Students Learning Outcomes

After studying this unit, the students will be able to:

- Know that a rational expression behaves like a rational number.
- Define a rational expression as the quotient $\frac{p(x)}{q(x)}$ of two polynomials $p(x)$ and $q(x)$ where $q(x)$ is not the zero polynomial.
- Examine whether a given algebraic expression is a
  - polynomial or not,
  - rational expression or not.
- Define $\frac{p(x)}{q(x)}$ as a rational expression in its lowest terms if $p(x)$ and $q(x)$ are polynomials with integral coefficients and having no common factor.
- Examine whether a given rational algebraic expression is in lowest from or not.
- Reduce a given rational expression to its lowest terms.
- Find the sum, difference and product of rational expressions.
- Divide a rational expression with another and express the result in it lowest terms.
- Find value of algebraic expression for some particular real number.
  Know the formulas
  $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$,
  $(a + b)^2 - (a - b)^2 = 4ab$
- Find the value of $a^2 + b^2$ and of ab when the values of $a + b$ and $a - b$ are known.
- Know the formulas
  $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.
- find the value of $a^2 + b^2 + c^2$ when the values of $a + b + c$ and $ab + bc + ca$ are given.
- find the value of $a + b + c$ when the values of $a^2 + b^2 + c^2$ and $ab + bc + ca$ are given.
- find the value of $ab + bc + ca$ when the values of $a^2 + b^2 + c^2$ and $a + b + c$ are given.

- know the formulas
  $(a + b)^3 = a^3 + 3ab(a + b) + b^3$,
  $(a - b)^3 = a^3 - 3ab(a - b) - b^3$,
- find the value of $a^3 \pm b^3$ when the values of $a \pm b$ and ab are given
- find the value of $x^3 \pm$ when the value of $x \pm$ is given.
- know the formulas
  $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$.
  - find the product of $x+\frac{1}{x}$ and $x^2+\frac{1}{x^2}-1$.
  - find the product of $x-\frac{1}{x}$ and $x^2+\frac{1}{x^2}+1$.
  - find the continued product of
    $(x + y)(x - y)(x^2 + xy + y^2)(x^2 - xy + y^2)$.
- recognize the surds and their application.
- explain the surds of second order. Use basic operations on surds of second order to rationalize the denominators and evaluate it.
- explain rationalization (with precise meaning) of real numbers of the types $\frac{1}{a+b\sqrt{x}}$, $\frac{1}{\sqrt{x}+\sqrt{y}}$ and their combinations where x and y are natural numbers and a and b integers.

4.1 Algebraic Expressions

Algebra is a generalization of arithmetic. Recall that when operations of addition and subtraction are applied to algebraic terms, we obtain an algebraic expression. For instance, $5x^2 - 3x + \frac{2}{\sqrt{x}}$ and $3xy + \frac{3}{x}$ $(x \neq 0)$ are algebraic expressions.

Polynomials

A polynomial in the variable x is an algebraic expression of the form $P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0$, $a_n \neq 0$ ...... (i) where n, the highest power of x, is a non-negative integer called the degree of the polynomial and each coefficient a_n, is a real number. The coefficient a_n of the highest power of x is called the *leading coefficient* of the polynomial. $2x^4y^3 + x^2y^2 + 8x$ is a polynomial in two variables x and y and has degree 7.

From the study of similar properties of integers and polynomials w.r.t. addition and multiplication, we may say that polynomials behave like integers.

Self Testing

Justify the following as polynomial or not a polynomial.

(i) $3x^2 + 8x + 5$ (ii) $x^3 + \sqrt{2}x^2 + 5x - 3$

(iii) $x^2 + \sqrt{x} - 4$ (iv) $\frac{3x^2 + 2x + 8}{3x + 4}$

4.1.1 Rational Expressions Behave like Rational Numbers

Let a and b be two integers, then $\frac{a}{b}$ is not necessarily an integer.

Therefore, number system is extended and $\frac{a}{b}$ is defined as a rational number where $a, b \in Z$ and $b \neq 0$.

Similarly, if $p(x)$ and $q(x)$ are two polynomials, the $\frac{p(x)}{q(x)}$ is not necessarily a polynomial, where $q(x) \neq 0$. Therefore, similar to the idea of rational numbers, concept of rational expressions is developed.

4.1.2 Rational Expression

The quotient $\frac{p(x)}{q(x)}$ of two polynomials, $p(x)$ and $q(x)$, where $q(x)$ is a non-zero polynomial, is called a rational expression.

For example, $\frac{2x+1}{3x+8}$, $3x + 8 \neq 0$ is a rational expression.

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In the rational expression $\frac{p(x)}{q(x)}$, $p(x)$ is called the numerator and $q(x)$ is known as the denominator of the rational expression $\frac{p(x)}{q(x)}$. The rational expression $\frac{p(x)}{q(x)}$ need not be a polynomial.

Note:

Every polynomial $p(x)$ can be regarded as a rational expression, since we can write $p(x)$ as $\frac{p(x)}{1}$.Thus, every polynomial is a rational expression, but every rational expression need not be a polynomial.

Self Testing

Identify the following as a rational expression or not a rational expression.

(i) $\frac{2x+6}{3x-4}$ (ii) $\frac{3x+8}{x^2+x+2}$ (iii) $\frac{x^2+4x+5}{x^2+3\sqrt{x}+4}$ (iv) $\frac{\sqrt{x}}{3x^2+1}$

4.1.3 Properties of Rational Expressions

The method for operations with rational expressions is similar to operations with rational numbers.

Let $p(x), q(x), r(x), s(x)$ be any polynomials such that all values of the variable that make a rational expression undefined are excluded from the domain. Then following properties of rational expressions hold under the supposition that they all are defined (i.e., denominator (s) $\neq 0$).

(i) $\frac{p(x)}{q(x)} = \frac{r(x)}{s(x)}$ if and only if $p(x)\,s(x) = q(x)\,r(x)$ (Equality)

(ii) $\frac{p(x)k}{q(x)k} = \frac{p(x)}{q(x)}$ (Cancellation)

(iii) $\frac{p(x)}{q(x)} + \frac{r(x)}{r(x)} = \frac{p(x)\,s(x) + q(x)\,r(x)}{q(x)\,s(x)}$ (Addition)

(iv) $\dfrac{p(x)}{q(x)} - \dfrac{r(x)}{s(x)} = \dfrac{p(x)\,s(x) - q(x)\,r(x)}{q(x)\,s(x)}$ (Subtraction)

(v) $\dfrac{p(x)}{q(x)} \cdot \dfrac{r(x)}{s(x)} = \dfrac{p(x)\,r(x)}{q(x)\,s(x)}$ (Multiplication)

(vi) $\dfrac{p(x)}{q(x)} \div \dfrac{r(x)}{s(x)} = \dfrac{p(x)}{q(x)} \cdot \dfrac{s(x)}{r(x)} = \dfrac{p(x)\,s(x)}{q(x)\,r(x)}$ (Division)

(vii) **Additive inverse** of $\dfrac{p(x)}{q(x)}$ is $-\dfrac{p(x)}{q(x)}$

(viii) **Multiplicative inverse** or reciprocal of $\dfrac{p(x)}{q(x)}$ is $\dfrac{q(x)}{p(x)}, p(x) \neq 0,$ $q(x) \neq 0.$

4.1.4 Rational Expression in its Lowest form

The rational expression $\dfrac{p(x)}{q(x)}$ is said to be in its lowest form, if *p(x)* and *q(x)* are polynomials with integral coefficients and have no common factor.

For example, $\dfrac{x+1}{x^2+1}$ is in its lowest form.

4.1.5 To examine whether a rational expression is in lowest form or not

To examine the rational expression $\dfrac{p(x)}{q(x)}$, find H.C.F of $p(x)$ and $q(x)$. If H.C.F is 1, then the rational expression is in lowest form.

For example, $\dfrac{x-1}{x^2+1}$ is in its lowest form as H.C.F. of $x - 1$ and $x^2 + 1$ is 1.

4.1.6 Working Rule to reduce a rational expression to its lowest terms

Let the given rational expression be $\dfrac{p(x)}{q(x)}$

Step I Factorize each of the two polynomials $p(x)$ and $q(x)$.
Step II Find H.C. F. of $p(x)$ and $q(x)$.
Step III Divide the numerator $p(x)$ and the denominator $q(x)$ by the H.C. F. of $p(x)$ and $q(x)$. The rational expression so obtained, is in its lowest terms.

In other words, an algebraic fraction can be reduced to its lowest form by first factorizing both the polynomials in the numerator and the denominator and then cancelling the common factors between them.

Example

Reduce the following algebraic fractions to their lowest form.

(i) $\dfrac{lx + mx - ly - my}{3x^2 - 3y^2}$ (ii) $\dfrac{3x^2 + 18x + 27}{5x^2 - 45}$

Solution

(i) $\dfrac{lx + mx - ly - my}{3x^2 - 3y^2} = \dfrac{x(l+m) - y(l+m)}{3(x^2 - y^2)}$

$= \dfrac{(l+m)\,(x-y)}{3(x+y)\,(x-y)}$ ...... (factorizing)

$= \dfrac{l+m}{3(x+y)}$ ...... (cancelling common factors)

which is in the lowest form

(ii) $\dfrac{3x^2 + 18x + 27}{5x^2 - 45} = \dfrac{3(x^2 + 6x + 9)}{5(x^2 - 9)}$ ...... *(monomial factors)*

$= \dfrac{3(x+3)\,(x+3)}{5(x+3)\,(x-3)}$ ...... *(factorizing)*

$= \dfrac{3(x+3)}{5(x-3)}$ ...... *(cancelling common factors)*

which is in the lowest form.

4.1.7 Sum, Difference and Product of Rational Expressions

For finding sum and difference of algebraic expressions containing rational expressions, we take the L.C.M. of the denominators and simplify as explained in the following examples by using properties stated in 4.1.3.

Example 1

Simplify (i) $\frac{1}{x-y}-\frac{1}{x+y}+\frac{2x}{x^2-y^2}$ (ii) $\frac{2x^2}{x^4-16}-\frac{x}{x^2-4}+\frac{1}{x+2}$

Solution

(i) $\frac{1}{x-y}-\frac{1}{x+y}+\frac{2x}{x^2-y^2}=\frac{1}{x-y}-\frac{1}{x+y}+\frac{2x}{(x+y)(x-y)}$

$$=\frac{x+y-(x-y)+2x}{(x+y)(x-y)} \quad \text{(L.C.M. of denominators)}$$

$$=\frac{x+y-x+y+2x}{(x+y)(x-y)}$$

$$=\frac{2x+2y}{(x+y)(x-y)} \quad \text{(simplifying)}$$

$$=\frac{2(x+y)}{(x+y)(x-y)}=\frac{2}{x-y} \quad \text{(cancelling common factors)}$$

(ii) $\frac{2x^2}{x^4-16}-\frac{x}{x^2-4}+\frac{1}{x+2}$

$$=\frac{2x^2}{(x^2+4)(x^2-4)}-\frac{x}{x^2-4}+\frac{1}{x+2} \quad \text{(difference of two squares)}$$

$$=\frac{2x^2}{(x^2+4)(x+2)(x-2)}-\frac{x}{(x+2)(x-2)}+\frac{1}{x+2}$$

$$=\frac{2x^2-x(x^2+4)+(x^2+4)(x-2)}{(x^2+4)(x+2)(x-2)}=\frac{2x^2-x^3-4x+x^3+4x-2x^2-8}{(x^2+4)(x+2)(x-2)}$$

$$=\frac{-8}{(x^2+4)(x+2)(x-2)} \quad \text{(on simplification)}$$

Example 2

Find the product $\frac{x+2}{2x-3y}\cdot\frac{4x^2-9y^2}{xy+2y}$ (in simplified form)

Solution

$$\frac{x+2}{2x-3y}\cdot\frac{4x^2-9y^2}{xy+2y}=\frac{(x+2)\,[(2x)^2-(3y)^2]}{(2x-3y)\,(x+2)y} \quad \text{(monomial factors)}$$

$$=\frac{(x+2)\,(2x+3y)\,(2x-3y)}{y(x+2)\,(2x-3y)} \quad \text{(factorizing)}$$

$$=\frac{2x+3y}{y} \quad \text{(reduced to the lowest forms)}$$

4.1.8 Dividing a Rational Expression with another Rational Expression

In order to divide one rational expression with another, we first invert for changing division to multiplication and simplify the resulting product to the lowest terms.

Example

Simplify $\frac{7xy}{x^2-4x+4}\div\frac{14y}{x^2-4}$

Solution

$$\frac{7xy}{x^2-4x+4}\div\frac{14y}{x^2-4}$$

$$=\frac{7xy}{x^2-4x+4}\cdot\frac{x^2-4}{14y} \quad \text{...(changing division into multiplication)}$$

$$=\frac{7xy}{(x-2)\,(x-2)}\cdot\frac{(x+2)\,(x-2)}{14y} \quad \text{...(factorizing)}$$

$$=\frac{x(x+2)}{2(x-2)} \quad \text{...(reduced to lowest forms)}$$

4.1.9 Evaluation of Algebraic Expression for some particular Real Number Definition

If specific numbers are substituted for the variables in an algebraic expression, the resulting number is called the value of the expression.

Example

Evaluate $\frac{3x^2\sqrt{y}+6}{5(x+y)}$ if $x = -4$ and $y = 9$

Solution

We have, by putting $x = -4$ and $y = 9$,

$$=\frac{3x^2\sqrt{y}+6}{5(x+y)}=\frac{3(-4)^2\sqrt{9}+6}{5(-4+9)}=\frac{3(16)(3)+6}{5(5)}=\frac{150}{25}=6$$

EXERCISE 4.1

1. Identify whether the following algebraic expressions are polynomials (Yes or No).

(i) $3x^2+\frac{1}{x}-5$ (ii) $3x^3-4x^2-x\sqrt{x}+3$

(iii) $x^2-3x+\sqrt{2}$ (iv) $\frac{3x}{2x-1}+8$

2. State whether each of the following expressions is a rational expression or not.

(i) $\frac{3\sqrt{x}}{3\sqrt{x}+5}$ (ii) $\frac{x^3-2x^2+\sqrt{3}}{2+3x-x^2}$

(iii) $\frac{x^2+6x+9}{x^2-9}$ (iv) $\frac{2\sqrt{x}+3}{2\sqrt{x}-3}$

3. Reduce the following rational expressions to the lowest form.

(i) $\frac{120x^2y^3z^5}{30x^3yz^2}$ (ii) $\frac{8a(x+1)}{2(x^2-1)}$

(iii) $\frac{(x+y)^2-4xy}{(x-y)^2}$ (iv) $\frac{(x^3-y^3)(x^2-2xy+y^2)}{(x-y)(x^2+xy+y^2)}$

(v) $\frac{(x+2)(x^2-1)}{(x+1)(x^2-4)}$ (vi) $\frac{x^2-4x+4}{2x^2-8}$

(vii) $\frac{64x^5-64x}{(8x^2+8)(2x+2)}$ (viii) $\frac{9x^2-(x^2-4)^2}{4+3x-x^2}$

4. Evaluate (a) $\frac{x^3y-2z}{xz}$ for

(i) $x = 3, y = -1, z = -2$ (ii) $x = -1, y = -9, z = 4$

(b) for $x = 4, y = -2, z = -1$

5. Perform the indicated operation and simplify.

(i) $\frac{15}{2x-3y}-\frac{4}{3y-2x}$ (ii) $\frac{1+2x}{1-2x}-\frac{1-2x}{1+2x}$

(iii) $\frac{x^2-25}{x^2-36}-\frac{x+5}{x+6}$ (iv) $\frac{x}{x-y}-\frac{y}{x+y}-\frac{2xy}{x^2-y^2}$

(v) $\frac{x-2}{x^2+6x+9}-\frac{x+2}{2x^2-18}$ (vi) $\frac{1}{x-1}-\frac{1}{x+1}-\frac{2}{x^2+1}-\frac{4}{x^4-1}$

6. Perform the indicated operation and simplify.

(i) $(x^2-49).\frac{5x+2}{x+7}$ (ii) $\frac{4x-12}{x^2-9}\div\frac{18-2x^2}{x^2+6x+9}$

(iii) $\frac{x^6-y^6}{x^2-y^2}\div(x^4+x^2y^2+y^4)$ (iv) $\frac{(x^3-y^3)(x^2-2xy+y^2)}{(x-y)(x^2+xy+y^2)}$

(v) $\frac{x^2-1}{x^2+2x+1}\cdot\frac{x+5}{1-x}$

4.2 Algebraic Formulae

4.2.1 Using the formulas

(i) $(a+b)^2+(a-b)^2 = 2(a^2+b^2)$ and $(a+b)^2-(a-b)^2 = 4ab$

The process of finding the values of a^2+b^2 and abis explained in the following examples.

$\Rightarrow$ $(7)^2-(3)^2 = 4ab$ ....(substituting given values)

$\Rightarrow$ $49-9 = 4ab$

Example

If $a + b = 7$ and $a - b = 3$, then find the value of **(a)** $a^2 + b^2$ **(b)** ab

Solution

We are given that $a + b = 7$ and $a - b = 3$

(a) To find the value of $(a^2 + b^2)$, we use the formula

$$(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$$

Substituting the values $a + b = 7$ and $a - b = 3$, we get

$$(7)^2 + (3)^2 = 2(a^2 + b^2)$$
$$\Rightarrow \quad 49 + 9 = 2(a^2 + b^2)$$
$$\Rightarrow \quad 58 = 2(a^2 + b^2) \quad \text{....(simplifying)}$$
$$\Rightarrow \quad 29 = a^2 + b^2 \quad \text{....(dividing by 2)}$$

(b) To find the value of ab, we make use of the formula

$$(a + b)^2 - (a - b)^2 = 4ab$$
$$(7)^2 - (3)^2 = 4ab$$
$$\Rightarrow \quad 49 - 9 = 4ab$$
$$\Rightarrow \quad 40 = 4ab \quad \text{....(simplifying)}$$
$$\Rightarrow \quad 10 = ab \quad \text{....(dividing by 4)}$$

Hence $a^2 + b^2 = 29$ and $ab = 10$.

(ii) $\mathbf{(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca}$

This formula, square of a trinomial, involves three expressions, namely; $(a + b + c)$, $(a^2 + b^2 + c^2)$ and $2(ab + bc + ca)$. If the values of two of them are known, the value of the third expression can be calculated. The method is explained in the following examples.

Example 1

If $a^2 + b^2 + c^2 = 43$ and $ab + bc + ca = 3$, then find the value of $a + b + c$.

Solution

We know that

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$
$$\Rightarrow \quad (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$

$$\Rightarrow \quad (a + b + c)^2 = 43 + 2 \times 3 \quad (\text{Putting } a^2 + b^2 + c^2 = 43 \text{ and } ab + bc + ca = 3)$$
$$\Rightarrow \quad (a + b + c)^2 = 49$$
$$\Rightarrow \quad a + b + c = \pm\sqrt{49}$$

Hence $a + b + c = \pm 7$

Example 2

If $a + b + c = 6$ and $a^2 + b^2 + c^2 = 24$, then find the value of $ab + bc + ca$.

Solution

We have

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$
$$\therefore \quad (6)^2 = 24 + 2(ab + bc + ca)$$
$$\Rightarrow \quad 36 = 24 + 2(ab + bc + ca)$$
$$\Rightarrow \quad 12 = 2(ab + bc + ca)$$

Hence $ab + bc + ca = 6$

Example 3

If $a + b + c = 7$ and $ab + bc + ca = 9$, then find the value of $a^2 + b^2 + c^2$.

Solution

We know that

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$
$$\Rightarrow \quad (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$
$$\Rightarrow \quad (7)^2 = a^2 + b^2 + c^2 + 2(9)$$
$$\Rightarrow \quad 49 = a^2 + b^2 + c^2 + 18$$
$$\Rightarrow \quad 31 = a^2 + b^2 + c^2$$

Hence $a^2 + b^2 + c^2 = 31$

(iii) $\mathbf{(a + b)^3 = a^3 + 3ab(a + b) + b^3}$

$\mathbf{(a - b)^3 = a^3 - 3ab(a - b) - b^3}$

Example 1

If $2x - 3y = 10$ and $xy = 2$, then find the value of $8x^3 - 27y^3$

Solution

We are given that $2x - 3y = 10$

$\Rightarrow (2x - 3y)^3 = (10)^3$

$\Rightarrow 8x^3 - 27y^3 - 3 \times 2x \times 3y(2x - 3y) = 1000$

$\Rightarrow 8x^3 - 27y^3 - 18xy(2x - 3y) = 1000$

$\Rightarrow 8x^3 - 27y^3 - 18 \times 2 \times 10 = 1000$

$\Rightarrow 8x^3 - 27y^3 - 360 = 1000$

Hence $8x^3 - 27y^3 = 1360$

Example 2

If $x+\frac{1}{x}=8$, then find the value of $x^3+\frac{1}{x^3}$

Solution

We have been given $x+\frac{1}{x}=8$

$\Rightarrow \left(x+\frac{1}{x}\right)^3 = (8)^3$

$\Rightarrow x^3+\frac{1}{x^3}+3\times x\times\frac{1}{x}\left(x+\frac{1}{x}\right) = 512$

$\Rightarrow x^3+\frac{1}{x^3}+3\times\left(x+\frac{1}{x}\right) = 512$

$\Rightarrow x^3+\frac{1}{x^3}+3\times 8 = 512$

$\Rightarrow x^3+\frac{1}{x^3}+24 = 512$

Hence $x^3+\frac{1}{x^3} = 488$

Example 3

If $x-\frac{1}{x}=4$, then find

Solution

We have

$\Rightarrow \left(x-\frac{1}{x}\right)^3 = 64$

$\Rightarrow x^3-\frac{1}{x^3}-3\left(x-\frac{1}{x}\right) = 64$

$\Rightarrow x^3-\frac{1}{x^3}-3(4) = 64$

$\Rightarrow x^3-\frac{1}{x^3}-12 = 64$

$\Rightarrow x^3-\frac{1}{x^3} = 64+12$

$\Rightarrow x^3-\frac{1}{x^3} = 76$

(iv) $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

The procedure for finding the products of $\left(x\pm\frac{1}{x}\right)$ and $x^2+\frac{1}{x^2}\mp 1$ is also explained in following examples.

Example 1

Factorize $64x^3 + 343y^3$

Solution

We have

$64x^3 + 343y^3 = (4x)^3 + (7y)^3$

$= (4x + 7y)[(4x)^2 - (4x)(7y) + (7y)^2]$

$= (4x + 7y)(16x^2 - 28xy + 49y^2)$

Example 2

Factorize $125x^3 - 1331y^3$

Solution

We have

$125x^3 - 1331y^3 = (5x)^3 - (11y)^3$

$= (5x - 11y)\,[(5x)^2 + (5x)(11y) + (11y)^2]$

$= (5x - 11y)(25x^2 + 55xy + 121y^2)$

Example 3

Find the product $\left(\frac{2}{3}x+\frac{3}{2x}\right)\left(\frac{4}{9}x^2-1+\frac{9}{4x^2}\right)$

Solution

$$\left(\frac{2}{3}x+\frac{3}{2x}\right)\left(\frac{4}{9}x^2-1+\frac{9}{4x^2}\right)$$

$$=\left(\frac{2}{3}x+\frac{3}{2x}\right)\left[\left(\frac{2}{3}x\right)^2-\left(\frac{2}{3}x\right)\left(\frac{3}{2x}\right)+\left(\frac{3}{2x}\right)^2\right]$$

$$=\left(\frac{2}{3}x\right)^3+\left(\frac{3}{2x}\right)^3$$

$$=\frac{8}{27}x^3+\frac{27}{8x^3}$$

Example 4

Find the product $\left(\frac{4}{5}x-\frac{5}{4x}\right)\left(\frac{16}{25}x^2+\frac{25}{16x^2}+1\right)$

Solution

$$\left(\frac{4}{5}x-\frac{5}{4x}\right)\left(\frac{16}{25}x^2+\frac{25}{16x^2}+1\right)$$

$$=\left(\frac{4}{5}x-\frac{5}{4x}\right)\left(\frac{16x^2}{25}+1+\frac{25}{16x^2}\right) \qquad \text{(rearranging)}$$

$$=\left(\frac{4}{5}x-\frac{5}{4x}\right)\left[\left(\frac{4}{5}x\right)^2+\left(\frac{4}{5}x\right)\left(\frac{5}{4x}\right)+\left(\frac{5}{4x}\right)^2\right]$$

$$=\left(\frac{4}{5}x\right)^3-\left(\frac{5}{4x}\right)^3=\frac{64}{125}x^3-\frac{125}{64x^3}$$

Example 5

Find the continued product of $(x + y)(x - y)(x^2 + xy + y^2)(x^2 - xy + y^2)$

Solution

$(x + y)(x - y)(x^2 + xy + y^2)(x^2 - xy + y^2)$

$= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$ (rearranging)

$= (x^3 + y^3)(x^3 - y^3) = (x^3)^2 - (y^3)^2 = x^6 - y^6$

EXERCISE 4.2

1. (i) If $a + b = 10$ and $a - b = 6$, then find the value of $(a^2 + b^2)$
   (ii) If $a + b = 5$, $a - b = \sqrt{17}$, then find the value of ab.
2. If $a^2 + b^2 + c^2 = 45$ and $a + b + c = -1$, then find the value of $ab + bc + ca$.
3. If $m + n + p = 10$ and $mn + np + mp = 27$, then find the value of $m^2 + n^2 + p^2$.
4. If $x^2 + y^2 + z^2 = 78$ and $xy + yz + zx = 59$, then find the value of $x + y + z$.
5. If $x + y + z = 12$ and $x^2 + y^2 + z^2 = 64$, then find the value of $xy + yz + zx$.
6. If $x + y = 7$ and $xy = 12$, then find the value of $x^3 + y^3$.
7. If $3x + 4y = 11$ and $xy = 12$, then find the value of $27x^3 + 64y^3$.
8. If $x - y = 4$ and $xy = 21$, then find the value of $x^3 - y^3$.
9. If $5x - 6y = 13$ and $xy = 6$, then find the value of $125x^3 - 216y^3$.
10. If $x+\frac{1}{x}=3$, then find the value of $x^3+\frac{1}{x^3}$
11. If $x-\frac{1}{x}=7$, then find the value of $x^3-\frac{1}{x^3}$
12. If $\left(3x+\frac{1}{3x}\right)=5$, then find the value of $\left(27x^3+\frac{1}{27x^3}\right)$
13. If $\left(5x-\frac{1}{5x}\right)=6$, then find the value of $\left(125x^3-\frac{1}{125x^3}\right)$
14. Factorize (i) $x^3 - y^3 - x + y$ (ii) $8x^3-\frac{1}{27y^3}$

15. Find the products, using formulas.

(i) $(x^2 + y^2)(x^4 - x^2y^2 + y^4)$ (ii) $(x^3 - y^3)(x^6 + x^3y^3 + y^6)$

(iii) $(x - y)(x + y)(x^2 + y^2)(x^2 + xy + y^2)(x^2 - xy + y^2)(x^4 - x^2y^2 + y^4)$

(iv) $(2x^2 - 1)(2x^2 + 1)(4x^4 + 2x^2 + 1)(4x^4 - 2x^2 + 1)$

4.3 Surds and their Application

4.3.1 Definition

An irrational radical with rational radicand is called a surd.

Hence the radical is a surd if

(i) a is rational, (ii) the result is irrational.

e.g., $\sqrt{3}, \sqrt{2/5}, \sqrt[3]{7}, \sqrt[4]{10}$ are surds.

But $\sqrt{\pi}$ and $\sqrt{2+\sqrt{17}}$ are not surds because π and $2+\sqrt{17}$ are not rational.

> Note that for the surd $\sqrt[n]{a}$, n is called surd index or the order of the surd and the rational number 'a' is called the radicand. $\sqrt[3]{7}$ is third order surd.

Every surd is an irrational number but every irrational number is not a surd. e.g., the surd $\sqrt[3]{5}$ is an irrational but the irrational number $\sqrt{\pi}$ is not a surd.

4.3.2 Operations on surds

(a) Addition and Subtraction of Surds

Similar surds (i.e., surds having same irrational factors) can be added or subtracted into a single term is explained in the following examples.

Example

Simplify by combining similar terms.

(i) $4\sqrt{3} - 3\sqrt{27} + 2\sqrt{75}$. (ii) $\sqrt[3]{128} - \sqrt[3]{250} + \sqrt[3]{432}$

Solution

(i) $4\sqrt{3} - 3\sqrt{27} + 2\sqrt{75}$

$= 4\sqrt{3} - 3\sqrt{9\times 3} + 2\sqrt{25\times 3} = 4\sqrt{3} - 3\sqrt{9}\sqrt{3} + 2\sqrt{25}\times\sqrt{3}$

$= 4\sqrt{3} - 9\sqrt{3} + 10\sqrt{3} = (4 - 9 + 10)\sqrt{3} = 5\sqrt{3}$

(ii) $\sqrt[3]{128} - \sqrt[3]{250} + \sqrt[3]{432}$

$= \sqrt[3]{64\times 2} - \sqrt[3]{125\times 2} + \sqrt[3]{216\times 2}$

$= \sqrt[3]{(4)^3\times 2} - \sqrt[3]{(5)^3\times 2} + \sqrt[3]{(6)^3\times 2}$

$= \sqrt[3]{(4)^3}\sqrt[3]{2} - \sqrt[3]{(5)^3}\sqrt[3]{2} + \sqrt[3]{(6)^3}\sqrt[3]{2}$

$= 4\sqrt[3]{2} - 5\sqrt[3]{2} + 6\sqrt[3]{2} = (4 - 5 + 6)\sqrt[3]{2} = 5\sqrt[3]{2}$

(b) Multiplication and Division of Surds

We can multiply and divide surds of the same order by making use of the following laws of surds

$$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

and the result obtained will be a surd of the same order.

If surds to be multiplied or divided are not of the same order, they must be reduced to the surds of the same order.

Example

Simplify and express the answer in the simplest form.

(i) $\sqrt{14}\sqrt{35}$ (ii) $\dfrac{\sqrt[6]{12}}{\sqrt{3}\sqrt[3]{2}}$

Solution

(i) $\sqrt{14}\sqrt{35} = \sqrt{14\times 35} = \sqrt{7\times 2\times 7\times 5} = \sqrt{(7)^2\times 2\times 5}$

$= \sqrt{(7)^2\times 10} = \sqrt{(7)^2}\times\sqrt{10} = 7\sqrt{10}$

(ii) We have $\dfrac{\sqrt[6]{12}}{\sqrt{3}\sqrt[3]{2}}$.

For $\sqrt{3}\sqrt[3]{2}$ the L.C.M. of orders 2 and 3 is 6.

Thus $\sqrt{3} = (3)^{1/2} = (3)^{3/6} = \sqrt[6]{3^3} = \sqrt[6]{27}$

and $\sqrt[3]{2} = (2)^{1/3} = (2)^{2/6} = \sqrt[6]{(2)^2} = \sqrt[6]{4}$

Hence $\dfrac{\sqrt[6]{12}}{\sqrt{3}\sqrt[3]{2}} = \dfrac{\sqrt[6]{12}}{\sqrt[6]{27}\sqrt[6]{4}} = \dfrac{\sqrt[6]{12}}{\sqrt[6]{108}} = \sqrt[6]{\dfrac{12}{108}} = \sqrt[6]{\dfrac{1}{9}}$

Its simplest form is

$$\sqrt[6]{\left(\frac{1}{3}\right)^2} = \left(\frac{1}{3}\right)^{2/6} = \left(\frac{1}{3}\right)^{1/3} = \sqrt[3]{\frac{1}{3}}$$

EXERCISE 4.3

1. Express each of the following surd in the simplest form.
   (i) $\sqrt{180}$ (ii) $3\sqrt{162}$
   (iii) $\frac{3}{4}\sqrt[3]{128}$ (iv) $\sqrt[5]{96x^6y^7z^8}$
2. Simplify
   (i) $\dfrac{\sqrt{18}}{\sqrt{3}\sqrt{2}}$ (ii) $\dfrac{\sqrt{21}\sqrt{9}}{\sqrt{63}}$
   (iii) $\sqrt[5]{243\,x^5\,y^{10}\,z^{15}}$ (iv) $\frac{4}{5}\sqrt[3]{125}$
   (v) $\sqrt{21}\times\sqrt{7}\times\sqrt{3}$
3. Simplify by combining similar terms.
   (i) $\sqrt{45} - 3\sqrt{20} + 4\sqrt{5}$ (ii) $4\sqrt{12} + 5\sqrt{27} - 3\sqrt{75} + \sqrt{300}$
   (iii) $\sqrt{3}\,(2\sqrt{3} + 3\sqrt{3})$ (iv) $2(6\sqrt{5} - 3\sqrt{5})$
4. Simplify
   (i) $(3+\sqrt{3})\,(3-\sqrt{3})$ (ii) $(\sqrt{5}+\sqrt{3})^2$
   (iii) $(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})$ (iv) $\left(\sqrt{2}+\frac{1}{\sqrt{3}}\right)\left(\sqrt{2}-\frac{1}{\sqrt{3}}\right)$

   (v) $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})(x+y)(x^2+y^2)$

4.4 Rationalization of Surds

(a) Definitions

(i) A surd which contains a single term is called a monomial surd. e.g., $\sqrt{2}, \sqrt{3}$ etc.

(ii) A surd which contains sum of two monomial surds or sum of a monomial surd and a rational number is called a binomial surd. e.g., $\sqrt{3}+\sqrt{7}$ or $\sqrt{2}+5$ or $\sqrt{11}-8$ etc.
We can extend this to the definition of a trinomial surd.

(iii) If the product of two surds is a rational number, then each surd is called the rationalizing factor of the other.

(iv) The process of multiplying a given surd by its rationalizing factor to get a rational number as product is called rationalization of the given surd.

(v) Two binomial surds of second order differing only in sign connecting their terms are called conjugate surds. Thus $(\sqrt{a}+\sqrt{b})$ and $(\sqrt{a}-\sqrt{b})$ are conjugate surds of each other.
The conjugate of $x+\sqrt{y}$ is $x-\sqrt{y}$.
The product of the conjugate surds $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$,
$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = (\sqrt{a})^2-(\sqrt{b})^2 = a-b$,
is a rational quantity independent of any radical.
Similarly, the product of $a+b\sqrt{m}$ and its conjugate $a-b\sqrt{m}$ has no radical. For example,
$(3+\sqrt{5})(3-\sqrt{5}) = (3)^2-(\sqrt{5})^2 = 9-5 = 4$, which is a rational number.

(b) Rationalizing a Denominator

Keeping the above discussion in mind, we observe that, in order to rationalize a denominator of the form $a+b\sqrt{x}$ (or $a-b\sqrt{x}$), we multiply both numerator and denominator by the conjugate factor $a-b\sqrt{x}$ (or $a+b\sqrt{x}$). By doing this we eliminate the radical and thus obtain a denominator free of any surd.

(c) Rationalizing Real Numbers of the Types $\frac{1}{a+b\sqrt{x}}, \frac{1}{\sqrt{x}+\sqrt{y}}$

For the expressions $\frac{1}{a+b\sqrt{x}}, \frac{1}{\sqrt{x}+\sqrt{y}}$ and their combinations, where *x, y* are natural numbers and *a, b* are integers, rationalization is explained with the help of following examples.

Example 1

Rationalize the denominator $\frac{58}{7-2\sqrt{5}}$

Solution

To rationalize the denominator, we multiply both the numerator and denominator by the conjugate $(7+2\sqrt{5})$ of $(7-2\sqrt{5})$, i.e.,

$$\frac{58}{7-2\sqrt{5}} = \frac{58}{7-2\sqrt{5}} \times \frac{7+2\sqrt{5}}{7+2\sqrt{5}} = \frac{58(7+2\sqrt{5})}{(7)^2-(2\sqrt{5})^2}$$

$= \frac{58(7+2\sqrt{5})}{49-20}$; (radical is eliminated in the denominator)

$$= \frac{58(7+2\sqrt{5})}{29} = 2(7+2\sqrt{5})$$

Example 2

Rationalize the denominator $\frac{2}{\sqrt{5}+\sqrt{2}}$

Solution

Multiply both the numerator and denominator by the conjugate $\sqrt{5}-\sqrt{2}$ of $\sqrt{5}+\sqrt{2}$, to get

$$\frac{2}{\sqrt{5}+\sqrt{2}} = \frac{2}{\sqrt{5}+\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{2\sqrt{5}-\sqrt{2}}{5-2}$$

$$= \frac{2(\sqrt{5}-\sqrt{2})}{3} = \frac{2(\sqrt{5}-\sqrt{2})}{3}$$

Example 3

Simplify $\frac{6}{2\sqrt{3}-\sqrt{6}} + \frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} - \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}$

Solution

First we shall rationalize the denominators and then simplify. We have

$$\frac{6}{2\sqrt{3}-\sqrt{6}} + \frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} - \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}$$

$$= \frac{6}{2\sqrt{3}-\sqrt{6}} \times \frac{2\sqrt{3}+\sqrt{6}}{2\sqrt{3}+\sqrt{6}} + \frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} \times \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} - \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}} \times \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}}$$

$$= \frac{6(2\sqrt{3}+\sqrt{6})}{(2\sqrt{3})^2-(\sqrt{6})^2} + \frac{\sqrt{6}(\sqrt{3}-\sqrt{2})}{(\sqrt{3})^2-(\sqrt{2})^2} - \frac{4\sqrt{3}(\sqrt{6}+\sqrt{2})}{(\sqrt{6})^2-(\sqrt{2})^2}$$

$$= \frac{6(2\sqrt{3}+\sqrt{6})}{12-6} + \frac{\sqrt{6}(\sqrt{3}-\sqrt{2})}{3-2} - \frac{4\sqrt{3}(\sqrt{6}+\sqrt{2})}{6-2}$$

$$= \frac{12\sqrt{3}+6\sqrt{6}}{6} + \frac{\sqrt{6}\sqrt{3}-\sqrt{6}\sqrt{2}}{1} - \frac{4\sqrt{3}\sqrt{6}+4\sqrt{3}\sqrt{2}}{4}$$

$$= 2\sqrt{3}+\sqrt{6}+3\sqrt{2}-2\sqrt{3}-3\sqrt{2}-\sqrt{6} = 0$$

Example 4

Find rational numbers *x* and *y* such that $\frac{4+3\sqrt{5}}{4-3\sqrt{5}} = x + y\sqrt{5}$

Solution

We have

$$\frac{4+3\sqrt{5}}{4-3\sqrt{5}} = \frac{4+3\sqrt{5}}{4-3\sqrt{5}} \times \frac{4+3\sqrt{5}}{4+3\sqrt{5}} = \frac{(4+3\sqrt{5})^2}{(4)^2-(3\sqrt{5})^2}$$

$$= \frac{16+24\sqrt{5}+45}{16-45} = \frac{61+24\sqrt{5}}{-29}$$

$$\Rightarrow \quad \frac{-61}{29} - \frac{24}{29}\sqrt{5} = x + y\sqrt{5} \qquad \text{(given)}$$

Hence, an comparing the two sides, we get

$$x=-\frac{61}{29},\ y=-\frac{24}{29}$$

Example 5

If $x=3+\sqrt{8}$, then evaluate

(i) $x+\frac{1}{x}$ and (ii) $x^2+\frac{1}{x^2}$

Solution

Since $x=3+\sqrt{8}$, therefore,

$$\frac{1}{x}=\frac{1}{3+\sqrt{8}}=\frac{1}{3+\sqrt{8}}\times\frac{3-\sqrt{8}}{3-\sqrt{8}}=\frac{3-\sqrt{8}}{(3)^2-(\sqrt{8})^2}$$
$$=\frac{3-\sqrt{8}}{9-8}=3-\sqrt{8}$$

(i) $x+\frac{1}{x}=3+\sqrt{8}+3-\sqrt{8}=6$

(ii) $\left(x+\frac{1}{x}\right)^2=36$

or $x^2+2x\times\frac{1}{x}+\frac{1}{x^2}=36$

or $x^2+\frac{1}{x^2}=34$

EXERCISE 4.4

1. Rationalize the denominator of the following.

(i) $\frac{3}{4\sqrt{3}}$ (ii) $\frac{14}{\sqrt{98}}$ (iii) $\frac{6}{\sqrt{8}\sqrt{27}}$ (iv) $\frac{1}{3+2\sqrt{5}}$

(v) $\frac{15}{\sqrt{31}-4}$ (vi) $\frac{2}{\sqrt{5}-\sqrt{3}}$ (vii) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$ (viii) $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$

2. Find the conjugate of $x+\sqrt{y}$.

(i) $3+\sqrt{7}$ (ii) $4-\sqrt{5}$ (iii) $2+\sqrt{3}$ (iv) $2+\sqrt{5}$

(v) $5+\sqrt{7}$ (vi) $4-\sqrt{15}$ (vii) $7-\sqrt{6}$ (viii) $9+\sqrt{2}$

3. (i) If $x=2-\sqrt{3}$, find $\frac{1}{x}$ (ii) If $x=4-\sqrt{17}$, find $\frac{1}{x}$

(iii) If $x=\sqrt{3}+2$, find $x+\frac{1}{x}$

4. Simplify

(i) $\frac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}}+\frac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}$ (ii) $\frac{1}{2+\sqrt{3}}+\frac{2}{\sqrt{5}-\sqrt{3}}+\frac{1}{2+\sqrt{5}}$

(iii) $\frac{2}{\sqrt{5}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{2}}-\frac{3}{\sqrt{5}+\sqrt{2}}$

5. (i) If $x=2+\sqrt{3}$, find the value of $x-\frac{1}{x}$ and $\left(x-\frac{1}{x}\right)^2$

(ii) If $x=\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$, find the value of $x+\frac{1}{x}, x^2+\frac{1}{x^2}$ and $x^3+\frac{1}{x^3}$

[Hint: $a^2+b^2=(a+b)^2-2ab$ and $a^3+b^3=(a+b)^3-3ab(a+b)$]

6. Determine the rational numbers a and b if $\frac{\sqrt{3}-1}{\sqrt{3}+1}+\frac{\sqrt{3}+1}{\sqrt{3}-1}=a+b\sqrt{3}$.

REVIEW EXERCISE 4

1. Multiple Choice Questions. Choose the correct answer.

2. Fill in the blanks.

(i) The degree of the polynomial $x^2y^2+3xy+y^3$ is ....

(ii) $x^2-4=$ ....

(iii) $x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)(.........)$

(iv) $2(a^2+b^2)=(a+b)^2+(.........)^2$

(v) $\left(x-\frac{1}{x}\right)^2=.........$

(vi) Order of surd $\sqrt[3]{x}$ is ....

(vii) $\frac{1}{2-\sqrt{3}}=.........$

3.If $x+\frac{1}{x}=3$ find (i) $x^2+\frac{1}{x^2}$ (ii) $x^4+\frac{1}{x^4}$

4.If $x-\frac{1}{x}=2$ find (i) $x^2+\frac{1}{x^2}$ (ii) $x^4+\frac{1}{x^4}$

5.Find the value of x^3+y^3 and xy if $x+y=5$ and $x-y=3$.

6.If $p=2+\sqrt{3}$ find

(i) $p+\frac{1}{p}$ (ii) $p-\frac{1}{p}$

(iii) $p^2+\frac{1}{p^2}$ (iv) $p^2-\frac{1}{p^2}$

7.If $q=\sqrt{5}+2$, find

(i) $q+\frac{1}{q}$ (ii) $q-\frac{1}{q}$

(iii) $q^2+\frac{1}{q^2}$ (iv) $q^2-\frac{1}{q^2}$

8. Simplifying

(i) $\frac{\sqrt{a^2+2}+\sqrt{a^2-2}}{\sqrt{a^2+2}-\sqrt{a^2-2}}$ (ii) $\frac{1}{a-\sqrt{a^2-x^2}}-\frac{1}{a+\sqrt{a^2-x^2}}$

SUMMARY

- An algebraic expression is that in which constants or variables or both are combined by basic operations.
- Polynomial means an expression with many terms.
- Degree of polynomial means highest power of variable.

- Expression in the form $\frac{p(x)}{q(x)}$, $(q(x)\neq 0)$ is called rational expression.
- An irrational radical with rational radicand is called a surd.
- In $\sqrt[n]{x}$, n is called surd index or surd order and rational number x is called radicand.
- A surd which contains a single term is called monomial surd.
- A surd which contains sum or difference of two surds is called binomial surd.
- Conjugate surd of is $\sqrt{x}+\sqrt{y}$ defined as $\sqrt{x}-\sqrt{y}$.

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